ON THE HYERS-ULAM STABILITY OF A DIFFERENTIABLE MAP

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Abstract. We consider a differentiable map \( f \) from an open interval \( I \) to a uniformly closed linear subspace \( A \) of \( C(X) \), the Banach space of all complex-valued bounded continuous functions on a topological space \( X \). Let \( \varepsilon \) be a non-negative real number, \( \lambda \) a complex number so that \( \Re \lambda \neq 0 \). Then we show that \( f \) can be approximated by the solution to \( A \)-valued differential equation \( x' (t) = \lambda x(t) \), if \( \| f'(t) - \lambda f(t) \| \leq \varepsilon \) holds for every \( t \in I \).

1. Introduction

In this paper, \( I \) denotes an open interval of the real number field \( \mathbb{R} \), unless the contrary is explicitly stated. That is \( I = (a, b) \) for some \(-\infty \leq a < b \leq +\infty \). The letters \( \varepsilon \) and \( \lambda \) denote a non-negative real number and a complex number, respectively. Let \( X \) be a topological space, \( C(X) \) a Banach space of all complex-valued bounded continuous functions on \( X \) with respect to the pointwise operations and the supremum norm \( \| \cdot \|_\infty \) on \( X \). Throughout this paper, \( A \) denotes a uniformly closed linear subspace of \( C(X) \).

Definition 1.1. Let \( B \) be a Banach space. If a map from \( I \) into \( B \). We say that \( f \) is differentiable, if for every \( t \in I \) there exists an \( f'(t) \in B \) so that

\[
\lim_{s \to 0} \left\| \frac{f(t + s) - f(t) - f'(t) s}{s} \right\|_B = 0,
\]

where \( \| \cdot \|_B \) denotes the norm on \( B \).

Let \( f \) be a differentiable function on \( I \) into \( \mathbb{R} \). Alsina and Ger [1] gave all the solutions to the inequality \( |f'(t) - f(t)| \leq \varepsilon \) for every \( t \in I \). Then they showed that each solution to the inequality above was approximated by a solution to the differential equation \( x'(t) = x(t) \). In accordance with [1], we define the Hyers-Ulam stability of Banach space valued differentiable map:

Definition 1.2. Let \( B \) be a Banach space, \( f \) a differentiable map on \( I \) into \( B \) so that

\[
\| f'(t) - \lambda f(t) \|_B \leq \varepsilon, \quad (t \in I).
\]

We say that the Hyers-Ulam stability holds for \( f \), if there exist a \( k \geq 0 \) and a differentiable map \( x \) on \( I \) into \( B \) such that

\[
x'(t) = \lambda x(t) \text{ and } \| f(t) - x(t) \|_B \leq k \varepsilon
\]

holds for every \( t \in I \).

Let \( C(X, \mathbb{R}) \) be the Banach space of all real-valued bounded continuous functions on \( X \) and \( C_0(X, \mathbb{R}) \) the Banach space of all functions of \( C(X, \mathbb{R}) \) which vanish at infinity. Let \( r \) be a non-zero real number. In [2], we considered a differentiable map \( f \) on \( I \) into \( C(X, \mathbb{R}) \).

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(resp. \( C_d(X, \mathbb{R}) \)) with the inequality \( \| f'(t) - r f(t) \|_\infty \leq \varepsilon \). Then we showed that the Hyers-Ulam stability held for \( f \). That is, \( f \) can be approximated by a solution to \( C(X, \mathbb{R}) \) (resp. \( C_d(X, \mathbb{R}) \)) valued differential equation \( x'(t) = rx(t) \).

In this paper, we consider a differentiable map \( f \) on \( I \) into \( A \) so that the inequality \( \| f'(t) - \lambda f(t) \|_\infty \leq \varepsilon \) holds for every \( t \in I \). Unless \( \text{Re} \lambda = 0 \), we show that the Hyers-Ulam stability holds for \( f \). If \( \text{Re} \lambda = 0 \), we give an example so that the Hyers-Ulam stability does not hold. Also we consider the Hyers-Ulam stability of an entire function.

2. Preliminaries

We give a characterization of the inequality \( \| f'(t) - \lambda f(t) \| \leq \varepsilon \).

**Proposition 2.1.** Let \( B \) be a Banach space, \( f \) a differentiable map on \( I \) into \( B \). Then the following conditions are equivalent.

(i) \( \| f'(t) - \lambda f(t) \|_B \leq \varepsilon \), \( (t \in I) \).

(ii) There exists a differentiable map \( g \) on \( I \) into \( B \) such that \( f(t) = g(t)e^{\lambda t} \) and \( \| g'(t) \|_B \leq \varepsilon e^{-(\text{Re} \lambda)t} \), for every \( t \in I \).

**Proof.** (i) \( \Rightarrow \) (ii) Put \( g(t) = f(t)e^{-\lambda t} \) for every \( t \in I \). Then we see that \( g \) is differentiable and

\[ g'(t) = \{ f'(t) - \lambda f(t) \} e^{-\lambda t}, \quad (t \in I). \]

By hypothesis, we have the inequality

\[ \| g'(t) \|_B \leq \varepsilon e^{-(\text{Re} \lambda)t} \]

for every \( t \in I \).

(ii) \( \Rightarrow \) (i) If \( f(t) = g(t)e^{\lambda t} \), we have

\[ f'(t) = \{ g'(t) + \lambda g(t) \} e^{\lambda t} = g'(t)e^{\lambda t} + \lambda f(t) \]

for every \( t \in I \). Since \( \| g'(t) \|_B \leq \varepsilon e^{-(\text{Re} \lambda)t} \),

\[ \| f'(t) - \lambda f(t) \|_B \leq \varepsilon \]

holds for every \( t \in I \).

In particular, if we consider the case where \( \varepsilon = 0 \), then we have a solution of Banach space valued differential equation \( f'(t) = \lambda f(t) \). For the completeness we give a proof.

**Proposition 2.2.** Let \( B \) be a Banach space, \( f \) a differentiable map on \( I \) into \( B \). Then the following conditions are equivalent.

(i) \( f'(t) = \lambda f(t) \), \( (t \in I) \).

(ii) There exists \( a \in B \) so that \( f(t) = ae^{\lambda t} \), \( (t \in I) \).

**Proof.** It is enough to show that the map \( g(t) \) given in the condition (ii) of Proposition 2.1 is constant, if \( g'(t) = 0 \) for every \( t \in I \). Fix any \( t_0 \in I \), then we define the function \( \hat{g} \) on \( I \) into \( \mathbb{R} \) as

\[ \hat{g}(t) = \| g(t) - g(t_0) \|_B, \quad (t \in I). \]

We see that \( \hat{g} \) is differentiable and \( \hat{g}'(t) = 0 \) for every \( t \in I \), since \( g'(t) = 0 \). Therefore, \( \hat{g} \) is a constant function. Since \( \hat{g}(t_0) = 0 \), we have \( g(t) = g(t_0) \). Thus \( g(t) \) is a constant function and this completes the proof.
3. ONE POINT CASE

The results below are proved in case where $\text{Re}\lambda > 0$, while corresponding ones hold in case where $\text{Re}\lambda < 0$ and we omit them. In this section we consider the case where $X$ is a singleton. In Lemma 3.1 and 3.2, $g$ denotes a differentiable function on $I$ into $\mathbb{C}$ so that

$$|g'(t)| \leq \varepsilon e^{-(\text{Re}\lambda)t}$$

for every $t \in I$. Let $u$ and $v$ be the real part and the imaginary part of $g$, respectively. Unless $\text{Re}\lambda = 0$, we define the functions $\hat{u}$ and $\hat{v}$ on $I$ into $\mathbb{C}$ as

$$\hat{u}(t) = u(t) - \frac{\varepsilon}{\text{Re}\lambda} e^{-(\text{Re}\lambda)t},$$

$$\hat{v}(t) = v(t) - \frac{\varepsilon}{\text{Re}\lambda} e^{-(\text{Re}\lambda)t}.$$  

**Lemma 3.1.** Let $\text{Re}\lambda \neq 0$ and $t_0 \in I$. Then we have the inequalities

$$0 \leq \hat{u}(s) - \hat{u}(t_0) \leq \frac{2\varepsilon}{\text{Re}\lambda} \left\{ e^{-(\text{Re}\lambda) t_0} - e^{-(\text{Re}\lambda) s} \right\},$$

$$0 \leq \hat{v}(s) - \hat{v}(t_0) \leq \frac{2\varepsilon}{\text{Re}\lambda} \left\{ e^{-(\text{Re}\lambda) t_0} - e^{-(\text{Re}\lambda) s} \right\}$$

for every $s \in I$ with $t_0 \leq s$.

**Proof.** Since $g'(t) = u'(t) + iv'(t)$, we have

$$|u'(t)|, |v'(t)| \leq |g'(t)| \leq \varepsilon e^{-(\text{Re}\lambda)t}$$

for every $t \in I$. By definition,

$$\hat{u}'(t) = u'(t) + \varepsilon e^{-(\text{Re}\lambda)t}, \quad (t \in I).$$

Hence, we obtain the inequality

$$0 \leq \hat{u}'(t) \leq 2\varepsilon e^{-(\text{Re}\lambda)t}$$

for every $t \in I$. We define the function $U$ on $I$ into $\mathbb{C}$ as

$$U(s) = \hat{u}(s) - \frac{2\varepsilon}{\text{Re}\lambda} e^{-(\text{Re}\lambda)s} + \hat{u}(t_0) + \frac{2\varepsilon}{\text{Re}\lambda} e^{-(\text{Re}\lambda)t_0}, \quad (s \in I).$$

Then $U$ is differentiable and

$$U'(s) = -\hat{u}'(s) + 2\varepsilon e^{-(\text{Re}\lambda)s} \geq 0$$

for every $s \in I$. Since $U(t_0) = 0$, we have $U(s) \geq 0$ if $s \geq t_0$. Since $\hat{u}'(s) \geq 0$, the inequality $\hat{u}(t_0) \leq \hat{u}(s)$ holds if $t_0 \leq s$. Therefore, we have

$$0 \leq \hat{u}(s) - \hat{u}(t_0) \leq \frac{2\varepsilon}{\text{Re}\lambda} \left\{ e^{-(\text{Re}\lambda)t_0} - e^{-(\text{Re}\lambda)s} \right\},$$

if $t_0 \leq s$. In a way similar to the above, we see that

$$0 \leq \hat{v}(s) - \hat{v}(t_0) \leq \frac{2\varepsilon}{\text{Re}\lambda} \left\{ e^{-(\text{Re}\lambda)t} - e^{-(\text{Re}\lambda)s} \right\}$$

holds, if $t_0 \leq s$ and a proof is omitted. \(\square\)

**Lemma 3.2.** Let $\text{Re}\lambda > 0$, then both $\lim_{s \to t, s \neq t} \hat{u}(s)$ and $\lim_{s \to t, s \neq t} \hat{v}(s)$ exist.
Proof. As a first step, we show that sup_{t \in I} \hat{u}(t) is finite. To this end fix any t_0 \in I, then by Lemma 3.1 we have the inequality
\[
\hat{u}(t) \leq \hat{u}(t_0) + \frac{2\varepsilon}{\text{Re} \lambda} \left\{ e^{-(\text{Re} \lambda) t_0} - e^{-(\text{Re} \lambda) t} \right\}
\]
< \hat{u}(t_0) + \frac{2\varepsilon}{\text{Re} \lambda} e^{-(\text{Re} \lambda) t_0},
if t_0 \leq t. Since \hat{u}'(t) \geq 0 for every t \in I, we obtain \hat{u}(t) \leq \hat{u}(t_0) if t < t_0. Therefore,
\[
\hat{u}(t) \leq \hat{u}(t_0) + \frac{2\varepsilon}{\text{Re} \lambda} e^{-(\text{Re} \lambda) t_0}
\]
holds for every t \in I. Thus sup_{t \in I} \hat{u}(t) is finite.

Next we show that \lim_{s \to \sup_{t \in I} \hat{u}} \hat{u}(s) = \sup_{t \in I} \hat{u}(t). In fact, for every \eta > 0 there exists an
s_0 \in I such that sup_{t \in I} \hat{u}(t) - \eta < \hat{u}(s_0). Since \hat{u}'(t) \geq 0 for every t \in I, we have
\[
\sup_{t \in I} \hat{u}(t) - \eta < \hat{u}(s) < \sup_{t \in I} \hat{u}(t) + \eta,
if s_0 \leq s. Therefore,
\[
\lim_{s \to \sup_{t \in I} \hat{u}} \hat{u}(s) = \sup_{t \in I} \hat{u}(t)
\]
holds. In a way similar to the above, we see that \lim_{s \to \sup_{t \in I} \hat{u}} \hat{u}(s) = \sup_{t \in I} \hat{u}(t) and a proof
is omitted. □

**Theorem 3.3.** Let \text{Re} \lambda > 0, f a differentiable function on I into \mathbb{C} so that
\[
|f'(t) - \lambda f(t)| \leq \varepsilon, \quad (t \in I).
\]
Then there exists a \theta \in \mathbb{C} such that
\[
|f(t) - \theta e^{\lambda t}| \leq \frac{\sqrt{2}\varepsilon}{\text{Re} \lambda}
\]
holds for every t \in I.

Proof. By Proposition 2.1, there exists a differentiable function g on I into \mathbb{C} such that
\[
f(t) = g(t)e^{\lambda t} \quad \text{and} \quad |g'(t)| \leq \varepsilon e^{-(\text{Re} \lambda) t}, \quad (t \in I).
\]
Let u and v be the real part and the imaginary part of g, respectively. We define the functions on I into \mathbb{C} as
\[
\hat{u}(t) = u(t) - \frac{\varepsilon}{\text{Re} \lambda} e^{-(\text{Re} \lambda) t},
\]
\[
\hat{v}(t) = v(t) - \frac{\varepsilon}{\text{Re} \lambda} e^{-(\text{Re} \lambda) t}.
\]
Then we see that both \lim_{s \to \sup_{t \in I} \hat{u}(t) and \lim_{s \to \sup_{t \in I} \hat{v}(t)} exist, by Lemma 3.2. Note that
for every t \in I we have
\[
0 \leq \hat{u}(s) - \hat{u}(t) < \frac{2\varepsilon}{\text{Re} \lambda} e^{-(\text{Re} \lambda) t},
\]
if t \leq s, by Lemma 3.1. Therefore, we obtain the inequality
\[
\left| u(t) - \lim_{s \to \sup_{t \in I} \hat{u}(s) \right| = \lim_{s \to \sup_{t \in I} \hat{u}(s) \left| \hat{u}(t) + \frac{\varepsilon}{\text{Re} \lambda} e^{-(\text{Re} \lambda) t} - \hat{u}(s) \right| \leq \frac{\varepsilon}{\text{Re} \lambda} e^{-(\text{Re} \lambda) t}
\]
for every t \in I. In a way similar to the above, we see that
\[
\left| v(t) - \lim_{s \to \sup_{t \in I} \hat{v}(s) \right| \leq \frac{\varepsilon}{\text{Re} \lambda} e^{-(\text{Re} \lambda) t}, \quad (t \in I).
\]
Therefore, we have the inequality
\[
\left| f(t) - \lim_{s \to \text{sup} I} \left\{ \tilde{u}(t) + i \tilde{v}(t) \right\} e^{\lambda t} \right| \\
= \sqrt{\left\{ u(t) - \lim_{s \to \text{sup} I} \tilde{u}(s) \right\}^2 + \left\{ v(t) - \lim_{s \to \text{sup} I} \tilde{v}(s) \right\}^2} e^{\text{Re} \lambda t} \\
\leq \frac{\sqrt{2} \varepsilon}{\text{Re} \lambda} e^{-(\text{Re} \lambda) t} e^{(\text{Re} \lambda) t} = \frac{\sqrt{2} \varepsilon}{\text{Re} \lambda}
\]
for every \( t \in I \). This completes the proof. \( \square \)

4. General Case

In this section we consider the case where \( X \) is any topological space.

**Theorem 4.1.** Let \( \text{Re} \lambda > 0 \), \( f \) a differentiable map on \( I \) into \( A \) so that
\[
\| f'(t) - \lambda f(t) \|_{\infty} \leq \varepsilon, \quad (t \in I).
\]
If \( A \) has constant functions, then there exists a \( \theta \in A \) such that
\[
\| f(t) - \theta e^{\lambda t} \|_{\infty} \leq \frac{\sqrt{2} \varepsilon}{\text{Re} \lambda}
\]
holds for every \( t \in I \). Unless \( A \) has constant functions, then there exists a \( \hat{\theta} \in A \) such that
\[
\| f(t) - \hat{\theta} e^{\lambda t} \|_{\infty} \leq \frac{2\sqrt{2} \varepsilon}{\text{Re} \lambda}
\]
for every \( t \in I \).

**Proof.** For every \( x \in X \) we define the induced function \( f_x \) on \( I \) into \( \mathbb{C} \) as
\[
f_x(t) = f(t)(x), \quad (t \in I).
\]
Then \( f_x \) is a differentiable function, and for every \( x \in X \)
\[
(f_x)'(t) = f'(t)(x), \quad (t \in I)
\]
holds, by definition. Therefore, for every \( x \in X \) we see that
\[
\| (f_x)'(t) - \lambda f_x(t) \| \leq \| f'(t) - \lambda f(t) \|_{\infty} \leq \varepsilon, \quad (t \in I).
\]
By Proposition 2.1, for every \( x \in X \) there corresponds a differentiable function \( g_x \) on \( I \) into \( \mathbb{C} \) such that
\[
f_x(t) = g_x(t) e^{\lambda t} \quad \text{and} \quad |(g_x)'(t)| \leq \varepsilon e^{-(\text{Re} \lambda) t}
\]
for every \( t \in I \). Let \( u_x \) and \( v_x \) be the real part and the imaginary part of \( g_x \), respectively. We define the functions on \( I \) into \( \mathbb{C} \) as
\[
u_x(t) = u_x(t) - \frac{\varepsilon}{\text{Re} \lambda} e^{-(\text{Re} \lambda) t}, \\
v_x(t) = v_x(t) - \frac{\varepsilon}{\text{Re} \lambda} e^{-(\text{Re} \lambda) t}.
\]
By the proof of Theorem 3.3, for every \( x \in X \) we have
\[
\left| f_x(t) - \lim_{s \to \text{sup} I} \{ u_x(s) + i v_x(s) \} e^{\lambda t} \right| \leq \frac{\sqrt{2} \varepsilon}{\text{Re} \lambda}, \quad (t \in I).
\]
We define the function \( \theta \) on \( X \) into \( \mathbb{C} \) as
\[
\theta(x) = \lim_{s \to \text{sup} I} \{ u_x(s) + i v_x(s) \}.
\]
By definition, the inequality
\[ \| f(t) - \theta e^{\lambda t} \|_{\infty} \leq \frac{\sqrt{\tau} \varepsilon}{\Re \lambda} \]
holds for every \( t \in I \).

Let \( \{ t_n \} \) be a sequence of \( I \) so that \( t_n \not\to \sup I \). Then we define the function \( \theta_n \) on \( X \) into \( \mathbb{C} \) as
\[ \theta_n(x) = \hat{u}_x(t_n) + i \hat{v}_x(t_n), \quad (x \in X). \]
Since \( g_x(t_n) = f_x(t_n) e^{-\lambda t_n} \), we see that the function \( x \mapsto g_x(t_n) \) belongs to \( A \) for every \( n \in \mathbb{N} \).

We show that \( \theta \) is an element of \( A \), if \( A \) has constant functions. In fact, \( \theta_n \) is an element of \( A \) for every \( n \in \mathbb{N} \) by the definition of \( \hat{u}_x \) and \( \hat{v}_x \). Note that
\[ |\hat{u}_x(s) - \hat{u}_x(t)|, |\hat{v}_x(s) - \hat{v}_x(t)| \leq \frac{2\varepsilon}{\Re \lambda} |e^{-(\Re \lambda) s} - e^{-(\Re \lambda) t}|, \]
if \( t \leq s \), by Lemma 3.1. Therefore, we have
\[ \| \theta(x) - \theta_n(x) \| = \lim_{s \not\to \sup I} \sqrt{|\hat{u}_x(s) - \hat{u}_x(t_n)|^2 + |\hat{v}_x(s) - \hat{v}_x(t_n)|^2} \leq \frac{2\sqrt{\tau} \varepsilon}{\Re \lambda} \left| \lim_{s \not\to \sup I} e^{-(\Re \lambda) s} - e^{-(\Re \lambda) t_n} \right| \]
for every \( x \in X \) and every \( n \in \mathbb{N} \). Hence \( \theta \) is a uniform limit of \( \{ \theta_n \} \subset A \). Since \( A \) is uniformly closed, \( \theta \) is an element of \( A \).

Next we consider the case where \( A \) does not have constant functions. We define the functions \( \hat{\theta} \) and \( \hat{\theta}_n \) on \( X \) into \( \mathbb{C} \) as
\[ \hat{\theta}_n(x) = \theta_n(x) + \frac{1 + i}{\Re \lambda} \lim_{s \not\to \sup I} e^{-(\Re \lambda) s}, \]
\[ \hat{\theta}_n(x) = \theta_n(x) + \frac{1 + i}{\Re \lambda} e^{-(\Re \lambda) t_n}. \]
Note that \( \hat{\theta}_n(x) = g_x(t_n) \) holds for every \( x \in X \) and every \( n \in \mathbb{N} \), hence \( \{ \hat{\theta}_n \} \subset A \). Then we have
\[ \| \hat{\theta}(x) - \hat{\theta}_n(x) \| \leq \| \theta(x) - \theta_n(x) \| + \frac{1 + i}{\Re \lambda} \left| \lim_{s \not\to \sup I} e^{-(\Re \lambda) s} - e^{-(\Re \lambda) t_n} \right| \]
for every \( x \in X \) and every \( n \in \mathbb{N} \). Since \( A \) is uniform closed, \( \hat{\theta} \) belongs to \( A \). Moreover,
\[ \| f(t) - \hat{\theta} e^{\lambda t} \|_{\infty} \leq \| f(t) - \theta e^{\lambda t} \|_{\infty} + \frac{1 + i}{\Re \lambda} \left| \lim_{s \not\to \sup I} e^{-(\Re \lambda) s} e^{\lambda t} \right| \]
holds for every \( t \in I \). This completes the proof.

**Corollary 4.2.** Let \( \Re \lambda > 0 \), \( f \) a differentiable map on \((a, +\infty)\), for some \(-\infty \leq a < +\infty\), into \( A \) so that
\[ \| f(t) - \lambda f(t) \|_{\infty} \leq \varepsilon, \quad (t \in (a, +\infty)). \]
Then \( f \) is uniquely approximated by a function of \( A \) in the sense of Theorem 4.1.
**Proof.** By Theorem 4.1, it is enough to show that if $\theta_1, \theta_2 \in A$ so that
\[
\|f(t) - \theta_j e^{\lambda t}\|_\infty \leq k_j \varepsilon, \quad (t \in (a, +\infty))
\]
for some $k_j \geq 0$, $(j = 1, 2)$ then $\theta_1 = \theta_2$. In fact,
\[
\|\theta_1 - \theta_2\|_\infty \leq \|\theta_1 - f(t)e^{\lambda t}\|_\infty + \|f(t)e^{\lambda t} - \theta_2\|_\infty \\
\leq (k_1 + k_2)\varepsilon e^{-(\Re \lambda)t} \to 0, \quad (t \to +\infty).
\]
Thus we have $\theta_1 = \theta_2$. This completes the proof. \hfill \Box

In general, the Hyers-Ulam stability does not hold if $\Re \lambda = 0$.

**Example 4.1.** Let $I = (0, +\infty)$, $\varepsilon > 0$ and $f$ be the function on $I$ into $\mathbb{C}$ defined by
\[
f(t) = \varepsilon t e^{i t}, \quad (t \in I).
\]
Then the inequality $|f'(t) - i f(t)| = \varepsilon$ holds for every $t \in I$. On the other hand, the Hyers-Ulam stability does not hold. In fact, assume to the contrary that there exist a $c \in \mathbb{C}$ and $k \geq 0$ such that
\[
|f(t) - ce^{i t}| \leq k \varepsilon, \quad (t \in I).
\]
By the triangle inequality
\[
|f(t)| \leq k \varepsilon + |c|
\]
holds for every $t \in I$. Though this is a contradiction, since $|f(t)| = \varepsilon t$ and since $I = (0, +\infty)$.

If we consider the case where $I$ is a finite interval, then the situation is different:

**Theorem 4.3.** Let $I = (a, b)$, where $-\infty < a < b < +\infty$, $\varepsilon \geq 0$ and $\lambda \in \mathbb{C}$ with $\Re \lambda = 0$. If $f$ is a differentiable map on $I$ into $A$ so that
\[
\|f'(t) - \lambda f(t)\|_\infty \leq \varepsilon, \quad (t \in I),
\]
then there exists a $\theta \in A$ such that
\[
\|f(t) - \theta e^{\lambda t}\|_\infty \leq \frac{(b - a)\varepsilon}{\sqrt{2}}
\]
holds for every $t \in I$.

**Proof.** Let $f_x, g_x, u_x$ and $v_x$ be the differentiable function on $I$ into $\mathbb{C}$, defined in the proof of Theorem 4.1. Then for every $x \in X$ we see that
\[
f_x(t) = g_x(t) e^{\lambda t} \quad \text{and} \quad \|g_x\|_\infty \leq \varepsilon, \quad (t \in I),
\]
by definition. Apply the mean value theorem to $u_x$ and $v_x$ respectively, then we have
\[
\left| g_x(t) - g_x \left( \frac{a + b}{2} \right) \right| = \left| (u_x)'(p) \left( t - \frac{a + b}{2} \right) + i(v_x)'(q) \left( t - \frac{a + b}{2} \right) \right| \\
< \sqrt{2} \epsilon \frac{b - a}{2} = \frac{(b - a)\varepsilon}{\sqrt{2}}
\]
for some $p, q \in I$. Since $\Re \lambda = 0$, the inequality
\[
\left\| f(t) - g \left( \frac{a + b}{2} \right) e^{\lambda t} \right\|_\infty \leq \frac{(b - a)\varepsilon}{\sqrt{2}}
\]
holds for every $t \in I$. \hfill \Box
5.HYERS-ULAM STABILITY OF AN ENTIRE FUNCTION

Recall that a function is entire if it is holomorphic in the whole plane $\mathbb{C}$. We may consider the Hyers-Ulam stability of an entire function.

**Theorem 5.1.** Let $f$ be an entire function so that

$$|f'(z) - \lambda f(z)| \leq \varepsilon, \quad (z \in \mathbb{C}).$$

Unless $\lambda = 0$, there exists a $\theta \in \mathbb{C}$ such that

$$|f(z) - \theta e^{\lambda z}| \leq \frac{\varepsilon}{|\lambda|},$$

holds for every $z \in \mathbb{C}$. If we consider the case where $\lambda = 0$, then the Hyers-Ulam stability holds for $f$ if and only if $f$ is a constant function.

**Proof.** In a way similar to the proof of Proposition 2.1, we see that the inequality $|f'(z) - \lambda f(z)| \leq \varepsilon$ holds for every $z \in \mathbb{C}$ if and only if there corresponds an entire function $g$ so that

$$f(z) = g(z) e^{\lambda z} \quad \text{and} \quad |g'(z)| \leq \varepsilon |e^{-\lambda z}|, \quad (z \in \mathbb{C}).$$

Therefore $g'(z) e^{\lambda z}$ is a bounded entire function. Thus $g'(z) e^{\lambda z}$ is constant, by Liouville’s theorem. Put $c_1 = g'(z) e^{\lambda z}$, then $|c_1| \leq \varepsilon$.

Unless $\lambda = 0$, there exists a $c_2 \in \mathbb{C}$ such that

$$g(z) = c_2 - \frac{c_1}{\lambda} e^{-\lambda z}, \quad (z \in \mathbb{C}).$$

Therefore, we have the equality

$$f(z) = c_2 e^{\lambda z} - \frac{c_1}{\lambda}$$

for every $z \in \mathbb{C}$. Hence

$$|f(z) - c_2 e^{\lambda z}| \leq \frac{\varepsilon}{|\lambda|}, \quad (z \in \mathbb{C}).$$

Next we consider the case where $\lambda = 0$. Then there exists a $c_3 \in \mathbb{C}$ so that

$$g(z) = c_1 z + c_3, \quad (z \in \mathbb{C}).$$

Therefore $f(z) = c_1 z + c_3$ for every $z \in \mathbb{C}$, since $\lambda = 0$. Then it is easy to see that the Hyers-Ulam stability holds for $f$, if and only if $f$ is a constant function, and a proof is omitted. 

**References**


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