

A NOTE ON THE FEASIBLE SETS OF LINEAR INEQUALITY SYSTEMS

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ABSTRACT. The aim of this paper is to study some properties of the feasible set associated with a consistent system defined on a locally convex space. The consequences of the boundedness and some aspects of the dimension of feasible set have been analyzed. Moreover, the differences between the finite dimensional and the infinite dimensional cases have been shown through illustrative examples.

1 Introduction and notation This paper deals with linear inequality systems of the form

$$\sigma := \{\langle x_j, \varphi \rangle \geq c_j, j \in J\},$$

where J denotes an arbitrary nonempty index set, x_j is a vector belonging to a given real locally convex space X and c_j is a real number for each $j \in J$, whereas the unknown φ ranges on the continuous dual space X^* . If there exists a continuous linear form φ satisfying $\langle x_j, \varphi \rangle := \varphi(x_j) \geq c_j$ for all $j \in J$, then σ is said to be consistent. The set of all solutions of σ will be denoted by F_σ and will be called the feasible set of σ . Obviously, F_σ is a convex and weakly closed subset of X^* .

On the other hand, we associate with σ the wedge

$$K_\sigma := \text{cone} \{(x_j, c_j), j \in J; (o, -1)\},$$

i.e., the set of all nonnegative linear combinations of elements (x_j, c_j) and the pair $(o, -1) \in X \times \mathbb{R}$, where o denotes the zero-vector in X . The wedge K_σ is called *characteristic cone* of σ in [1-2]. The consistency of σ is closely related to the properties of either its characteristic cone K_σ or its closure $\text{cl}K$, which is called the *reference cone* of σ in [5].

We shall say that a system $\sigma := \{\langle x_j, \varphi \rangle \geq c_j, j \in J\}$ is nontrivial if there exists at least some $j \in J$ such that $x_j \neq o$ and $c_j > 0$.

The main purpose of the paper is to provide conditions for the weak boundedness of the feasible set associated with a given system σ , in the infinite dimensional case. This will allow us to prove that Theorem 2.1 in [2] is only valid under the assumption of X to be finite dimensional. Finally, we study the dimensionality of F_σ by means of two relevant sets, namely, the lineality space of the reference cone and the affine hull of the feasible set (see, e.g., [3, p.33] and [4], respectively).

2 Boundedness of the feasible set Recall that, given a consistent system $\sigma := \{\langle x_j, \varphi \rangle \geq c_j, j \in J\}$, its feasible set F_σ is convex and closed for the weak topology on X^* .

The next theorem provides some necessary conditions for the boundedness of F_σ for the mentioned topology, but its proof requires some additional notation.

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Given a convex set B , we denote by $C_B := \{x \in X : x + B \subset B\}$ the so-called *recession cone* of B (see [3, p.34]). In other words,

$$(1) \quad C_B := \{x \in X : b + tx \in B \text{ for all } b \in B \text{ and } t \geq 0\}.$$

Theorem 1. *Let $\sigma := \{\langle x_j, \varphi \rangle \geq c_j, j \in J\}$ be a consistent system posed on a locally convex space X . Consider the following conditions:*

- (i) F_σ is weakly bounded;
- (ii) the unique solution of the homogeneous associated system $\sigma_0 := \{\langle x_j, \varphi \rangle \geq 0, j \in J\}$ is o ; and
- (iii) $\text{cl}P = X$, where $P := \text{cone}\{x_j, j \in J\}$.

Then, (i) implies (ii) and the conditions (ii) and (iii) are equivalent.

Proof. (i) \Rightarrow (ii) Taking $B = F_\sigma$ in 1 we have

$$(2) \quad C_{F_\sigma} := \{\varphi \in X^* : \psi + t\varphi \in F_\sigma \text{ for all } \psi \in F_\sigma \text{ and } t \geq 0\}$$

and, since F_σ is weakly bounded, its recession cone $C_{F_\sigma} = \{o\}$.

On the other hand, from 2, it is immediate to check that

$$C_{F_\sigma} = \{\varphi \in X^* : \langle x_j, \varphi \rangle \geq 0, j \in J\} = F_{\sigma_0}.$$

Hence $F_{\sigma_0} = \{o\}$.

(ii) \Rightarrow (iii) First, we shall prove that

$$(3) \quad A := \{x \in X : \langle x, \varphi \rangle \geq 0, \text{ for all } \varphi \in F_{\sigma_0}\} = \text{cl}P.$$

In fact, for each $x \in A$ the pair $(x, 0)$ is a consequent relation of σ_0 and, according to Theorem 2 in [5], we get

$$(x, 0) \in \text{cl}K_{\sigma_0} := \text{cl}[\text{cone}\{(x_j, 0), j \in J; (o, -1)\}],$$

so that $x \in \text{cl}[\text{cone}\{x_j, j \in J\}] = \text{cl}P$.

Reciprocally, let x be an arbitrary element of $\text{cl}P$ and let $\{y_\delta : \delta \in D\}$ be a net convergent to x and contained in P . Then, for any $\varphi \in F_{\sigma_0}$ we have $\langle y_\delta, \varphi \rangle \geq 0$ and consequently, $\langle x, \varphi \rangle \geq 0$. Thus $x \in A$.

Now, since $F_{\sigma_0} = \{o\}$, the set $A = X$. Hence, from 3, we obtain $\text{cl}P = X$.

(iii) \Rightarrow (ii) Assume the existence of $\varphi_0 \in F_{\sigma_0}$ such that $\varphi_0 \neq 0$. Then there exists $x_0 \neq o$ satisfying $\langle x_0, \varphi_0 \rangle \neq 0$. We shall obtain a contradiction in both possible cases:

If $\langle x_0, \varphi_0 \rangle < 0$, then $x_0 \notin \text{cl}P$, with $\text{cl}P = X$, and this is a contradiction. If, alternatively, $\langle x_0, \varphi_0 \rangle > 0$, we obtain the same contradiction just replacing x_0 with $-x_0$.

Therefore, $F_{\sigma_0} = \{o\}$ and the theorem follows.

We have just seen that Theorem 1 is valid for locally convex spaces but it is important to remark that, in finite dimension, the three conditions of the statement are equivalent to each other (see [2, p. 80]). The next example provides a consistent systems σ whose feasible set F_σ is not bounded whereas the unique solution of its corresponding homogeneous systems σ_0 is 0. This shows that the equivalence between conditions (i) and (ii) (or (iii)) does not hold for infinite dimensional spaces.

Example 2. Consider, in the Hilbert space $X = l^2$, the set

$$A = \{ \eta = (y_n)_{n=1,2,\dots} \in l^2 : |y_n| \leq n \}.$$

It can be easily realized that A is not bounded by taking the sequence $(\eta_n)_{n=1,2,\dots}$ defined as $\eta_n = (1, 2, \dots, n, \frac{1}{n+1}, \frac{1}{n+2}, \dots)$, with norm $\|\eta_n\| > n$ for each $n = 1, 2, \dots$. On the other hand, it is obvious that A is a closed and convex set, so it can be expressed as the intersection of all closed halfspaces containing it (see, for instance, Theorem 20.7.5 in [4]). Since the dual space $X^* = X$, each halfspace is defined by a vector, say \varkappa , and a real number c_\varkappa . Therefore, denoting by B the set of all the characteristic vectors \varkappa , we have

$$A = \bigcap_{\varkappa \in B} \{ \eta : \langle \varkappa, \eta \rangle \geq c_\varkappa \}.$$

In this way, A is nothing else than the feasible set of the system

$$\sigma := \{ \langle \varkappa, \eta \rangle \geq c_\varkappa, \varkappa \in B \}.$$

We have just to show that F_{σ_0} reduces to zero. In fact, as we have shown in Theorem 1, the recession cone of the feasible set is $C_{F_\sigma} = F_{\sigma_0}$. Moreover, since $F_\sigma = A$, we get

$$C_A = \{ \eta : \xi + t\eta \in A \text{ for all } \xi \in A \text{ and } t \geq 0 \}.$$

Taking an arbitrary element η in C_A , then $\xi + t\eta \in A$ for all $\xi \in A$ and for all real $t \geq 0$. If $\eta = (y_n)_{n=1,2,\dots}$ and $\xi = (x_n)_{n=1,2,\dots}$ then, for each $n = 1, 2, \dots$ we must have

$$(4) \quad |x_n + ty_n| \leq n,$$

with y_n being fixed, and this for all $(x_n)_{n=1,2,\dots} \in A$ and for all $t \geq 0$. Inequality 4 clearly implies that $y_n = 0$ for each n and so $\eta = 0$. Hence $F_{\sigma_0} = C_A = \{0\}$.

3 Codimension Given a nonvoid closed and convex set A its *lineality space* L_A is defined as $L_A := \{x \in X : x + A = A\}$. In particular, given a consistent system σ , the lineality space of $\text{cl}K_\sigma$ will be merely denoted by L , i.e., $L := L_{\text{cl}K_\sigma}$. On the other hand, the *affine hull* of a convex set A is the minimal affine subspace which contains A , so that it can be expressed as $\text{aff}(A) = x + [A - A]$, where $[A - A]$ represents the linear span of the set $A - A$ and x is an arbitrary element of A . Finally, the weak closure of $[F_\sigma - F_\sigma]$ will be denoted by M . The next result shows that both sets L and M are complementary from a dimensional point of view.

Theorem 3. *Let σ be a consistent system posed on a locally convex space X . If $\text{codim}M < +\infty$, then $\dim L = \text{codim}M$.*

Proof. Assume $\text{codim}M = n$. Then, according to Theorem 15.8.2 in [4], there exists a finite dimensional subspace, say F , which is the topological complement (for the weak topology on X^*) of M . Thus, $X^* = M \oplus F$ and the orthogonals M^\perp and F^\perp constitute a topological decomposition for the weak topology in X (recall [4, 20.5.4]). Moreover, by [4, 9.2.7a], the subspace M^\perp has dimension n and so, if $\{y_1, y_2, \dots, y_n\}$ is a basis of M^\perp , then

$$(5) \quad [F_\sigma - F_\sigma] \subset M \subset M^{\perp\perp} = \bigcap_{i=1}^n [y_i; 0],$$

where $[y_i; 0]$ denotes the hyperplane defined by the linear form y_i and the scalar 0.

From 5 we deduce that there exist real numbers $\{d_1, d_2, \dots, d_n\}$ such that the hyperplanes $[y_i; d_i], i = 1, \dots, n$, satisfy $F_\sigma \subset \bigcap_{i=1}^n [y_i; d_i]$, i.e., $\pm(y_i, d_i), i = 1, \dots, n$, are consequent relations of σ . Hence, applying again Theorem 2 in [5], we get $\pm(y_i, d_i) \in \text{cl}K_\sigma$, so that $\{(y_i, d_i) : i = 1, \dots, n\} \subset L$.

On the other hand, since $\{y_1, y_2, \dots, y_n\}$ is a basis of M^\perp , the vectors $\{(y_i, d_i) : i = 1, \dots, n\}$ are linearly independent and consequently,

$$(6) \quad \dim L \geq n.$$

Now, consider p linearly independent vectors $\{(z_j, e_j) : j = 1, \dots, p\}$ from L . Clearly the weak closure of $\text{aff}(F_\sigma)$ is included in $\bigcap_{j=1}^p [z_j; e_j]$ and so

$$\text{codim}M = \text{codim}(\text{claff}(F_\sigma)) \geq \text{codim}(\bigcap_{j=1}^p [z_j; e_j]) = p.$$

Hence $n \geq p$, which entails $n \geq \dim L$ which combined with 6 yields $n = \dim L$. The proof is complete.

Notice that, by definition, the dimension of an arbitrary convex set A is given by $\dim A := \dim [A - A]$ (see [3, p.9]), i.e., the dimension (codim) of A is the dimension (codim) of the subspace associated with its affine hull. Thus, under the hypothesis of Theorem 3, we have

Corollary 4. *If $\text{codim}F_\sigma = 0$, then $L = \{(0, 0)\}$.*

Moreover, since in finite dimensional spaces all subspaces are closed, we get the following result:

Corollary 5. *If $\dim X = n$, then $\dim L = \text{codim}F_\sigma = n - \dim F_\sigma$. In particular, if $\dim F_\sigma = n$, then K_σ is a cone.*

Remark 6. Theorem 3 generalizes the first statement in Corollary 5, which was proved by Zhu in [5] for the finite case. Next, we shall show that the hypothesis $\text{codim}M < +\infty$ (i.e., the finite dimension of the weak closure of $\text{aff}(F_\sigma)$) in Theorem 3 is essential.

Example 7. Consider the space $X = l^1$ and its dual $X^* = l^\infty$. Define the set

$$B = \{x \in l^1 : x = (0, b_2, 0, b_4, \dots) \text{ with } b_{2n} = 0 \text{ except for a finite quantity}\},$$

and let J be an index set having the same cardinality as B . This means that each element of B can be denoted as x_j for a unique index $j \in J$.

Consider the system $\sigma = \{\langle x_j, \varphi \rangle \geq 0, j \in J\}$, whose feasible set is

$$F_\sigma = \{\varphi \in l^\infty : \varphi = (\varphi_1, 0, \varphi_3, 0, \dots)\}.$$

It can be easily realized that $\text{cl}K_\sigma \subset B \times]-\infty, 0]$. Moreover, since the maximal subspace contained in $B \times]-\infty, 0]$ is $B \times \{0\}$, the lineality space L associated with $\text{cl}K_\sigma$ satisfies $L \subset B \times \{0\}$ and so

$$(7) \quad \dim L \leq \dim B = \aleph_0.$$

On the other hand, since F_σ is a closed subspace of l^∞ , the quotient space l^∞/F_σ verifies

$$(8) \quad \text{codim}F_\sigma = \dim(l^\infty/F_\sigma) > \aleph_0.$$

From 7 and 8 we conclude that $\dim L \neq \text{codim}F_\sigma$ and this proves that Theorem 3 can fail for those systems σ whose feasible sets F_σ have infinite codimension.

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