SECTIONAL REPRESENTATIONS OF GELFAND-MAZUR ALGEBRAS

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Abstract. It is shown that if a topologically semisimple algebra $A$, which has at least one closed maximal left (or right) ideal, is a locally pseudocovex Waelbroek algebra, a locally $A$-pseudocovex algebra, a locally pseudocovex Fréchet algebra, an exponentially gaëted algebra with bounded elements, or a Gelfand-Mazur algebra, for which there exists at least one closed 2-sided ideal, which is maximal as left (or right) ideal, then $A$ is representable as a subalgebra of a section algebra.

1. Introduction

Let $\mathbb{C}$ be the field of complex numbers. A linear topological space $(\mathcal{A}, \tau)$ over $\mathbb{C}$ is called a topological algebra over $\mathbb{C}$ (shortly, a topological algebra) if there has been defined an associative separately continuous multiplication such that $\mathcal{A}$ is an algebra. It means that for each neighbourhood $\mathcal{O}$ of zero and each $a \in \mathcal{A}$ there exists a neighbourhood $\mathcal{U}$ of zero such that $\forall l \subset \mathcal{O}$ and $\forall a \subset \mathcal{U}$.

We say that $\mathcal{A}$ is a locally pseudocovex algebra if $\mathcal{A}$ has a base of neighbourhoods of zero consisting of balanced pseudocovex sets. The set $U$ is called pseudocovex if $U + U \subset \nu U$ for some $\nu > 0$. A locally pseudocovex algebra $\mathcal{A}$ is called a locally absorbingly pseudocovex algebra (shortly, a locally $A$-pseudocovex algebra) if $\mathcal{A}$ has a base $B$ of neighbourhoods of zero consisting of balanced pseudocovex sets which satisfies the following condition: for each $U \in B$ and for each $a \in \mathcal{A}$ there exists $\nu = \nu(a, U) > 0$ such that $\forall l \subset \mathcal{O}$ and $\forall a \subset \mathcal{U}$.

We also say that a topological algebra $\mathcal{A}$ is an exponentially galëted algebra if for each neighbourhood $\mathcal{O}$ of zero of $\mathcal{A}$ there exists another neighbourhood $\mathcal{U}$ of zero such that

$$\left\{ \sum_{k=0}^{n} \frac{a_k}{2^k} : a_0, \ldots, a_n \in \mathcal{U} \right\} \subset \mathcal{O}$$

for each $n \in \mathbb{N}$ (see [1], p. 65) and a Fréchet algebra if $\mathcal{A}$ is complete and metrizable.

Let $\mathcal{A}$ be a topological algebra with unit $e_\mathcal{A}$, $m(\mathcal{A})$ denote the set of all closed two-sided ideals of $\mathcal{A}$ which are maximal as left (right) ideals. In case when the quotient algebra $\mathcal{A}/M$ (in the quotient topology) is topologically isomorphic with $\mathbb{C}$ for each $M \in m(\mathcal{A})$, then $\mathcal{A}$ is called a Gelfand-Mazur algebra. The term Gelfand-Mazur algebra is initially applied, independently of each other, by Mati Abel ([2], [3]) and A. Mallios [7]. Since then this terminology has been extensively employed.

An element $a$ of a topological algebra $\mathcal{A}$ is called to be bounded in $\mathcal{A}$ if there exists a number $\lambda \in \mathbb{C} \setminus \{0\}$ such that the set $\left\{ \left( \frac{\lambda}{n} \right)^n : n \in \mathbb{N} \right\}$ is bounded in $\mathcal{A}$ and $\mathcal{A}$ is called to be a topological algebra with bounded elements if every element of $\mathcal{A}$ is bounded in $\mathcal{A}$.

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An element \( a \in A \) is called quasi-invertible in \( A \) if there exists \( b \in A \) such that \( a + b - ab = \theta_A \) (where \( \theta_A \) is the zero element of \( A \)). A topological algebra \( A \) is a \( Q \)-algebra if the set of all quasi-invertible elements \( Q_{\text{inv}}A \) is open in \( A \). It is known that a topological algebra \( A \) with unit \( 1_A \) is a \( Q \)-algebra if and only if the set of its invertible elements \( \text{Inv}A \) is open in \( A \) (since \( Q_{\text{inv}}A = e_A - \text{Inv}A \)). A topological algebra with unit is called a Warbbrock algbrm if it is both a \( Q \)-algebra and a topological algebra with continuous inversion (see [7], p. 54).

Let \( \mathcal{M} \) be a maximal regular left (right) ideal of an algebra \( A \) and let \( \mathcal{P} = \{ a \in A : aA \subseteq \mathcal{M} \} \) (\( \mathcal{P} = \{ a \in A : Aa \subseteq \mathcal{M} \} \) respectively). Then we say that \( \mathcal{P} \) is a primitive ideal of \( A \) (with respect to \( \mathcal{M} \)). Let now \( A \) be a topological algebra with unit and let \( \mathcal{Z}(A) = \{ z \in A : za = az \text{ for each } a \in A \} \) be the center of \( A \). An ideal \( \mathcal{M} \in m(\mathcal{Z}(A)) \) is called an extendible ideal in \( A \) if

\[
\mathcal{I}(\mathcal{M}) = \text{cl}_A\{ \sum_{k=1}^n a_k m_k : n \in \mathbb{N}, a_1, a_2, ..., a_n \in A, m_1, m_2, ..., m_n \in \mathcal{M} \} \neq \mathcal{A}.
\]

(Here \( \text{cl}_A(\mathcal{U}) \) stands for the closure of the set \( \mathcal{U} \) in the topology of \( A \).) We know (see [8], p. 169) that for every two-sided ideal \( \mathcal{I} \) of \( A \) for which \( \text{cl}_A(\mathcal{I}) \neq A \) the set \( \text{cl}_A(\mathcal{I}) \) is a two-sided ideal of \( A \). So \( \mathcal{I}(\mathcal{M}) \) is also a two-sided ideal in \( A \). Let \( m(\mathcal{Z}(A)) = \{ \mathcal{M} \in m(\mathcal{Z}(A)) : \mathcal{M} \text{ is an extendible ideal in } A \} \).

Let \( A \) be again a topological algebra. The set \( \mathcal{R} = \bigcap\{ \mathcal{M} : \mathcal{M} \text{ is a closed maximal regular right ideal of } A \} \) is called the topological radical of the algebra \( A \). The topological algebra \( A \) is said to be topologically semi-simple if its topological radical \( \mathcal{R} = \{ \theta_A \} \).

Let now \( B \) and \( X \) be topological spaces and \( \pi : B \to X \) a continuous and open surjection. Then the complex \( (B, \pi, X) \) is called a fiber bundle. The mapping \( f : X \to B \) is said to be a section of the fiber bundle \( (B, \pi, X) \) (shortly, a section of \( \pi \)) if and only if \( \pi f(x) = x \) for every \( x \in X \). Let \( (B, \pi, X) \) be a fiber bundle for which the fibers

\[
B_x = \{ b \in B : \pi(b) = x \}
\]

are topological algebras, for every \( x \in X \). Then the set of all continuous sections of \( \pi \) is denoted by \( \Gamma(\pi) \). Defining algebraic operations in \( \Gamma(\pi) \) point-wise and topology by giving the subbase of \( f_0 \in \Gamma(\pi) \) by

\[
\mathcal{B}(f_0) = \{ U_O(f_0) : O \in \mathcal{B}(\theta_P) \}
\]

where \( \mathcal{B}(\theta_P) \) is a base of neighbourhoods of zero of the algebra

\[
P = \prod_{x \in X} B_x
\]

in the product topology and \( U_O(f_0) = \{ f \in \Gamma(\pi) : (f \circ f_0)(x) \subseteq X \in O \} \), we see that \( \Gamma(\pi) \) is a topological algebra which is called a section algbrm.

Let now \( f \) be a representation of a topological algebra \( A \) in another topological algebra \( B \), that is \( f \) is a continuous homomorphism from \( A \) into \( B \). In case when \( B \) is a section algebra, then \( f \) is called a sectional representation of \( A \). The aim of this paper is to generalize sectional representations of Banach algebras with unit given in [4] to the case of topological algebras and find the possible general conditions for a topological algebra with unit to be representable as a subalgebra of a section algebra.

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2. Properties of the quotient algebra and its center

To describe properties of a quotient algebra and its center, we need the following results.

**Theorem 1.** Let \( A \) be a topological algebra and \( \mathcal{I} \) a two-sided ideal of \( A \). If \( A \) is a Gelfand-Mazur algebra for which \( m(A) \neq 0 \), then there exists such a topology \( \tau \) on \( A \) that \( A/\mathcal{I} \) in the quotient topology (defined by \( \tau \)) and \( \mathcal{Z}(A/\mathcal{I}) \) in the subspace topology are exponentially galbed algebras with bounded elements.

**Proof.** It is known (see [2], p. 123, Theorem 2) that any topological algebra for which \( m(A) \neq 0 \) is a Gelfand-Mazur algebra iff there exists a topology \( \tau \) on \( A \) such that \((A, \tau)\) is an exponentially galbed algebra with bounded elements.

Let \( \tau_\mathcal{I} \) be the quotient topology on \( A/\mathcal{I} \) and \( \pi : A \to A/\mathcal{I} \) the canonical homomorphism. Then \( \tau_\mathcal{I} = \{ T' \cap \mathcal{Z}(A/\mathcal{I}) : T' \in \tau_\mathcal{T} \} \) is the topology on \( \mathcal{Z}(A/\mathcal{I}) \) generated by \( \tau_\mathcal{T} \).

Let \( \mathcal{O}' \) be a neighbourhood of zero in \((A/\mathcal{I}, \tau_\mathcal{I})\). Since \( \pi \) is a continuous open mapping (see [6], p 104), then \( \mathcal{O} = \pi^{-1}(\mathcal{O}') \) is a neighbourhood of zero in \((A, \tau)\). Now we can find a neighbourhood of zero \( \mathcal{U} \) in \((A, \tau)\) such that

\[
\left\{ \sum_{k=0}^{n} \frac{a_k}{2^k} : a_0, \ldots, a_n \in \mathcal{U} \right\} \subset \mathcal{O}
\]

for every \( n \in \mathbb{N} \). Taking \( \mathcal{V}' = \pi(\mathcal{U}) \), we can see that

\[
\left\{ \sum_{k=0}^{n} \frac{x_k}{2^k} : x_0, x_1, \ldots, x_n \in \mathcal{V}' \right\} \subset \mathcal{O}'
\]

for every \( n \in \mathbb{N} \) which implies that \( A/\mathcal{I} \) is exponentially galbed.

Every neighbourhood of zero in \( \mathcal{Z}(A/\mathcal{I}) \) is representable in the form \( \mathcal{O}'' = \mathcal{O}' \cap \mathcal{Z}(A/\mathcal{I}) \) where \( \mathcal{O}' \) is a neighbourhood of zero in \( A/\mathcal{I} \). As above we can find the neighborhood \( \mathcal{V}' \) of zero. Taking now \( \mathcal{V}'' = \mathcal{V}' \cap \mathcal{Z}(A/\mathcal{I}) \), we can see that

\[
\left\{ \sum_{k=0}^{n} \frac{y_k}{2^k} : y_0, y_1, \ldots, y_n \in \mathcal{V}'' \right\} \subset \mathcal{O}''
\]

for every \( n \in \mathbb{N} \) which implies that \( \mathcal{Z}(A/\mathcal{I}) \) is also exponentially galbed.

It is easy to see that the elements of \( A/\mathcal{I} \) and \( \mathcal{Z}(A/\mathcal{I}) \) are bounded.

**Corollary 1.** Let \( A \) be a topological algebra and \( \mathcal{I} \) a two-sided ideal of \( A \). If \( A \) is a Gelfand-Mazur algebra for which \( m(A) \neq 0 \), then \( A/\mathcal{I} \) is a Gelfand-Mazur algebra. If hereby \( m(A/\mathcal{I}) \neq 0 \) then \( \mathcal{Z}(A/\mathcal{I}) \) is a Gelfand-Mazur algebra too.

**Lemma 1.** Let \( A \) be a locally pseudoconvex (a locally A-pseudoconvex) algebra and \( \mathcal{I} \) a two-sided ideal of \( A \). Then \( A/\mathcal{I} \) and \( \mathcal{Z}(A/\mathcal{I}) \) are also locally pseudoconvex (locally A-pseudoconvex) algebras.

**Proof.** Since \( A \) is a locally pseudoconvex algebra, we can find a base \( B \) of neighbourhoods of zero of \( A \) consisting of balanced pseudoconvex neighbourhoods of zero. It is easy to see that \( B' = \pi(B) \) and

\[
B'' = \{ \mathcal{U}' \cap \mathcal{Z}(A/\mathcal{I}) : \mathcal{U}' \in B' \}
\]

are suitable bases of neighbourhoods of zero for \( A/\mathcal{I} \) and \( \mathcal{Z}(A/\mathcal{I}) \) respectively.

**Lemma 2.** Let \( A \) be a Fréchet (a unital locally pseudoconvex Fréchet) algebra and \( \mathcal{I} \) a closed two-sided ideal. Then \( A/\mathcal{I} \) and \( \mathcal{Z}(A/\mathcal{I}) \) are also Fréchet (unital locally pseudoconvex Fréchet) algebras.
Proof. According to our hypothesis (see also [6], p. 138, Theorem 2), $A/I$ is a Fréchet algebra. Since $\mathcal{Z}(A/I)$ is a closed linear subspace of $A/I$, then $\mathcal{Z}(A/I)$ is also complete and metrizable, that is, a Fréchet algebra too.

Using Lemmas 1 and 3, we can get the following.

**Lemma 3.** Let $A$ be a (locally pseudoconvex) Waelbroeck algebra with unit and $I$ a closed two-sided ideal. Then $A/I$ and $\mathcal{Z}(A/I)$ are also (locally pseudoconvex) Waelbroeck algebras with unit.

**Proof.** It is shown in [11] that $A/I$ is a Waelbroeck algebra with unit. This implies that $\text{Inv} A/I$ is an open set in $A/I$. Since $\text{Inv} \mathcal{Z}(A/I) = \text{Inv} A/I \cap \mathcal{Z}(A/I)$, then $\text{Inv} \mathcal{Z}(A/I)$ is an open set in $\mathcal{Z}(A/I)$. We can see that the inversion is continuous in $\mathcal{Z}(A/I)$ (because $\text{Inv} \mathcal{Z}(A/I)$ is a subset of $\text{Inv} A/I$). Hence, $\mathcal{Z}(A/I)$ is a Waelbroeck algebra too.

Using [2], p. 120-122 (Theorem 1 and Corollary 1), see also [3], the following result can be proved:

**Lemma 4.** Let $A$ be a topological division algebra for which at least one of the following statements holds:

a) there exists a topology $\tau$ on $A$ such that $(A, \tau)$ is a locally pseudoconvex Hausdorff algebra with continuous inversion;

b) $A$ is a locally $A$-pseudoconvex Hausdorff algebra;

c) $A$ is a locally pseudoconvex Fréchet algebra;

d) there exists a topology $\tau$ on $A$ such that $(A, \tau)$ is an exponentially gaubed Hausdorff algebra with bounded elements.

Then $A$ is topologically isomorphic to $\mathbb{C}$.

**Theorem 3.** Let $A$ be a topological algebra with unit, $P$ a closed primitive ideal of it and let one of the following statements be true:

a) $A$ is a locally pseudoconvex Waelbroeck algebra;

b) $A$ is a locally $A$-pseudoconvex algebra;

c) $A$ is a locally pseudoconvex Fréchet algebra;

d) $A$ is an exponentially gaubed algebra with bounded elements;

e) $A$ is a Gel’fand-Mazur algebra for which $m(A) \neq 0$.

Then $\mathcal{Z}(A/P)$ is topologically isomorphic to $\mathbb{C}$.

**Proof.** Since $P$ is a primitive ideal of $A$ then $\mathcal{Z}(A/P)$ is a field (see [5], p. 136 (Proposition 9) and [9], p. 61 (Corollary 2.4.5)). Thus $\mathcal{Z}(A/P)$ is a division Hausdorff algebra.

In case a) we obtain from Lemma 3 that $\mathcal{Z}(A/P)$ is a locally pseudoconvex Waelbroeck algebra. Since every Waelbroeck algebra is an algebra with continuous inversion, then $\mathcal{Z}(A/P)$ satisfies condition a) of Lemma 4. In case b) we obtain from Lemma 1 that $\mathcal{Z}(A/P)$ is a locally $A$-pseudoconvex algebra and thus satisfies the condition b) of Lemma 4. In case c) we obtain from Lemma 2 that $\mathcal{Z}(A/P)$ satisfies the condition c) of Lemma 4. In cases d) and e) we obtain from Theorem 1 that $\mathcal{Z}(A/P)$ has a topology in which $\mathcal{Z}(A/P)$ is an exponentially gaubed algebra with bounded elements and thus satisfies condition d) of Lemma 4.

**Corollary 2.** Let $A$ be a topological algebra with unit and $M$ be a closed maximal left (right) ideal of $A$. If at least one of the statements a) - e) of the Theorem 3 is true, then the following statements are also true:

1) every $b \in \mathcal{Z}(A)$ defines such $\lambda \in \mathbb{C}$ that $b - \lambda e_A \in M$;

2) $M \cap \mathcal{Z}(A) \in m(\mathcal{Z}(A))$.

**Proof.** 1) Let $b \in \mathcal{Z}(A)$, $P$ be a primitive ideal in $A$ with respect to $M$ and $\pi: A \to A/P$ the canonical homomorphism. Then $\mathcal{Z}(A/P)$ is topologically isomorphic to $\mathbb{C}$, according Theorem 3. We will denote this isomorphism by $\mu$. Since $\pi(b) \in \mathcal{Z}(A/P)$, we can find $\lambda \in \mathbb{C}$ such that $\mu(\pi(b)) = \lambda = \mu(\pi(\lambda e_A))$. Therefore $\pi(b) = \pi(\lambda e_A)$ which implies that $b - \lambda e_A \in P \subseteq M$.
2) Let $\mathcal{M}_z = \mathcal{M} \cap \mathcal{I}(\mathcal{A})$. Then $\mathcal{M}_z$ is a closed linear subspace of $\mathcal{A}$ because $\mathcal{M}$ and $\mathcal{I}(\mathcal{A})$ were closed subsets of $\mathcal{A}$. Since $e_A \notin \mathcal{M}$ then $\mathcal{M}_z \neq \mathcal{I}(\mathcal{A})$. Let now $z \in \mathcal{M}_z$. Then $z \in \mathcal{M}$ which implies $za \in \mathcal{M}$ for every $a \in \mathcal{A}$. Hence $za \in \mathcal{M}_z$ for every $a \in \mathcal{A}$ which implies that $\mathcal{M}_z$ is a closed ideal of $\mathcal{I}(\mathcal{A})$. Let now $I$ be an ideal of $\mathcal{I}(\mathcal{A})$ such that $\mathcal{M}_z \subseteq \mathcal{I}$. If $I \neq \mathcal{M}_z$ then there exists a $b \in \mathcal{I} \setminus \mathcal{M}_z$ and according to 1) there exists $\lambda \in \mathbb{C}$ such that $b - \lambda e_A \in \mathcal{M}_z$. Since $b \notin \mathcal{M}_z$ we see that $\lambda \neq 0$ and therefore there exists $\lambda^{-1}$. Since $b - \lambda e_A \in \mathcal{M}_z \subseteq \mathcal{I}$ we have $e_A = \lambda^{-1}[b - (b - \lambda e_A)] \in \mathcal{I}$ and therefore $\mathcal{I} = \mathcal{I}(\mathcal{A})$. So $\mathcal{M}_z \in m(\mathcal{I}(\mathcal{A}))$. Since $\mathcal{M}_z \subseteq \mathcal{M}$ and $\mathcal{M}$ is a closed subset, we have $\mathcal{I}(\mathcal{M}_z) \subseteq \mathcal{M} \neq \mathcal{A}$. Hence $\mathcal{M}_z \in m_e(\mathcal{I}(\mathcal{A}))$.

3. Sectional representations

Let $\mathcal{A}$ be a topological algebra with unit for which at least one of the conditions a) - e) of Theorem 3 is true and which has at least one closed left (right) maximal ideal. Then $\mathcal{I}(\mathcal{A})$ is a (commutative) Gelfand-Mazur algebra by Theorem 1 and Lemmas 1, 2 and 3 (see also [2], p. 125-126, Corollaries 2 and 3) and $m_e(\mathcal{I}(\mathcal{A})) \neq \emptyset$, by Corollary 2.

For every $\mathcal{M} \in m_e(\mathcal{I}(\mathcal{A}))$ let $\mathcal{A}_M = \mathcal{A}/\mathcal{I}(\mathcal{M})$ and $\kappa_M : \mathcal{A} \rightarrow \mathcal{A}_M$ the canonical homomorphism, $\kappa_M(\mathcal{M}) = \kappa_M(a)$ for every $\mathcal{M} \in m_e(\mathcal{I}(\mathcal{A}))$ and

$$\mathcal{B} = \bigcup_{\mathcal{M} \in m_e(\mathcal{I}(\mathcal{A}))} \mathcal{A}_M.$$ 

Then $\kappa$ maps $m_e(\mathcal{I}(\mathcal{A}))$ into $\mathcal{B}$.

Let now $\pi : \mathcal{B} \rightarrow m_e(\mathcal{I}(\mathcal{A}))$ be such mapping, which will assign to every $b \in \mathcal{B}$ such ideal $\mathcal{M} \in m_e(\mathcal{I}(\mathcal{A}))$ that $b \in \mathcal{A}_M$ i.e. $b = \kappa_M(a)$ for some $a \in \mathcal{A}$. It is easy to see that $\pi$ is well defined. Indeed, if $b = \kappa_M(a_1)$ (i.e. $\pi(b) = \mathcal{M}_1$) and $b = \kappa_M(a_2)$ (i.e. $\pi(b) = \mathcal{M}_2$), then $a_1 + \mathcal{I}(\mathcal{M}_1) = a_2 + \mathcal{I}(\mathcal{M}_2)$. Since $a_1 = a_1 + \theta_\mathcal{M} \in a_1 + \mathcal{I}(\mathcal{M}_1) = a_2 + \mathcal{I}(\mathcal{M}_2)$, we can find $d_2 \in \mathcal{I}(\mathcal{M}_2)$ such that $a_1 = a_2 + d_2$ or $a_1 - a_2 = d_2 \in \mathcal{I}(\mathcal{M}_2)$.

Let $c_1 \in \mathcal{I}(\mathcal{M}_1)$ be an arbitrary element. Then

$$a_1 + c_1 \in a_1 + \mathcal{I}(\mathcal{M}_1) = a_2 + \mathcal{I}(\mathcal{M}_2)$$

and we can find $c_2 \in \mathcal{I}(\mathcal{M}_2)$ such that $a_1 + c_1 = a_2 + c_2$. Since $a_1 - a_2 \in \mathcal{I}(\mathcal{M}_2)$ and $c_2 \in \mathcal{I}(\mathcal{M}_2)$ then $c_2 = (a_1 - a_2) \in \mathcal{I}(\mathcal{M}_2)$, so that $\mathcal{I}(\mathcal{M}_1) \subseteq \mathcal{I}(\mathcal{M}_2)$. Analogously we get that $\mathcal{I}(\mathcal{M}_2) \subseteq \mathcal{I}(\mathcal{M}_1)$. Hence $\mathcal{I}(\mathcal{M}_1) = \mathcal{I}(\mathcal{M}_2)$ and

$$\mathcal{M}_1 = \mathcal{I}(\mathcal{M}_1) \cap \mathcal{I}(\mathcal{A}) = \mathcal{I}(\mathcal{M}_2) \cap \mathcal{I}(\mathcal{A}) = \mathcal{M}_2.$$ 

The last result proves that $\pi$ is well defined and $\mathcal{A}_M \cap \mathcal{A}_{M_0} \neq \emptyset$ if and only if $\mathcal{M}_1 = \mathcal{M}_2$.

The set $m_e(\mathcal{I}(\mathcal{A}))$ is endowed with the topology $\tau$, a subbase of neighbourhoods of $\mathcal{M}_0 \in m_e(\mathcal{I}(\mathcal{A}))$ of which consists of sets

$$O(\mathcal{M}_0) = \{ \mathcal{M} \in m_e(\mathcal{I}(\mathcal{A})) : |(\varphi_\mathcal{M} - \varphi_{\mathcal{M}_0})(z)| < \varepsilon \}$$

where $\varepsilon > 0$ and $z \in \mathcal{I}(\mathcal{A})$ vary, while $\varphi_\mathcal{M}$ denotes such nontrivial homomorphism $\mathcal{I}(\mathcal{A}) \rightarrow \mathbb{C}$ for which $\ker \varphi_\mathcal{M} = \mathcal{M}$. On the algebras $\mathcal{A}_M$ we shall consider quotient topologies $\tau_\mathcal{M}$ and on $\mathcal{B}$ the topology $\tau_\mathcal{B} = \{ \pi^{-1}(U) : U \in \tau \}$. Then $(\mathcal{B}, \pi, m_e(\mathcal{I}(\mathcal{A})))$ is a fiber bundle and $a \in \Gamma(\pi)$ for every $a \in \mathcal{A}$.

Next we define a mapping $\Lambda : \mathcal{A} \rightarrow \Gamma(\pi)$, such that $\Lambda(a) = a\pi$ for every $a \in \mathcal{A}$. It is easy to see that $\Lambda$ is a continuous homomorphism. Hence, $\Lambda$ is a sectional representation of the topological algebra $\mathcal{A}$. 


Lemma 5. Let $\mathcal{A}$ be a topological algebra with unit, which satisfies at least one of the conditions a) - e) of the Theorem 3 and $I$ be a closed maximal left (right) ideal of $\mathcal{A}$. Then we can find such $\mathcal{M} \in m_*(\mathcal{Z}(\mathcal{A}))$ that $\kappa_{\mathcal{M}}(I) = \{x^-(\mathcal{M}) : x \in I\}$ is a left (right) ideal of $\mathcal{A}_{\mathcal{M}}$.

Proof. See [4], p. 197-198 (Theorem 2.8 (iii)).

Theorem 4. Let $\mathcal{A}$ be a topological algebra with unit, which satisfies at least one of the conditions a) - e) of the Theorem 3 and $I$ a closed maximal left (right) ideal of $\mathcal{A}$. Then $I = \kappa_{\mathcal{M}}^{-1}(J) = \{a \in A : a^-(\mathcal{M}) \in J\}$ for some $\mathcal{M} \in m_*(\mathcal{Z}(\mathcal{A}))$ and for some closed maximal left (right) ideal $J$ of $\mathcal{A}_{\mathcal{M}}$.

Proof. Let $I$ be a closed maximal left (right) ideal of $\mathcal{A}$.

According to Lemma 5, we can find $\mathcal{M} \in m_*(\mathcal{Z}(\mathcal{A}))$ such that $J = \{a^-(\mathcal{M}) : a \in I\}$ is a left (right) ideal of $\mathcal{A}_{\mathcal{M}}$. Next we can find a maximal left (right) ideal $I_{\mathcal{M}}$ of $\mathcal{A}_{\mathcal{M}}$ such that $J \subset I_{\mathcal{M}}$. If there exists $j \in I_{\mathcal{M}} \setminus J$, then it is possible to find $b \in \mathcal{A}\setminus I$ such that $b^-(\mathcal{M}) = j$. Hence $\kappa_{\mathcal{M}}^{-1}(I_{\mathcal{M}}) \supset \{b\} \cup I \cup I$. Let $c \in \mathcal{A}$ and $d \in \kappa_{\mathcal{M}}^{-1}(I_{\mathcal{M}})$. Then $\kappa_{\mathcal{M}}(cd) = \kappa_{\mathcal{M}}(c)\kappa_{\mathcal{M}}(d) \in I_{\mathcal{M}}$ which implies that $cd \in \kappa_{\mathcal{M}}^{-1}(I_{\mathcal{M}})$ (for a right ideal). Analogously, we get $c + d, \lambda c \in \kappa_{\mathcal{M}}^{-1}(I_{\mathcal{M}})$. Since $I$ is a maximal left (right) ideal of $\mathcal{A}$, we have $\kappa_{\mathcal{M}}^{-1}(I_{\mathcal{M}}) = I_{\mathcal{M}}$ which implies that $a^-(\mathcal{M}) \in I_{\mathcal{M}}$ for every $a \in A$. Hence $I_{\mathcal{M}} = I_{\mathcal{M}}$ which is not possible. So we have shown that $J$ is a maximal left (right) ideal of $\mathcal{A}_{\mathcal{M}}$. According to the definition of $J$, it is clear that $I = \kappa_{\mathcal{M}}^{-1}(J)$.

Next we show that $J$ is closed. If $c_{\mathcal{M}}(J) \neq A_{\mathcal{M}}$ then $c_{\mathcal{M}}(J)$ is also an ideal in $A_{\mathcal{M}}$ (see [8], p. 169) and $J = c_{\mathcal{M}}(J)$. Let us suppose that $c_{\mathcal{M}}(J) = A_{\mathcal{M}}$ and let $O'$ be a neighbourhood of zero in $A$. Then $\kappa_{\mathcal{M}}(O') = O$ is a neighbourhood of zero in $A_{\mathcal{M}}$. Since $\kappa_{\mathcal{M}}(x) \in c_{\mathcal{M}}(J)$, then there exists a family $(x_{\lambda})_{\lambda \in \Lambda} \in J$ such that $\kappa_{\mathcal{M}}(i\lambda)_{\lambda \in \Lambda} \rightarrow \kappa_{\mathcal{M}}(x)$. Now we can find such a $\mu \in \Lambda$ that $\kappa_{\mathcal{M}}(i\lambda - x) \in O$ for every $\lambda > \mu$. If $\lambda > \mu$ then $i\lambda - x \in \kappa_{\mathcal{M}}^{-1}(\kappa_{\mathcal{M}}(O')) = I_{\mathcal{M}} + O' \subset I + O'$ (because $\mathcal{M} = I \cap \mathcal{Z}(\mathcal{A})$, see proof of Theorem 2.8 (iii) in [4]) and therefore

$$e_{\mathcal{A}} = (e_{\mathcal{A}} - i_{\lambda_0}) + i_{\lambda_0} \in I + O' + I \subset I + O'$$

Hence

$$e_{\mathcal{A}} \in \{I + O' : O' \text{ is a neighbourhood of a base of zero in } A\} = c_{\mathcal{M}}(J) = I$$

(see [10], p. 13). Thus $I = A$ which is not possible. Therefore, $J$ is a closed maximal left (right) ideal.

It is easy to verify that the following statement holds.

Lemma 6. Let $\mathcal{A}$ be a topological algebra with unit, $\mathcal{M} \in m_*(\mathcal{Z}(\mathcal{A}))$ and $J$ be an arbitrary closed left (right) ideal of $\mathcal{A}_{\mathcal{M}}$. Then $\kappa_{\mathcal{M}}^{-1}(J)$ is a left (right) ideal of $\mathcal{A}$.

Lemma 7. Let $\mathcal{A}$ be a topological algebra with unit for which at least one of the conditions a) - e) of Theorem 3 holds. Suppose that there exists a closed maximal left (right) ideal in $\mathcal{A}$. If we denote the topological radicals of the algebras $\mathcal{A}$ and $\mathcal{A}_{\mathcal{M}}$, where $\mathcal{M} \in m_*(\mathcal{Z}(\mathcal{A}))$, by $\mathcal{R}$ and $\mathcal{R}_{\mathcal{M}}$ respectively, then

$$\mathcal{R} = \bigcap \{\kappa_{\mathcal{M}}^{-1}(\mathcal{R}_{\mathcal{M}}) : \mathcal{M} \in m_*(\mathcal{Z}(\mathcal{A}))\}$$

Proof. Suppose that $x \in \bigcap \{\kappa_{\mathcal{M}}^{-1}(\mathcal{R}_{\mathcal{M}}) : \mathcal{M} \in m_*(\mathcal{Z}(\mathcal{A}))\}$ and $I$ is an arbitrary closed maximal left (right) ideal of $\mathcal{A}$. According to Theorem 4, we can find $\mathcal{M} \in m_*(\mathcal{Z}(\mathcal{A}))$ and a closed maximal left (right) ideal $J$ of $\mathcal{A}_{\mathcal{M}}$ such that $I = \kappa_{\mathcal{M}}^{-1}(J)$. Since $\kappa_{\mathcal{M}}(x) \in \mathcal{R}_{\mathcal{M}} \subset J$ then $x \in I$, for any closed maximal left (right) ideal $I$ of $\mathcal{A}$. Therefore, $x \in \mathcal{R}$ such that

$$\mathcal{R} \supset \bigcap \{\kappa_{\mathcal{M}}^{-1}(\mathcal{R}_{\mathcal{M}}) : \mathcal{M} \in m_*(\mathcal{Z}(\mathcal{A}))\}.$$
Suppose now that \( y \in \mathcal{R} \) and \( \mathcal{M} \in m_\epsilon(\mathcal{Z}(\mathcal{A})) \). If \( \mathcal{J} \) is an arbitrary closed maximal left (right) ideal of \( \mathcal{A}_\mathcal{M} \), then \( \kappa_\mathcal{M}^{-1}(\mathcal{J}) \) is a closed left (right) ideal of \( \mathcal{A} \) by Lemma 6. It is clear that \( \mathcal{I}(\mathcal{M}) \subseteq \kappa_\mathcal{M}^{-1}(\mathcal{J}) \). Suppose that \( \kappa_\mathcal{M}^{-1}(\mathcal{J}) \subseteq \mathcal{H} \) for a left (right) ideal \( \mathcal{H} \) of \( \mathcal{A} \). Then \( \mathcal{J} \subseteq \kappa_\mathcal{M}(\mathcal{H}) \). If \( \kappa_\mathcal{M}(\mathcal{H}) = \mathcal{A}_\mathcal{M} \), then we can find such a \( h \in \mathcal{H} \) that \( \kappa_\mathcal{M}(h) = e_{\mathcal{A}_\mathcal{M}} \) (here \( e_{\mathcal{A}_\mathcal{M}} \) denotes the unit element of \( \mathcal{A}_\mathcal{M} \)). But this means that \( h - e_{\mathcal{A}} \in \mathcal{I}(\mathcal{M}) \subseteq \kappa_\mathcal{M}^{-1}(\mathcal{J}) \subseteq \mathcal{H} \), which implies \( e_{\mathcal{A}} \in \mathcal{H} \). Hence \( \mathcal{H} = \mathcal{A} \) which contradicts the assumption that \( \mathcal{H} \) is an ideal of \( \mathcal{A} \). Since \( \mathcal{J} \) is a maximal left (right) ideal of \( \mathcal{A}_\mathcal{M} \), then \( \mathcal{J} = \kappa_\mathcal{M}(\mathcal{H}) \), that is, \( \mathcal{H} = \kappa_\mathcal{M}^{-1}(\mathcal{J}) \).

Hence, \( \kappa_\mathcal{M}^{-1}(\mathcal{J}) \) is a closed maximal left (right) ideal of \( \mathcal{A} \) and \( \kappa_\mathcal{M}(\mathcal{M}(y)) \in \mathcal{J} \) for every closed maximal left (right) ideal \( \mathcal{J} \) of \( \mathcal{A}_\mathcal{M} \). Therefore \( \kappa_\mathcal{M}(y) \in \mathcal{R}_\mathcal{M} \) if \( \mathcal{M} \in m_\epsilon(\mathcal{Z}(\mathcal{A})) \), so that

\[
\mathcal{R} \subseteq \bigcap \{ \kappa^{-1}_\mathcal{M}(\mathcal{R}_\mathcal{M}) : \mathcal{M} \in m_\epsilon(\mathcal{Z}(\mathcal{A})) \}
\]

which completes the proof.

**Corollary 3.** Let \( \mathcal{A} \) be a topologically semisimple algebra, which satisfies the conditions of Lemma 7. Then the mapping \( A \) is one-to-one.

**Proof.** Since \( \mathcal{A} \) is topologically semisimple, its topological radical \( \mathcal{R} = \{ e_{\mathcal{A}} \} \). Hence

\[
\ker A = \bigcap \{ \kappa^{-1}_\mathcal{M}(\theta_{\mathcal{A}}) : \mathcal{M} \in m_\epsilon(\mathcal{Z}(\mathcal{A})) \} \subseteq
\]

\[
\bigcap \{ \kappa^{-1}_\mathcal{M}(\mathcal{R}_\mathcal{M}) : \mathcal{M} \in m_\epsilon(\mathcal{Z}(\mathcal{A})) \} = \mathcal{R} = \{ e_{\mathcal{A}} \}
\]

one obtains that \( A \) is one-to-one.

Now we formulate the main result of this paper.

**Theorem 5.** Let \( \mathcal{A} \) be a topologically semisimple algebra with unit, having at least one closed maximal left (right) ideal. If one of the following statements is true:

a) \( \mathcal{A} \) is a locally pseudoconnex Waelbroeck algebra;

b) \( \mathcal{A} \) is a locally \( A \)-pseudoconnex algebra;

c) \( \mathcal{A} \) is a locally pseudoconnex Fréchet algebra;

d) \( \mathcal{A} \) is an exponentially gelled algebra with bounded elements;

e) \( \mathcal{A} \) is a Gelfand-Mazur algebra for which \( m(\mathcal{A}) \neq \emptyset \),

then \( \mathcal{A} \) can be considered as a subalgebra of the section algebra \( \Gamma(\pi) \).

**Proof.** Since \( \mathcal{A} \) is a one-to-one representation of \( \mathcal{A} \) in \( \Gamma(\pi) \), we can consider \( \mathcal{A} \) as a subalgebras of \( \Gamma(\pi) \).

**BIBLIOGRAPHY**


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