THERE ARE NO CODIMENSION 1 LINEAR ISOMETRIES ON THE BALL AND POLYDISK ALGEBRAS

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Abstract. Let $A$ be the ball algebra or the polydisk algebra in $\mathbb{C}^n$. When $n > 1$, there are no codimension 1 linear isometries on $A$. (via LaTeX2e)

1. Introduction.

Let $X$ be a compact Hausdorff space and $C(X)$ the Banach algebra of all complex-valued continuous functions on $X$ with the supremum norm. A uniformly closed subalgebra of $C(X)$ is called a function algebra on $X$ if it separates the points of $X$ and contains the constants. Let $B_n$ be the open unit ball of $\mathbb{C}^n$ and $S_n$ be the boundary of $B_n$. Let $D$ and $T$ stand for $B_1$ and $S_1$ respectively. Let $A(S_n)$ be the space of all $f \in C(S_n)$ which can be extended holomorphically on $B_n$. The algebra $A(S_n)$ is called the ball algebra. When $n = 1$, the algebra $A(T)$ is called the disk algebra.

Let $D^n$ be the unit polydisk and $T^n$ be the torus. Let $A(T^n)$ be the space of all $f \in C(T^n)$ which can be extended holomorphically on $D^n$. The algebra $A(T^n)$ is called the polydisk algebra. We note that $A(S_n)$ is a function algebra on $S_n$ and $A(T^n)$ is a function algebra on $T^n$.

Let $H^\infty(D)$ be the Banach algebra of all bounded holomorphic functions on $D$. Let $H^\infty$ be the space of radial limits of functions in $H^\infty(D)$. Let $L^\infty$ be the algebra of all essentially bounded measurable functions on $T$. Then $H^\infty$ is an essential supremum norm closed subalgebra of $L^\infty$. A closed subalgebra of $L^\infty$ containing $H^\infty$ is said to be a Douglas algebra.

Let $E$ be a Banach space. A linear isometry $T : E \to E$ is said to be of codimension 1 if the range of $T$ has codimension 1 in $E$. In [2], Araujo and Font studied codimension 1 linear isometries on function algebras and on Douglas algebras. And they conjectured that there are no codimension 1 linear isometries on proper Douglas algebras. In [4], Izuchi gave a characterization of codimension 1 linear isometries of Douglas algebras. Also in [8], Takayama and Wada characterized codimension 1 linear isometries on the disk algebra.

In this paper, we studied codimension 1 linear isometries on the ball and polydisk algebras. Our theorem is the following.

Theorem Let $A$ be $A(S_n)$ or $A(T^n)$. When $n > 1$, there are no codimension 1 linear isometries on $A$.

2. Proof.

Suppose that $T : A \to A$ is a codimension 1 linear isometry. We denote by $\partial A$ the Shilov boundary of $A$. We say that the range of $T$ separates strongly the points of $\partial A$, if

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for given two elements of $\partial A$, $x_1$ and $x_2$, there exists $f \in T(A)$ such that $|f(x_1)| \neq |f(x_2)|$.

By [2, p. 2277], Araujo and Font classified codimension 1 linear isometries $T$ on function algebras into three types:

**Type I.** The range of $T$ separates strongly the points of $\partial A$, except two of them.

**Type II.** The range of $T$ separates strongly the points of $\partial A$ and there exists an element $x_0 \in \partial A$ such that $f(x_0) = 0$ for all $f \in T(A)$.

**Type III.** The range of $T$ separates strongly the points of $\partial A$ and, for each $x \in \partial A$, there exists $f \in T(A)$ such that $f(x) \neq 0$.

We shall prove that $T$ is a codimension 1 linear isometry of type III. To prove this, suppose first that $T$ is a codimension 1 linear isometry of type I and let $x_1$ and $x_2$ be the points which cannot be separated strongly. Then by [1, Corollary 5.1 and Lemma 2.1], $\partial A$ is homeomorphic to a quotient space of $\partial A$ identifying with $x_1$ and $x_2$ in $\partial A$. But $S_n$ and $T^n$ do not satisfy this condition. This is a contradiction.

Next suppose that $T$ is a codimension 1 linear isometry of type II. Let $x_0$ be the point in $\partial A$ such that $f(x_0) = 0$ for all $f \in T(A)$. By [2, Theorem 6.1], $x_0$ is isolated in $\partial A$. This is absurd.

Hence, $T$ is a codimension 1 linear isometry of type III. By [2, theorem 4], there exists a homeomorphism $\varphi$ of $\partial A$ onto $\partial A$ and a continuous map $\psi: \partial A \to \mathbb{C}$ such that $|\psi(x)| = 1$ for all $x \in \partial A$, and

$$(Tf)(x) = \psi(x)f(\varphi(x)) \text{ for all } x \in \partial A \text{ and } f \in A.$$ 

Since $T1 = \psi \in A$, $\psi$ is an inner function in $A$.

**Case** $A = A(S_n)$

Since there is no non-constant inner function extends continuously to $S_n$, $TA = A \circ \varphi \subseteq A$. Since the codimension of $A \circ \varphi$ in $A$ is 1,

$$A = A \circ \varphi + \mathbb{C}g \text{ for some } g \not\in A \circ \varphi.$$ 

Therefore

$$(2) \quad A \circ \varphi^{-1} = A + \mathbb{C}g \circ \varphi^{-1}, \quad g \circ \varphi^{-1} \not\in A.$$ 

By the above, $A \circ \varphi^{-1}$ is a function algebra on $S_n$ and $A$ is a proper subalgebra of $A \circ \varphi^{-1}$. For a function $f$ on $S_n$ and $\zeta \in S_n$, put $f_\zeta(\lambda) = f(\lambda \zeta), \lambda \in T$. Since $g \circ \varphi^{-1} \not\in A$, there exists a point $\zeta_0$ in $S_n$ such that $(g \circ \varphi^{-1})_{\zeta_0} \not\in A(T)$, see [6, p.6]. Put $(A \circ \varphi^{-1})_{\zeta_0} = \{f_{\zeta_0}(\lambda) : f \in (A \circ \varphi^{-1})\}$. Then $(A \circ \varphi^{-1})_{\zeta_0}$ is a closed subalgebra of $C(T)$. Since $A(S)_{\zeta_0} = A(T), A(T) \subseteq (A \circ \varphi^{-1})_{\zeta_0} \subseteq C(T)$. By Wermer's maximality theorem [3, p.214], $C(T) = (A \circ \varphi^{-1})_{\zeta_0}$. Therefore

$$(3) \quad C(T) = A(T) + \mathbb{C}(g \circ \varphi^{-1})_{\zeta_0}.$$ 

Hence $\tilde{z} = h_1 + a(g \circ \varphi^{-1})_{\zeta_0}$ and $\tilde{z}^2 = h_2 + b(g \circ \varphi^{-1})_{\zeta_0}$ for some $h_1, h_2 \in A(T), a, b \in \mathbb{C}$. Since $(g \circ \varphi^{-1})_{\zeta_0} \not\in A(T), a \neq 0$. Then $\tilde{z}^2 - \frac{a}{b} \tilde{z} = h_2 - \frac{a}{b}h_1$. The right-hand side belongs to $A(T)$, but the left-hand side does not. This is a contradiction. Hence, when $n > 1$, there are no codimension 1 linear isometries on $A(S_n)$. 

Case $A = A(T^n)$

Let $\mathbb{Z}$ be the set of all integers and $\mathbb{Z}_+$ the set of all nonnegative integers. Let $\mathbb{Z}^n$ and $\mathbb{Z}_+^n$ be the sets of all $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \mathbb{Z}$ and $\alpha_i \in \mathbb{Z}_+$ for every $1 \leq i \leq n$, respectively. Let $\hat{f}(k)$ be the $k$-th Fourier coefficient of a function $f$ on $T^n$, that is

$$\hat{f}(k) = \int_{T^n} f(w)\tilde{w}^k \, dm_n(w) \quad (k \in \mathbb{Z}^n)$$

where $\tilde{w}^k = \tilde{w}_1^{k_1} \cdots \tilde{w}_n^{k_n}$ and $dm_n = \frac{1}{(2\pi)^n} \, d\theta_1 \cdots d\theta_n$.

By (1),

(4) \quad $TA = \psi(A \circ \varphi) \subset A$.

Furthermore

(5) \quad $A \circ \varphi \subset A$.

To prove this, let $B$ be a closed subalgebra of $C(T^n)$ generated by $A$ and $A \circ \varphi$. By (4), $\psi B \subset A$. Suppose $A \circ \varphi \not\subset A$. Then there exists a function $f_0$ in $B$ such that $f_0$ does not belong to $A$. By [5, Theorem 2.2.1], there exists $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n \mathbb{Z}_+^n$ such that $\hat{f}_0(k) \neq 0$. We may assume $k_1 < 0$. Then there exists a point $w_0 \in T^{n-1}$ such that $f_0(\lambda, w_0) \not\in A(T)$. To see this, suppose that $f_0(\lambda, w) \in A(T)$ for all $w \in T^{n-1}$. Then

$$\int_T f_0(\lambda, w)\lambda^{k_1} \, dm_1(\lambda) = 0.$$  

Now integrate this with respect to $w$ and conclude that $\hat{f}_0(k_1, \ldots, k_n) = 0$. This is a contradiction.

For a subspace $L$ of $C(T^n)$, let $L_{w_0} = \{ f(\lambda, w_0) : f \in L \}$. Then $B_{w_0}$ is a closed subalgebra of $C(T)$. Since $A(T^n)_{w_0} = A(T)$, $A(T) \not\subset B_{w_0}$. By Wermer's maximality theorem, $B_{w_0} = C(T)$. Since $\psi B \subset A$, then $\psi_{w_0} C(T) = A(T)$, where $\psi_{w_0}(\lambda) = \psi(\lambda, w_0)$. Since $\psi$ is an inner function, $|\psi_{w_0}| = 1$ on $T$. Hence $\psi_{w_0} C(T) = C(T)$. This is a contradiction. Hence (5) holds.

First, suppose that $\psi$ is invertible in $A$. Since $\psi$ is inner, $\psi$ is a constant function. By (4), the codimension of $A \circ \varphi$ in $A$ is 1. Then

$$A = A \circ \varphi + \mathbb{C} g \text{ for some } g \in A, \quad g \not\in A \circ \varphi.$$  

Therefore

$$A \circ \varphi^{-1} = A + \mathbb{C} g \circ \varphi^{-1}, \quad g \circ \varphi^{-1} \not\in A.$$  

Hence $A \circ \varphi^{-1}$ is a function algebra on $T^n$, and $A$ is a proper subalgebra of $A \circ \varphi^{-1}$. In the same way as the proof of (5), there exist a point $w_0 \in T^{n-1}$ such that

$$(A + \mathbb{C} g \circ \varphi^{-1})_{w_0} = C(T).$$

Therefore

$$A(T) + \mathbb{C}(g \circ \varphi^{-1})_{w_0} = C(T).$$

This leads a contradiction as the case $A = A(S_n)$.

Hence $\psi$ is not invertible in $A$. Then there exists a point $x_0 \in \hat{D}^n \setminus T^n$ such that $\psi(x_0) = 0$. By (4) and (5), $f(x_0) = 0$ for every $f \in TA$. Let $A_{x_0}$ be the set of all $f \in A$ such that $f(x_0) = 0$. Then $TA \subset A_{x_0}$ and $A_{x_0}$ has codimension 1 in $A$. Since the codimension of $TA$ in $A$ is 1, $TA = A_{x_0}$. Therefore $\psi A \subset A_{x_0} = TA = \psi(A \circ \varphi) \subset \psi A$. Hence the codimension of $\psi A$ in $A$ is 1, so that $A = \psi A + \mathbb{C} h$ for some $h \in A$ and $h \not\in \psi A$. 

Then
\[ \tilde{\psi} A = A + \mathbb{C}\tilde{\psi} h, \quad \tilde{\psi}^2 A = A + \mathbb{C}\tilde{\psi} h, \quad \text{and} \quad \tilde{\psi} h \notin A. \]

Hence
\[ \tilde{\psi} = h_1 + a\tilde{\psi} h, \quad \text{for some} \ h_1 \in A \text{ and } a \in \mathbb{C}, \]
\[ \tilde{\psi}^2 = h_2 + b\tilde{\psi} h, \quad \text{for some} \ h_2 \in A \text{ and } b \in \mathbb{C}. \]

Since \( \tilde{\psi} h \notin A, a \neq 0, \) so that
\[ \tilde{\psi}^2 - \frac{b}{a} \tilde{\psi} = h_2 - \frac{b}{a} h_1. \]

The right-hand side belongs to \( A. \) Since \( \psi \) is an inner function in \( A \) and \( \frac{b}{a} \in \mathbb{C}, \) the left-hand side does not. This is a contradiction. Hence when \( n > 1, \) there are no codimension 1 linear isometries on \( A(T^n). \) \( \square \)

REFERENCES


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