DIMENSION ESTIMATE FOR A SET OBTAINED FROM A
THREE-DIMENSIONAL NON-PERIODIC SELF-AFFINE TILING

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Abstract. Some computational results on the three-dimensional Pisot tiling generated by the roots of $x^4 - x^3 - x^2 - x - 1 = 0$ are shown. We show an upper bound of the Hausdorff dimension of the set which is a projection of the intersection of three-tiles to the plane.

1 Introduction A tile is a compact subset of $\mathbb{R}^n$ which is equal to the closure of its interior. A set of tiles $\mathcal{T}$ is a tiling of $\mathbb{R}^n$, if $\mathcal{T}$ is a covering of $\mathbb{R}^n$ such that the intersection of interiors of any two tiles in $\mathcal{T}$ is empty. A tiling $\mathcal{T}$ is called self-affine, if there is an affine map such that the image of a tile in $\mathcal{T}$ is a union of tiles of $\mathcal{T}$.

Self-affine tilings are of special interest because of their relation to several topics of recent research, Markov partitions for toral automorphisms [11, 4, 1], wavelet theory [7, 10] and real quasi-crystal [8, 13]. There are numerous studies on self-affine tilings. All of the examples used there are about one or two-dimensional cases. Explicit examples of higher than or equal to three-dimensional cases have not been studied.

In this paper, we show some explicit computational results on the 3-dimensional non-periodic self-affine tiling generated by the roots of the equation $x^4 - x^3 - x^2 - x - 1 = 0$. We construct a Mauldin-Williams graph [9] of intersection of three tiles. The tiling we treat here is a Pisot tiling. Here we do not give the precise and general definition of the Pisot tilings. For further details, see [2, 15, 14, 3]. The tiling is constructed as follows: The polynomial $x^4 - x^3 - x^2 - x - 1$ is irreducible over $\mathbb{Q}$ and has four distinct roots,

$$
\begin{align*}
\beta & = 1.927561975482925304261905 \cdots, \\
\gamma & = -0.77400113215438540924032 \cdots, \\
\alpha & = -0.0763789311337457250847 \cdots - 0.8147036471703856268416 \cdots i, \\
\pi & = -0.0763789311337457250847 \cdots + 0.8147036471703856268416 \cdots i.
\end{align*}
$$

So $\beta$ is a Pisot number, that is, $\beta$ is an algebraic integer greater than 1 and all of its Galois conjugates over $\mathbb{Q}$ are strictly inside the unit circle. Let $w = d_{-l}d_{-l+1} \cdots d_{-1}$ be a word over $\{0, 1\}$. A tile $T(w)$ is defined as follows:

$$
T(w) = \left\{ \left( \sum_{i=-l}^{\infty} a_i \alpha^i, \sum_{i=-l}^{\infty} a_i \gamma^i \right) : \begin{array}{l}
\forall i \in \{0, 1\}, a_i \times a_{i+1} \times a_{i+2} \times a_{i+3} = 0, \\
a_{-l}a_{-l+1} \cdots a_{-1} = w \end{array} \right\},
$$

which is a subset of $\mathbb{C} \times \mathbb{R} \simeq \mathbb{R}^3$. The tiling $\mathcal{T}$ is defined by

$$
\mathcal{T} := \{ T(w) : w \in \{0, 1\}^* \},
$$

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where \( \{0,1\}^* \) denotes the set of all of the words over \( \{0,1\} \) (including the empty word \( \epsilon \)).

An automaton is a (directed labeled) graph. For an automaton \( M \), we denote by \( V(M) \) the vertex set of \( M \) and by \( E(M) \) the edge set of \( M \). Every edge \( e \in E(M) \) has the starting point \( s(e) \in V(M) \) and the end point \( t(e) \in V(M) \), and carry a label \( I(e) \in \Sigma(M) \) where \( \Sigma(M) \) is a finite set called the alphabet of \( M \). A sequence of edges \( e_1 \cdots e_t \) is called a path of \( M \) if \( t(e_i) = s(e_{i+1}) \). An automaton has a special vertex \( i_M \) called the initial state of \( M \). \( \Sigma^* \) denotes the set of all of the words over an alphabet \( \Sigma \). A word \( w = a_1 \cdots a_t \in \Sigma^* \) is accepted by \( M \) if there exists a path \( p = e_1 \cdots e_t \) starting from \( i_M \) such that \( I(e_1) \cdots I(e_t) = w \). An infinite word \( (a_i)_{i \geq 0} \) over \( \Sigma \) is accepted by \( M \) if \( a_0 \cdots a_h \) is accepted by \( M \) for all \( h \geq 0 \). We denote by \( L(M,i) \) the set of infinite words accepted by \( M \) with the initial state \( i \).

**Example 1** The automaton shown in Figure 1 accepts the words over \( \{0,1\} \) which does not include 11 as a subword.

![Figure 1](image)

The intersection of tiles is determined by automata. See [12] for the proof of the following theorem.

**Theorem 1** (Sadahiro) The intersection of tiles are represented by automata: For any \( n \) tiles \( T(w_1), T(w_2), \ldots, T(w_n) \), there exists an automaton \( m \) with the following property.

\[
\left( \sum_{i=-l}^{\infty} a_i \alpha^i, \sum_{i=-l}^{\infty} a_i \gamma^i \right) \in T(w_1) \cap T(w_2) \cap \cdots \cap T(w_n)
\]

if and only if \( a_{-l} \cdots a_{-1} = w_1 \) and \( a_{-l} a_{-l+1} \cdots a_h \) is accepted by \( m \) for any \( h \geq 0 \).

An infinite word accepted by the automaton in the theorem above determines a point in the intersection. For example, the automaton which represents \( T(0) \cap T(1) \cap T(11) \cap T(111) \) is shown in Figure 2, from which we can see \( T(0) \cap T(1) \cap T(11) \cap T(111) \) consists of only one point \(( -1, -1 ) \in \mathbb{C} \times \mathbb{R} \).

In fact, the following four presentations of \(( -1, -1 ) \) exist:

\[
( -1, -1 ) = \left( \sum_{n=0}^{\infty} \alpha^n (\alpha + \alpha^2 + \alpha^3), \sum_{n=0}^{\infty} \gamma^n (\gamma + \gamma^2 + \gamma^3) \right) \in T(0)
\]

\[
= \left( \frac{1}{\alpha} + \sum_{n=0}^{\infty} \alpha^n (\alpha + \alpha^2 + \alpha^4), \frac{1}{\gamma} + \sum_{n=0}^{\infty} \gamma^n (\gamma + \gamma^2 + \gamma^4) \right) \in T(1)
\]
\[
= \left( \frac{1}{\alpha^2} + \frac{1}{\alpha} + \sum_{n=0}^{\infty} \alpha^{4n}(\alpha + \alpha^3 + \alpha^4), \\
\quad \frac{1}{\gamma^2} + \frac{1}{\gamma} + \sum_{n=0}^{\infty} \gamma^{4n}(\gamma + \gamma^3 + \gamma^4) \right) (\in T(11))
\]

Figure 2:

\[
= \left( \frac{1}{\alpha^3} + \frac{1}{\alpha^2} + \frac{1}{\alpha} + \sum_{n=0}^{\infty} \alpha^{4n}(\alpha^2 + \alpha^3 + \alpha^4), \\
\qquad \frac{1}{\gamma^3} + \frac{1}{\gamma^2} + \frac{1}{\gamma} + \sum_{n=0}^{\infty} \gamma^{4n}(\gamma^2 + \gamma^3 + \gamma^4) \right) (\in T(11)).
\]

2 Dimension of \( T(0) \cap T(1) \cap T(11) \) We will study the following set in \( \mathbb{C} \):

\[
E = \left\{ \sum_{i=0}^{\infty} a_i \alpha^i : \left( \sum_{i=0}^{\infty} a_i \alpha^i, \sum_{i=0}^{\infty} a_i \gamma^i \right) \in T(0) \cap T(1) \cap T(11) \right\}.
\]

Figure 3 shows \( E \). The automaton which accepts words determining points in \( T(0) \cap T(1) \cap T(11) \) is shown in the appendix.

A cycle is a directed graph \( H \) for which there is a closed path which passes into every vertex exactly once and such that every edge of \( H \) is an edge of this path. A directed graph \( H \) is strongly connected provided that whenever each of \( x \) and \( y \) is a vertex of \( H \), then there is a path from \( x \) to \( y \).

A strongly connected component of \( G \) is a maximal subgraph \( H \) of \( G \) such that \( H \) is strongly connected. It is clear that the strongly connected components of \( G \) are pairwise disjoint. A vertex is not considered to be strongly connected unless it is looped on itself.

The automaton which represents \( T(0) \cap T(1) \cap T(11) \) is decomposed into strongly connected components as is shown in Figure 4. Every strongly connected components except a special component \( X \) consists of one cycle. Figure 5 shows the component \( X \). All of the infinite paths which do not remain in \( X \) end up in cycles and they are a countable set. The dimension of \( E \) is equal to the dimension of the set which consists of points determined by the infinite words accepted by \( X \) fixing a vertex as the initial state. Let \( a, b \) and \( c \) be the states shown in Figure 5. Let \( A = L(X,a), B = L(X,b), C = L(X,c) \) be the sets of the infinite words accepted by \( X \) with the initial satetes, \( a, b, c \), respectively. Then \( A, B, C \) satisfy the following set-equations, namely we obtain a graph iterated function system [5].

\[
\begin{align*}
A &= f_1(A) \cup f_2(A) \cup f_3(B), \\
B &= g_1(A) \cup g_2(B) \cup g_3(C), \\
C &= h_1(A) \cup h_2(A) \cup h_3(C).
\end{align*}
\]
cycles

initial state

Figure 4: decomposition to strongly connected components

Figure 5: \( X \)
The Mauldin-Williams graph $G$ for this system is shown in Figure 6. The dimension $s$

\begin{align*}
  f_1(x) &= \alpha^4x + \alpha^2 + \alpha + 1 \\
  f_2(x) &= \alpha^4x + \alpha^2 + \alpha \\
  f_3(x) &= \alpha^5x + \alpha^4 + \alpha^2 + \alpha \\
  g_1(x) &= \alpha^7x + \alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + 1 \\
  g_2(x) &= \alpha^4x + \alpha^3 + \alpha^2 + 1 \\
  g_3(x) &= \alpha^5x + \alpha^4 + \alpha^3 + \alpha \\
  h_1(x) &= \alpha^{10}x + \alpha^8 + \alpha^7 + \alpha^6 + \alpha^4 + \alpha^3 + \alpha^2 + 1 \\
  h_2(x) &= \alpha^5x + \alpha^3 + \alpha^2 + 1 \\
  h_3(x) &= \alpha^4x + \alpha^3 + \alpha^2
\end{align*}

Figure 6: Mauldin-Williams graph for $E$

associated to $G$ is computed as follows [5, 9]:

\[
\det \begin{pmatrix}
  2|\alpha|^{14} - 1 & |\alpha|^{16} & 0 \\
  |\alpha|^{17} & |\alpha|^{19} - 1 & |\alpha|^{15} \\
  |\alpha|^{18} + |\alpha|^{20} & 0 & |\alpha|^{14} - 1
\end{pmatrix} = 0.
\]

Thus $|\alpha|^4$ can take the following four real values,

\[-1, \quad -0.9704028248572908964456 \cdots, \quad -0.82652335536233482093 \cdots, \quad 0.7925434937573255006271 \cdots.\]
Since $s > 0$, we obtain $|\alpha|^s = 0.792543498375732550062 \ldots$ and $s = 1.15931959819470279575 \ldots$ By using the result of [5], this value is the upper bound of the Hausdorff dimension of $A, B, C$.

**Theorem 2** $E$ has the Hausdorff dimension smaller than or equal to $1.15931959819470279575 \ldots$.

From numerical experiments, the upper bound in the theorem above seems to be exactly equal to the Hausdorff dimension of $E$.

**Conjecture 1** Each of the sets, $f_1(A) \cap f_2(A), f_2(A) \cap f_3(B), f_3(B) \cap f_1(A), g_1(A) \cap g_2(B), g_2(B) \cap g_3(C), g_3(C) \cap g_1(A), h_1(A) \cap h_2(A), h_2(A) \cap h_3(C), h_3(C) \cap h_1(A)$ consists of only one point.

$A, B$ and $C$ has the same Hausdorff dimension $s$, and each of the $s$-dimensional Hausdorff measure of $A, B, C$ is positive and finite. (See the first half of the proof of Corollary 3.5 in [6].) Regarding this conjecture to be true, we have

$$
\mathcal{H}^s(A) = \mathcal{H}^s(f_1(A)) + \mathcal{H}^s(f_2(A)) + \mathcal{H}^s(f_3(B)) = |\alpha|^4s \mathcal{H}^s(A) + |\alpha|^{4s} \mathcal{H}^s(A) + |\alpha|^{5s} \mathcal{H}^s(B).
$$

where $\mathcal{H}^s(A)$ denotes the $s$-dimensional Hausdorff measure of $A$. In the same way, we obtain

$$
\begin{pmatrix}
\mathcal{H}^s(A) \\
\mathcal{H}^s(B) \\
\mathcal{H}^s(C)
\end{pmatrix}
= \begin{pmatrix}
2|\alpha|^{4s} & |\alpha|^{5s} & 0 \\
|\alpha|^{7s} & |\alpha|^{4s} & |\alpha|^{5s} \\
|\alpha|^{5s} + |\alpha|^{10s} & 0 & |\alpha|^{4s}
\end{pmatrix}
\begin{pmatrix}
\mathcal{H}^s(A) \\
\mathcal{H}^s(B) \\
\mathcal{H}^s(C)
\end{pmatrix}
$$

and we have (2).

Figure 7 shows the points $\{z : (z, w) \in T(0) \cap T \cap T', T, T' \in \mathcal{T}\}$, which seems to have the same dimension as that of $T(0) \cap T(1) \cap T(11)$.

## 3 Appendix

The transition function of the automaton $M$ are shown below. The notation

\[
\begin{align*}
m &= 0 \Rightarrow n \\
0 &= 0 \Rightarrow 1 \\
1 &= 0 \Rightarrow k
\end{align*}
\]

means that there are edges from the state $m$, one to the state $n$ labeled by 0, one to $l$ labeled by 0, and one to $k$ labeled by 1.
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