ON PRIME IDEALS OF GROUPOIDS-ORDERED GROUPOIDS

Niovi Kehayopulu and Michael Tsingelis

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Abstract. In a groupoid (resp. ordered groupoid) $G$, the non-empty intersection of the elements of a chain of prime ideals, is a prime ideal of $G$. As a consequence, each prime ideal of a groupoid (resp. ordered groupoid) $G$ containing a non-empty subset $K$ of $G$, contains a prime ideal $P^*$ of $S$ having the property: If $T$ is a prime ideal of $G$ such that $K \subseteq T \subseteq P^*$, then $T = P^*$. As a result, in a groupoid (resp. ordered groupoid) $G$ with zero, each prime ideal of $G$ contains a minimal prime ideal of $G$. Some further results on prime ideals of groupoids (resp. ordered groupoids) are also given.

If $(G, \cdot, \leq)$ is an ordered groupoid, a non-empty subset $I$ of $G$ is called an ideal of $G$ if $1) G1 \subseteq I$ and $IG \subseteq I$ and $2) a \in I$, $G \ni b \leq a$ implies $b \in I$ [2]. If $G$ is a groupoid, an ideal of $G$ is a non-empty subset $I$ of $G$ such that $GI \subseteq I$ and $IG \subseteq I$. An ideal $I$ of a groupoid (resp. ordered groupoid) $G$ is called prime if $a, b \in G$ such that $ab \in I$ implies $a \in I$ or $b \in I$. Equivalent Definition: $A, B \subseteq G$ such that $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$ [2]. If $(G, \cdot, \leq)$ is an ordered groupoid, a zero of $G$ is an element $0$ of $G$ such that $0x = x0 = 0$ and $0 \leq x$ for every $x \in G$ [1]. If $G$ is a groupoid, a zero of $G$ is an element $0$ of $G$ such that $0x = x0 = 0$ for every $x \in G$.

1. Each prime ideal of $G$ contains a minimal prime ideal of $G$

Proposition 1. Let $G$ be a groupoid (resp. ordered groupoid) and $B$ a chain (under set inclusion) of prime ideals of $G$. If the intersection $\cap \{B \mid B \in B\}$ is non-empty, then it is a prime ideal of $G$.

Proof. Since $\cap \{B \mid B \in B\} \neq \emptyset$, the set $\cap \{B \mid B \in B\}$ is an ideal of $G$. Let $a, b \in G$, $ab \in \cap \{B \mid B \in B\}$, $a \notin \cap \{B \mid B \in B\}$ and $b \notin \cap \{B \mid B \in B\}$. Let $B_1, B_2 \in B$ such that $a \notin B_1$ and $b \notin B_2$. Since $ab \in B_1$, $a \notin B_1$ and $b \notin B_2$. Since $ab \in B_1$, $b \notin B_2$ and $b \notin B_2$ prime, we have $a \notin B_2$. Since $b \in B_1$ and $b \notin B_2$, we have $B_1 \not\subseteq B_2$. Then, since $B$ is a chain, we have $B_2 \not\subseteq B_1$. Then $a \in B_1$. Impossible.

Proposition 2. Let $G$ be a groupoid (resp. ordered groupoid), $\emptyset \neq K \subseteq G$ and $P$ a prime ideal of $G$ such that $K \subseteq P$. Then, there exists a prime ideal $P^*$ of $G$ having the properties:

1) $P^* \subseteq P$.

2) For each prime ideal $T$ of $G$ such that $K \subseteq T \subseteq P^*$, we have $T = P^*$.

Proof. Let $A := \{A \mid A$ prime ideal of $G, K \subseteq A \subseteq P\}$.

Since $P \in A$, we have $A \neq \emptyset$.

Then, the set $A$ with the relation $\leq$ on $A$ defined by:

$\geq := \{(A, B) \in A \times A \mid B \subseteq A\}$

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is an ordered set.

Let $(B, \subseteq)$ be a chain in $\mathcal{A}$. The set $\cap\{ B \mid B \in B \}$ is an upper bound of $B$ in $\mathcal{A}$. In fact:

The set $\cap\{ B \mid B \in B \}$ is a prime ideal of $G$. Indeed:

$B$ is a prime ideal of $G$ for every $B \in \mathcal{A} \supseteq B$. Since $K \subseteq B$ for every $B \in \mathcal{A} \supseteq B$, we have $\emptyset \neq K \subseteq \cap\{ B \mid B \in B \}$, then $\{ B \mid B \in B \} \neq \emptyset$.

The set $B$ is a chain of prime ideals of $G$ and $\cap\{ B \mid B \in B \} \subseteq P$. By Proposition 1, the set $\cap\{ B \mid B \in B \}$ is a prime ideal of $G$.

Moreover, $K \subseteq \cap\{ B \mid B \in B \} \subseteq P$. Indeed: Since $K \subseteq B \subseteq P$ for every $B \in \mathcal{A} \supseteq B$, we have $K \subseteq \cap\{ B \mid B \in B \} \subseteq P$.

By Zorn’s Lemma, the set $\mathcal{A}$ has a maximal element, say $P^*$. For the set $P^*$, we have the following:

1) $P^* \subseteq P$ (since $P^* \in \mathcal{A}$).

2) Let $T$ be a prime ideal of $G$ such that $K \subseteq T \subseteq P^*$ ($\Rightarrow T = P^*$ ?)

Since $K \subseteq T \subseteq P^*$, we have $K \subseteq T \subseteq P$ (by 1)). Then, since $T$ is a prime ideal of $G$, we have $T \in \mathcal{A}$. Since $T, P^* \in \mathcal{A}, T \subseteq P^*$, we have $P^* \not\subseteq T$. Since $P^* \subseteq T \in \mathcal{A}$ and $P^*$ is a maximal in $\mathcal{A}$, we have $P^* = T$. \hspace{1cm} \Box

Let $G$ be a groupoid (resp. ordered groupoid). A prime ideal $P$ of $G$ is called a minimal prime ideal of $G$ if

For every prime ideal $T$ of $G$ such that $T \subseteq P$, we have $T = P$.

**Proposition 3.** Let $G$ be a groupoid (resp. ordered groupoid) with zero and $P$ a prime ideal of $G$. Then there exists a minimal prime ideal $P^*$ of $G$ such that $P^* \subseteq P$.

**Proof.** The set $P$ is a prime ideal of $G$ and $\{0\} \subseteq P$. By Proposition 2, there exists a prime ideal $P^*$ of $G$ having the properties:

1) $P^* \subseteq P$.

2) For each prime ideal $T$ of $G$ such that $\{0\} \subseteq T \subseteq P^*$, we have $T = P^*$.

The set $P^*$ is a minimal prime ideal of $G$. In fact: Let $T$ be a prime ideal of $G$ such that $T \subseteq P^*$. Since $T$ is an ideal of $G$, we have $\{0\} \subseteq T$. Then $\{0\} \subseteq T \subseteq P^*$. Then, by 2), $T = P^*$.

2. Some further remarks on prime ideals

Let $G$ be a groupoid (resp. ordered groupoid) with 0. We say that $G$ does not contain divisors of zero if

$a, b \in G, \ ab = 0 \ implies \ a = 0 \ or \ b = 0$.

**Remark 1.** A groupoid (resp. ordered groupoid) $G$ does not contain divisors of zero if and only if the set $\{0\}$ is a prime ideal of $G$.

**Lemma 1.** Let $G$ be a groupoid (resp. ordered groupoid) and $\{I_i \mid i \in I\}$ a (non-empty) family of ideals of $G$. Then the set $\bigcup_{i \in I} I_i$ is an ideal of $G$. \hspace{1cm} \Box

When we speak about a family, we always consider that it is non-empty.

**Proposition 4.** Let $G$ be a groupoid (resp. ordered groupoid) and $\{P_i \mid i \in I\}$ a family of prime ideals of $G$. Then the set $\bigcup_{i \in I} P_i$ is a prime ideal of $G$. 

Proof. By Lemma 1, the set \( \bigcup_{i \in I} P_i \) is an ideal of \( G \). Let \( a, b \in G \), \( ab \in \bigcup_{i \in I} P_i \). Let \( j \in I \) such that \( ab \in P_j \). Since \( P_j \) is prime, we have \( a \in P_j \subseteq \bigcup_{i \in I} P_i \) or \( b \in P_j \subseteq \bigcup_{i \in I} P_i \).

Lemma 2. Let \( G \) be a groupoid (resp. ordered groupoid) and \( \{ I_i \mid i \in I \} \) a family of ideals of \( G \). If \( \bigcap_{i \in I} I_i \neq \emptyset \), then the set \( \bigcap_{i \in I} I_i \) is an ideal of \( G \).

Corollary 1. Let \( S \) be a semigroup (resp. ordered semigroup). If \( I_i \) is an ideal of \( S \) for every \( i = 1, 2, \ldots, n \), then the set \( \bigcap_{i=1}^n I_i \) is an ideal of \( S \).

Proof. By Lemma 2, it is enough to prove that \( \bigcap_{i=1}^n I_i \neq \emptyset \). We have \( \emptyset \neq I_i \subseteq S \) for every \( i = 1, 2, \ldots, n \), so \( I_1 I_2 \cdots I_n \neq \emptyset \). Since \( I_i \) is an ideal of \( S \), we have \( I_1 I_2 \cdots I_n \subseteq I_i \) for every \( i = 1, 2, \ldots, n \), then \( I_1 I_2 \cdots I_n \subseteq \bigcap_{i=1}^n I_i \). Hence we have \( \bigcap_{i=1}^n I_i \neq \emptyset \). \( \square \)

As in Corollary 1, we prove the

Corollary 2. Let \( G \) be a groupoid (resp. ordered groupoid). If \( I_1, I_2 \) are ideals of \( G \), then the set \( I_1 \cap I_2 \) is an ideal of \( G \).

Proposition 5. Let \( G \) be a groupoid (resp. ordered groupoid), \( P_1, P_2 \) ideals of \( G \) such that \( P_1 \cap P_2 \) be a prime ideal of \( G \). Then \( P_1 \subseteq P_2 \) or \( P_2 \subseteq P_1 \).

Proof. Let \( P_1 \nsubseteq P_2 \) and let \( a \in P_2 \). Let \( b \in P_1 \) and \( b \notin P_2 \). We have \( ab \in P_2 G \subseteq P_2 \), \( ab \in GP_1 \subseteq P_1 \). Then \( ab \in P_1 \cap P_2 \). Since \( P_1 \cap P_2 \) is prime, we have \( a \in P_1 \cap P_2 \) or \( b \in P_1 \cap P_2 \). Since \( b \notin P_2 \), we have \( b \notin P_1 \cap P_2 \). Then \( a \in P_1 \cap P_2 \) and \( a \in P_1 \).

Remark 2. Let \( G \) be a groupoid (resp. ordered groupoid), \( P_1 \) a prime ideal of \( G \) and \( P_2 \) an ideal of \( G \) such that \( P_1 \subseteq P_2 \). Then the set \( P_1 \cap P_2 \) is a prime ideal of \( G \). \( \square \)

By Proposition 5 and Remark 2, we have the following:

Proposition 6. Let \( G \) be a groupoid (resp. ordered groupoid), \( P_1, P_2 \) prime ideals of \( G \). The following are equivalent:
1) \( P_1 \subseteq P_2 \) or \( P_2 \subseteq P_1 \).
2) \( P_1 \cap P_2 \) is a prime ideal of \( G \).

Proposition 7. Let \( G \) be a groupoid (resp. ordered groupoid), \( \{ P_i \mid i \in I \} \) a family of prime ideals of \( G \) which is a chain. If \( \bigcap_{i \in I} P_i \neq \emptyset \), then the set \( \bigcap_{i \in I} P_i \) is a prime ideal of \( G \).

Proof. By Lemma 2, the set \( \bigcap_{i \in I} P_i \) is an ideal of \( G \). Let \( a, b \in G \), \( ab \in \bigcap_{i \in I} P_i \), \( a \notin \bigcap_{i \in I} P_i \) and \( b \notin \bigcap_{i \in I} P_i \). Let \( j, k \in I \) such that \( a \notin P_j \) and \( b \notin P_k \).
We have $P_j \subseteq P_k$ or $P_k \subseteq P_j$.
Let $P_j \subseteq P_k$. Since $ab \in P_j$, $P_j$ prime and $a \not\in P_j$, we have $b \in P_j \subseteq P_k$.
Impossible.
The case $P_k \subseteq P_j$ is also impossible.

**Proposition 8.** Let $(S, \cdot, \leq)$ be an ordered semigroup, $I$ an ideal of $S$ and $P$ a prime ideal of $I$. Then $P$ is an ideal of $S$.

**Proof.** First of all, $\emptyset \neq P \subseteq I \subseteq S$.
Let $a \in S$, $b \in P$. Since $b \in I$, we have $ab \in SI \subseteq I$, $aba \in IS \subseteq I$, $(ab)^2 = (aba)b \in IP \subseteq P$. Since $P$ is prime, we have $ab \in P$. Similarly, $PS \subseteq P$.
Let $a \in P$, $S \ni b \leq a$. Since $S \ni b \leq a \in I$, $I$ an ideal of $S$, we have $b \in I$. Since $I \ni b \leq a \in P$, $P$ an ideal of $I$, we have $b \in P$. □

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**References**


University of Athens, Department of Mathematics,
*Mailing (home) address*: Niovi Kehayopulu, Nikomidas 18, 161 22 Kesariani, Greece
e-mail: nkehayop@cc.uoa.gr