ON LINEAR OPERATORS FROM ORLICZ SPACES INTO LOCALLY CONVEX LINEAR-TOPOLOGICAL SPACES

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Received May 15, 2000

Abstract. Let \( Y \) be a sequentially complete, locally convex linear-topological space, \((E, \Sigma, \mu)\), a non-atomic measure space and \( \varphi \) a real nondecreasing and continuous for \( u \geq 0 \) function, equal to 0 for \( u = 0 \). We prove the identity of the class of linear and pseudomodular continuous operators from the Orlicz space \( L^{\infty}_e(E, \Sigma, \mu) \) into \( Y \) with the class of similar operators from the Orlicz space \( L^{\infty}_r(E, \Sigma, \mu) \) into \( Y \), where

\[
\varphi(u) = \int_0^u p(t) dt \quad \text{for} \quad u \geq 0 \quad \text{and} \quad p(t) = \inf_{t \leq s} \frac{\varphi(s)}{s} \quad \text{for} \quad t \geq 0.
\]

Also, we show that the class of linear and pseudonorm continuous operators from the space of finite elements \( L^{\infty}_e(E, \Sigma, \mu) \) into \( Y \) is the same as from the space \( L^{\infty}_r(E, \Sigma, \mu) \) into \( Y \). The other proof of this fact, using Rademacher functions, one can find in [1] (th. 2.2 b). Our result is a little bit more general than the one mentioned above. But first of all, our proof is simple and essentially different from that in [1].

1. The Orlicz Space

Definition 1.1 A \( \varphi \)-function we call a real, nondecreasing and continuous for \( u \geq 0 \) function, equal to 0 for \( u = 0 \). A \( \varphi \)-function \( \varphi \) is called convex if it satisfies the Jensen inequality

\[
\varphi(\alpha u + \beta v) \leq \alpha \varphi(u) + \beta \varphi(v) \quad \text{for} \quad u, v, \alpha, \beta \geq 0, \quad \text{where} \quad \alpha + \beta = 1.
\]

As usual in the theory of the Orlicz spaces ([2], [3], [4]), a \( \varphi \)-function we call a real, nondecreasing and continuous for \( u \geq 0 \) function, equal to 0 only for \( u = 0 \) and tending to \( \infty \) as \( u \to \infty \). In this paper the use of the \( \varphi \)-functions instead of the \( \varphi \)-functions simplifies our considerations.

1.2. Let \((E, \Sigma, \mu)\) denote a measure space. Obviously, we assume that the measure \( \mu \) is \( \sigma \)-finite on \( E \). By \( S = S(E, \Sigma, \mu) \) we denote the space of \( \mu \)-measurable, real or complex-valued functions defined on \( E \). The measure space \((E, \Sigma, \mu)\) we call non-atomic if for every set \( G \in \Sigma \) there exists a set \( F \subset G \), \( F \in \Sigma \) with \( \mu(F) = \frac{1}{2} \mu(G) \). Then we say also that the measure \( \mu \) and the space \( S \) are non-atomic.

1.3. Let \( \varphi \) be a \( \varphi \)-function and \((E, \Sigma, \mu)\) a measure space. For \( f \in S \) we write

\[
\rho_\varphi(f) = \int_E \varphi(|f(x)|) d\mu.
\]

1991 Mathematics Subject Classification. 46E10, 46E30.

Key words and phrases. Orlicz spaces, continuous linear operators, locally convex linear topological spaces.
In the space $S$ the functional $\rho_\varphi(\cdot)$ is a pseudomodular in the Musielak-Orlicz sense \[6\].

By $L^{1,\varphi}_f = L^{1,\varphi}_f(E, \Sigma, \mu)$ we denote the class of those functions $f \in S$ for which $\rho_\varphi(\lambda f) < \infty$ for some $\lambda = \lambda(f) > 0$ and by $L^{0,\varphi}_f = L^{0,\varphi}(E, \Sigma, \mu)$ the class of those functions $f \in S$ for which $\rho_\varphi(\lambda f) < \infty$ for all $\lambda > 0$. The classes $L^{1,\varphi}_f$ and $L^{0,\varphi}_f$ are linear subspaces of $S$ and $L^{0,\varphi}_f \subseteq L^{1,\varphi}_f$.

The class $L^{1,\varphi}_f$ we call the Orlicz space and the class $L^{0,\varphi}_f$ the space of finite elements ([2],[3]).

The spaces $L^{1,\varphi}_f$ and $L^{0,\varphi}_f$ we call non-atomic if the measure space $(E, \Sigma, \mu)$ is non-atomic.

1.4. In the Orlicz space $L^{1,\varphi}_f$ the functional $\| \cdot \|_\varphi$ defined by the formula

$$\| f \|_\varphi = \inf \{ \varepsilon > 0 : \rho_\varphi(\frac{f}{\varepsilon}) \leq \varepsilon \} \quad (f \in L^{1,\varphi}_f),$$

is an $F$-pseudonorm. In the case when $\varphi$ is a convex $\bar{\varphi}$-function, the formula

$$\| f \|_\varphi = \inf \{ \varepsilon > 0 : \rho_\varphi(\frac{f}{\varepsilon}) \leq 1 \} \quad (f \in L^{1,\varphi}_f),$$

determines a $B$-pseudonorm in $L^{1,\varphi}_f$ equivalent to $\| \cdot \|_\varphi([2],[3],[6]).$

**Definition 1.5** We say that the sequence $(f_n)$ in $L^{1,\varphi}_f$ is pseudonorm convergent to $f$ in $L^{1,\varphi}_f$ if $\| f_n - f \|_\varphi \to 0$ as $n \to \infty$.

It is easily verified that the sequence $(f_n)$ in $L^{1,\varphi}_f$ is pseudonorm convergent to $f$ in $L^{1,\varphi}_f$ if and only if for any $\lambda > 0$ there holds $\rho_\varphi(\lambda(f_n - f)) \to 0$ as $n \to \infty$. The space $L^{0,\varphi}_f$ is a closed subspace of the Orlicz space $L^{1,\varphi}_f$ with respect on the pseudonorm convergence.

Also, we say that the sequence $(f_n)$ in $L^{1,\varphi}_f$ is pseudomodular convergent or $\varphi$-convergent to $f \in L^{1,\varphi}_f$ and write $f_n \overset{\varphi}{\rightarrow} f$, if $\rho_\varphi(\lambda(f_n - f)) \to 0$ as $n \to \infty$ for some $\lambda > 0$ depending on $(f_n)$, ([3]).

1.6. In the case when $\varphi$ is a $\varphi$-function, the suffix "pseudo" in 1.3, 1.4 and 1.5 we omit, because then we have the classical Orlicz spaces ([2],[3]).

2. Convex $\bar{\varphi}$-functions $\bar{\varphi}$ and $\bar{\bar{\varphi}}$ generated by $\varphi$.

2.1. Let $\varphi$ be a $\bar{\varphi}$-function. We define

$$\varphi(u) = \int_0^u p(t)dt \quad \text{for} \quad u \geq 0 , \quad \text{where} \quad p(t) = \inf_{s > t} \frac{\varphi(s)}{s} \quad \text{for} \quad t \geq 0.$$ 

Since the function $p$ is non-negative and nondecreasing for $t \geq 0$, so the function $\bar{\varphi}$ is a convex $\bar{\varphi}$-function, ([2]). Also, we observe that if $\varphi$ is such that $\lim_{u \to \infty} \inf_{u > 0} \frac{\varphi(u)}{u} = 0$, then $\varphi(u) = 0$ for $u \geq 0$, and if $\varphi$ is a $\varphi$-function with the property $\lim_{u \to \infty} \inf_{u > 0} \frac{\varphi(u)}{u} > 0$, then $\bar{\varphi}$ is a convex $\varphi$-function.

Further, by $\Psi_\varphi$ we denote the class of those convex $\bar{\varphi}$-function $\psi$ for which the inequality $\psi(u) \leq \varphi(u)$ for $u \geq 0$ holds. The class $\Psi_\varphi$ is not empty, because the function $\psi_\varphi(u) = 0$ for $u \geq 0$ belongs to $\Psi_\varphi$. We define
\( \varphi(u) = \sup \{ \psi(u) : \psi \in \Psi \} \) for \( u \geq 0 \).

Obviously, \( \varphi \) is a function satisfying the inequality \( 0 \leq \varphi(u) \leq \varphi(2u) \) for \( u \geq 0 \).

The function \( \varphi \) is nondecreasing and convex for \( u \geq 0 \), because it is a supremum of nondecreasing and convex functions for \( u \geq 0 \). The Jensen inequality for \( \varphi \) guarantees the continuity of \( \varphi \) for \( u > 0 \) and the inequality \( 0 \leq \varphi \leq \varphi(u) \) the continuity of \( \varphi \) for \( u = 0 \). Hence \( \varphi \) is the greatest convex \( \varphi \)-function satisfying the inequality \( \varphi(u) \leq \varphi(2u) \) for \( u \geq 0 \).

**Theorem 2.2** For any \( \varphi \)-function \( \varphi \) the following inequality holds

\[ \varphi(u) \leq \varphi(u) \leq \varphi(2u) \quad \text{for} \quad u \geq 0. \]

**Proof.** From the definition of \( \varphi \) we get for \( u > 0 \)

\[ \varphi(u) \leq u \inf_{u \leq s} \frac{\varphi(s)}{s} \leq u \frac{\varphi(u)}{u} = \varphi(u). \]

Since \( \varphi \) is a convex \( \varphi \)-function, so it must be \( \varphi(u) \leq \varphi(u) \) for \( u \geq 0 \).

On the other hand, from the Jensen inequality for \( \varphi \) and the fact that \( \varphi(0) = 0 \) it follows that the quotient \( \frac{\varphi(u)}{u} \) is a nondecreasing function for \( u > 0 \). Therefore by virtue of the inequality \( \varphi \leq \varphi \) we have for \( u > 0 \)

\[ \varphi(u) = u \frac{\varphi(u)}{u} = u \inf_{u \leq s} \frac{\varphi(s)}{s} \leq u \inf_{u \leq s} \frac{\varphi(s)}{s} = u \varphi(u) \leq \int_0^u p(t) \, dt \leq \varphi(2u). \]

Hence the inequality \( \varphi(u) \leq \varphi(2u) \) for \( u \geq 0 \) is also true. \( \square \)

**2.3** If the \( \varphi \)-function \( \varphi \) satisfies the condition \( \lim_{u \to \infty} \frac{\varphi(u)}{u} = \infty \) then the following equality holds \( \varphi = (\varphi^*)^* \), where

\[ \varphi^*(v) = \sup \{ uw - \varphi(u) : u \geq 0 \} \quad \text{for} \quad v \geq 0. \]

This theorem for \( \varphi \)-functions one can find in [4]. Since the proof of our theorem is analogical, we omit it.

**2.4** For arbitrary \( \varphi \)-function \( \varphi \) the following equality holds

\[ \varphi(u) = \inf_{k=1}^m \alpha_k \varphi(u_k) \quad (u \geq 0), \]

where infimum is taken over all convex combinations

\[ u = \sum_{k=1}^m \alpha_k u_k, \quad \text{where} \quad \alpha_k, u_k \geq 0 \quad \text{for} \quad k = 1, \ldots, m \quad \text{and} \quad \sum_{k=1}^m \alpha_k = 1. \]

**Proof.** By \( \varphi \) we denote the function defined by the right side of the equality \( (\ast) \). Let us take

\[ u = \sum_{k=1}^m \alpha_k u_k, \quad \text{where} \quad \alpha_k, u_k \geq 0 \quad \text{for} \quad k = 1, \ldots, m \quad \text{and} \quad \sum_{k=1}^m \alpha_k = 1. \]
Then by virtue of the inequality $\mathcal{P} \leq \varphi$ and the convexity of $\mathcal{P}$ we have

$$\mathcal{P}(u) \leq \sum_{k=1}^{m} \alpha_k \varphi(u_k) \leq \sum_{k=1}^{m} \alpha_k \varphi(u_k).$$

From this we deduce that $\mathcal{P}(u) \leq \varphi(u)$ dla $u \geq 0$.

On the other hand, from the definition of $\mathcal{P}$ we get immediately the inequality $\mathcal{P}(u) \leq \varphi(u)$ for $u \geq 0$. We shall show that $\mathcal{P}$ is a convex $\varphi$-function. Let us take arbitrary $\varepsilon > 0$, $u', u'', \alpha, \beta \geq 0, \alpha + \beta = 1$. We observe that there exist $\alpha'_{k}, u'_{k} \geq 0$, where $k = 1, \ldots, m'$, such that

$$\sum_{k=1}^{m'} \alpha'_{k} = 1, \quad u' = \sum_{k=1}^{m'} \alpha'_{k} u'_{k}, \quad \text{and} \quad \sum_{k=1}^{m'} \alpha'_{k} \varphi(u'_{k}) \leq \varphi(u') + \varepsilon,$$

and $\alpha''_{k}, u''_{k} \geq 0$, where $k = 1, \ldots, m''$ such that

$$\sum_{k=1}^{m''} \alpha''_{k} = 1, \quad u'' = \sum_{k=1}^{m''} \alpha''_{k} u''_{k}, \quad \text{and} \quad \sum_{k=1}^{m''} \alpha''_{k} \varphi(u''_{k}) \leq \varphi(u'') + \varepsilon.$$

Let us denote $m = m' + m''$, $\alpha_{k} = \alpha'_{k}, u_{k} = u'_{k}$ for $k = 1, \ldots, m'$ and $\alpha_{m'+k} = \beta \alpha''_{k}, u_{m'+k} = u''_{k}$ for $k = 1, \ldots, m''$. Since $\alpha_{k}, u_{k} \geq 0$ for $k = 1, \ldots, m$,

$$\sum_{k=1}^{m} \alpha_{k} = \sum_{k=1}^{m'} \alpha'_{k} + \beta \sum_{k=1}^{m''} \alpha''_{k} = 1 \quad \text{and}$$

$$\sum_{k=1}^{m} \alpha_{k} u_{k} = \sum_{k=1}^{m'} \alpha'_{k} u'_{k} + \beta \sum_{k=1}^{m''} \alpha''_{k} u''_{k} = \alpha u' + \beta u'',$$

so we get

$$\mathcal{P}(\alpha u' + \beta u'') \leq \sum_{k=1}^{m} \alpha_{k} \varphi(u_{k}) = \alpha \sum_{k=1}^{m'} \alpha'_{k} \varphi(u'_{k}) + \beta \sum_{k=1}^{m''} \alpha''_{k} \varphi(u''_{k})$$

$$\leq \alpha (\varphi(u') + \varepsilon) + \beta (\varphi(u'') + \varepsilon) = \alpha \varphi(u') + \beta \varphi(u'') + \varepsilon.$$

From this we obtain the Jensen inequality for $\mathcal{P}$ and thus the function $\mathcal{P}$ is convex for $u \geq 0$. Now, the convexity of $\varphi$ implies the continuity of $\mathcal{P}$ for $u > 0$ and the inequality $\mathcal{P} \leq \varphi \leq \varphi$ the continuity of $\mathcal{P}$ for $u = 0$ and $\mathcal{P}(0) = 0$. The function $\mathcal{P}$ is also nondecreasing for $u \geq 0$, because for $0 \leq u_{1} < u_{2}$ we have

$$\mathcal{P}(u_{1}) = \mathcal{P}(u_{1} u_{2} + (1 - u_{1} u_{2})0) \leq \frac{u_{1}}{u_{2}} \mathcal{P}(u_{2}) \leq \mathcal{P}(u_{2}).$$

Hence $\mathcal{P}$ is a convex $\varphi$-function. This fact and the inequality $\mathcal{P} \leq \varphi \leq \varphi$ imply finally the equality $\mathcal{P} = \varphi$. $\square$

2.5. Remark In 2.4 we may assume that the numbers $\alpha_{k}$ are of finite binary representation.

Proof. According to the formula 2.4 (*) for arbitrary $u \geq 0$ and $\varepsilon > 0$ there exist such $\alpha_{k}, u_{k} \geq 0$, where $k = 1, \ldots, m$ that

$$\sum_{k=1}^{m} \alpha_{k} = 1, \quad u = \sum_{k=1}^{m} \alpha_{k} u_{k} \quad \text{and} \quad \mathcal{P}(u) \leq \sum_{k=1}^{m} \alpha_{k} \varphi(u_{k}) \leq \mathcal{P}(u) + \varepsilon.$$
If $m = 1$, then $\alpha_1 = 1$. Therefore let us suppose $m > 1$. Since $\alpha_k \geq 0$ for $k = 1, \ldots, m$ and $\alpha_1 + \cdots + \alpha_m = 1$, so it must be $\alpha_k \geq \frac{1}{m}$ for some index $k_0$. We may assume that $k_0 = m$. Let $M$ be a positive number such that $u_k \leq M$ and $\varphi(u_k) \leq M$ for $k = 1, \ldots, m$. Since the function $\varphi$ is continuous in the point $u_m$, so there exists $\delta$ satisfying $0 < \delta < \varepsilon$ and such that $|\varphi(u_m) - \varphi(v)| \leq \frac{\varepsilon}{4}$ if $v \geq 0$ and $|u_m - v| \leq \delta$. We choose numbers $\beta_k$, $k = 1, \ldots, m - 1$, of finite binary representation such that $0 \leq \beta_k \leq \alpha_k$ and $\beta_k - \beta_{k+1} \leq \frac{\delta}{2mM}$ for $k = 1, \ldots, m - 1$.

We observe that the number $\beta_m = 1 - (\beta_1 + \cdots + \beta_{m-1})$ is of finite binary representation and satisfies the inequalities

$$\frac{1}{m} \leq \alpha_m \leq \beta_m \leq 1 \quad \text{and} \quad \beta_m - \alpha_m = \sum_{k=1}^{m-1} (\alpha_k - \beta_k) \leq \frac{\delta}{2mM}.$$ 

Let us put $v_k = u_k$ for $k = 1, \ldots, m - 1$,

$$v_m = \frac{1}{\beta_m} \left( \sum_{k=1}^{m-1} (\alpha_k - \beta_k) u_k + \alpha_m u_m \right).$$

We see that $v_k \geq 0$ for $k = 1, \ldots, m$,

$$\sum_{k=1}^{m} \beta_k v_k = \sum_{k=1}^{m} \alpha_k u_k = u,$$

and

$$|u_m - v_m| \leq \frac{1}{\beta_m} \left( \sum_{k=1}^{m-1} (\alpha_k - \beta_k) u_k + (\beta_m - \alpha_m) u_m \right) \leq \frac{\delta}{2mM} M + \frac{\delta}{2mM} M \leq \delta.$$

Therefore

$$\sum_{k=1}^{m} \beta_k \varphi(v_k) \leq \sum_{k=1}^{m} \beta_k \varphi(v_k) - \sum_{k=1}^{m} \alpha_k \varphi(u_k) + \varphi(u) + \frac{\varepsilon}{2} \leq \beta_m \varphi(v_m) - \alpha_m \varphi(u_m) + \frac{\varepsilon}{2} \leq \beta_m (\varphi(v_m) - \varphi(u_m)) + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{2mM} M + \varphi(u) + \frac{\varepsilon}{2} \leq \varphi(u) + \varepsilon$$

and we see that the remark is true.

3. Lemmas on simple functions.

**Definition 3.1** A simple function we call a function $g \in S$ of the form

$$(+) \quad g = \sum_{i=1}^{n} a_i \chi_{E_i},$$

where $a_i$ denote numbers and $\chi_{E_i}$ characteristic functions of sets $E_i \in \Sigma$ with finite measures. We will assume that the sets $E_i$ are pairwise disjoint.

It is well known that for every function $f \in S$ there exists a sequence of simple functions $(g_n)$ convergent to $f$ everywhere on $E$ and such that
\[ |g_n(x)| \leq |f(x)| \text{ and } |f(x) - g_n(x)| \leq |f(x)| \text{ for } x \in E \text{ and } n = 1, 2, \ldots. \] From this fact we get immediately the following lemma.

**Lemma 3.2** For every function \( f \in L^{p_0} \) with \( \rho_{p_0}(f) < \infty \) there exists a sequence of simple functions \( (g_n) \) such that \( \rho_{p_0}(g_n) \leq \rho_{p_0}(f) \) for \( n = 1, 2, \ldots \) and \( \rho_{p_0}(f - g_n) \to 0 \) as \( n \to \infty \). Moreover, for every function \( f \in L^{p_0} \) there exists a sequence of simple functions \( (g_n) \) such that \( \|g_n\|_{p_0} \leq \|f\|_{p_0} \) for \( n = 1, 2, \ldots \) and \( \|f - g_n\|_{p_0} \to 0 \) as \( n \to \infty \).

Now we prove our fundamental lemma.

**Lemma 3.3** Let \((E, \Sigma, \mu)\) be a non-atomic measure space, \( \varphi \) a \( \check{\varphi} \)-function, \( \delta > 0 \) and \( g \) a simple function such that \( \rho_{\varphi}(g) \leq \delta \). Then there exist simple functions \( g_1, \ldots, g_{2^k} \) such that \( \rho_{\varphi}(g_i) \leq 2 \delta \) for \( i = 1, \ldots, 2^k \) and

\[
g = \frac{1}{2^k}(g_1 + \cdots + g_{2^k}).
\]

**Proof.** Let the simple functions \( g \) be of the form 3.1 (+). We denote \( u_i = |a_i| \) for \( i = 1, \ldots, n \). By virtue of 2.4 for every \( i \) there exist \( u_{i,j} \geq 0 \) and \( \alpha_{i,j} > 0 \), where \( j = 1, \ldots, m_i \), such that

\[
\sum_{j=1}^{m_i} \alpha_{i,j} = 1 \quad u_i = \sum_{j=1}^{m_i} \alpha_{i,j} u_{i,j}
\]

and

\[
\sum_{j=1}^{m_i} \alpha_{i,j} \varphi(u_{i,j}) \leq \overline{\varphi}(u_i) + \delta((\sum_{i=1}^{n} \mu(E_i))^{-1}.
\]

On virtue of 2.5 we may assume that the the numbers \( \alpha_{i,j} \) are of finite binary representation. Let \( k \) be a non-negative integer such that all numbers

\[
k_{i,j} = 2^k \alpha_{i,j}, \quad (j = 1, \ldots, m_i, i = 1, \ldots, n),
\]

are positive integers. Next, we denote by \( K_{i,j,l} \), where the indices may assume the values \( j = 1, \ldots, m_i, i = 1, \ldots, n \) and \( l = 1, \ldots, 2^k \), the set of those positive integers \( r \leq 2^k \) for which there holds

\[
k_{i,j,l+1} < r \leq k_{i,j,l} \quad \text{or} \quad k_{i,j,l-1} < r + 2^k \leq k_{i,j,l},
\]

where \( k_{i,0,l} = l \) and \( k_{i,l,t} = l + \sum_{i=1}^{j} k_{i,l} \).

The measure space \((E, \Sigma, \mu)\) is non-atomic, therefore every set \( E_i \) may be divided on \( 2^k \) pairwise disjoint sets \( E_{i,r} \) with measures \( \mu(E_{i,r}) = \frac{1}{2^k} \mu(E_i) \) for \( r = 1, \ldots, 2^k \).

We set

\[
g_l = \sum_{i=1}^{n} \sum_{j=1}^{m_i} u_{i,j} \text{sign} a_i \sum_{r \in K_{i,j,l}} \chi_{E_{i,r}} \quad \text{for} \quad l = 1, \ldots, 2^k.
\]

The set \( K_{i,j,l} \) possesses \( k_{i,j} \) elements and therefore we have
\[\rho_\varphi(g_i) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \varphi(u_{i,j}) \sum_{r \in K_{i,j}} \mu(E_{i,r}) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \varphi(u_{i,j})k_{i,j} \frac{1}{2^k} \mu(E_i) = \]
\[= \sum_{i=1}^{n} \sum_{j=1}^{m_i} \alpha_{i,j} \varphi(u_{i,j}) \mu(E_i) \leq \sum_{i=1}^{n} \varphi(u_i) \mu(E_i) + \delta = \rho_\varphi(g) + \delta \leq 2\delta.\]

Finally, let us observe that

\[\frac{1}{2^k} \sum_{i=1}^{2^k} g_i = \frac{1}{2^k} \sum_{i=1}^{m_i} \sum_{j=1}^{m_i} u_{i,j} \text{sign} \alpha_i \sum_{r \in K_{i,j}} \chi_{E_{i,r}} = \]
\[= \frac{1}{2^k} \sum_{i=1}^{n} \sum_{j=1}^{m_i} k_{i,j} u_{i,j} \chi_{E_i} \text{sign} \alpha_i = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \alpha_{i,j} u_{i,j} \chi_{E_i} \text{sign} \alpha_i = \]
\[= \sum_{i=1}^{n} u_i \chi_{E_i} \text{sign} \alpha_i = g.\]

3.4 Let \((E, \Sigma, \mu) \delta \text{ be as in 3.3 and let } g \text{ be a simple function such that } \|g\|_\varphi \leq \delta. \text{ Then there exist simple functions } g_1, \ldots, g_{2^k} \text{ such that } \|g_l\|_\varphi \leq 2\delta \text{ for } l = 1, \ldots, 2^k \text{ and } g = \frac{1}{2^k} (g_1 + \cdots + g_{2^k}).\]

Proof. Let us observe that the inequality \(\|g\|_\varphi \leq \delta \) implies \(\rho_\varphi(\frac{g}{\delta}) \leq \delta \).

We apply 3.3 to the simple function \(h = \frac{g}{\delta}. \) So, there exist simple functions \(h_1, \ldots, h_{2^k}\) such that

\[\rho_\varphi(h_l) \leq 2\delta \text{ for } l = 1, \ldots, 2^k \text{ and } h = 2^{-k}(h_1 + \cdots + h_{2^k}).\]

We set \(g_l = \delta h_l\) for \(l = 1, \ldots, 2^k\) and observe that \(g_l\) are simple functions such that \(g = 2^{-k}(g_1 + \cdots + g_{2^k})\) and \(\|g_l\|_\varphi \leq 2\delta\), because \(\rho_\varphi(\frac{g_l}{2\delta}) = \rho_\varphi(h_l) \leq 2\delta. \)

4. The locally convex spaces

Definition 4.1 Let \(X\) be a linear space over field \(K\) real or complex numbers and let \(\tau\) be a topology in the set \(X\). A space \(\langle X, \tau \rangle\) is called a linear-topological space when the operations \(+ : X \times X \to X\) and \(- : K \times X \to X\) are continuous functions from \(X \times X\) into \(X\) and \(K \times X \to X\) respectively.

The linear-topological spaces with a topology defined by the metric in linear spaces are normed spaces, countable-normed spaces and \(F^*\)-spaces, because we have

\[|((x + y) - (x_0 + y_0)) \leq |x - x_0| + |y - y_0| \quad \text{and} \quad |ax - a_0 x_0| \leq |a(x - x_0)| + |(a - a_0)x_0|,\]
and it implies the continuity of the operators \(+\) and \(-\).
4.2. A neighbourhood base of the point 0 of linear-topological space has the following properties:
  a) for every $V \in \mathcal{V}_0$ there exists $U \in \mathcal{V}_0$ such that $U + U \subset V$
  b) if $V \in \mathcal{V}_0$, $\alpha \in K$ and $|\alpha| \leq 1$, then $\alpha V \subset V$
  c) if $V \in \mathcal{V}_0$ and $x \in X$, then there exists a number $\alpha_0 > 0$ such that $\alpha_0 x \in V$.

4.3. Let $\langle X, \tau \rangle$ be a linear-topological space and let $x_0 \in X$. If $V \in \mathcal{V}_0$ is a neighbourhood base of the point 0 in the linear-topological space $\langle X, \tau \rangle$ and $x_0 \in X$, then a family $\mathcal{V}_{x_0}$ of the sets of the form $V_{x_0} = V + x_0$, when $V \in \mathcal{V}_0$ is the neighbourhood base of the point $x_0$ in $\langle X, \tau \rangle$.

4.4. For any subset $A$ of linear-topological space $X$ and any base of the neighbourhood $\mathcal{V}_0$ of 0 $\in X$ the formula $\overline{A} = \bigcap_{V \in \mathcal{V}_0} (A + V)$ defines the closure of the set $A$.

The proof of this fact one can find in [8].

From 4.4 and the property (1) of base of the neighbourhood of 0 it follows that for every $V \in \mathcal{V}_0$ there exists $U \in \mathcal{V}_0$ such that

$$\overline{U} = \bigcap_{W \in \mathcal{V}_0} (U + W) \subset U + U \subset V.$$ 

From this it follows that there exists a neighbourhood base of 0 which consists of closed sets.

Definition 4.5 The set $A$ of the linear space $X$ is called a convex set if from the conditions $x, y \in A, \alpha, \beta \geq 0, \alpha + \beta = 1$ it follows that $\alpha x + \beta y \in A$.

4.6. If the sets $A$ and $B$ are convex in a linear space over the field $K$ of the real or complex numbers and $a \in K$, then the sets $aA, A + B, A - B$ are convex in $X$.

Definition 4.7 The linear-topological space $\langle X, \tau \rangle$ is called a locally convex if there exists a neighbourhood base $\mathcal{V}_0$ in $X$ of the point 0 which consists of convex sets.

5. Linear operators from the Orlicz spaces into locally convex linear-topological spaces

5.1. In the Orlicz space $L_\rho^{\varphi}$ and also in its subspace $L^{\varphi}$ we have two types of convergences, one is the pseudonorm convergence and the other the pseudomodular convergence. Therefore let us denote by $\alpha(L^{\varphi}, Y)$ the class of pseudonorm continuous linear operators from $L^{\varphi}$ into $Y$, where $Y$ is a locally convex linear-topological space and by $\alpha(L_\rho^{\varphi}, Y)$ the class of pseudomodular continuous linear operators from $L_\rho^{\varphi}$ into $Y$. Obviously, the linear operator $A$ from $L^{\varphi}$ into $Y$ is pseudonorm continuous, i.e., belongs to $\alpha(L^{\varphi}, Y)$, if and only if for every $V \in \mathcal{V}_0$ there exists $\delta > 0$ such that the conditions $f \in L^{\varphi}$ and $\|f\|_{\varphi} \leq \delta$ imply $A(f) \in V$. Let us observe that the linear operator $A$ from $L_\rho^{\varphi}$ into $Y$ is pseudomodular continuous, i.e., belongs to $\alpha(L_\rho^{\varphi}, Y)$, if and only if for every $V \in \mathcal{V}_0$ there exists $\delta > 0$ such that the conditions $f \in S$ and $\rho_\varphi(f) \leq \delta$ imply $A(f) \in V$. 
5.2. Let $\varphi$ be a $\tilde{\varphi}$-function and $\bar{\varphi}$ its convex $\tilde{\varphi}$-function as in 2.1. For these functions the inequality $\bar{\varphi}(u) \leq \varphi(u)$ for $u \geq 0$ holds. Hence we have $\rho_{\bar{\varphi}}(f) \leq \rho_{\varphi}(f)$ for $f \in S$ and next the inclusions $L^p_{\varphi} \subset L^p_{\bar{\varphi}}, L^{\infty}_{\varphi} \subset L^{\infty}_{\bar{\varphi}}$ and the inequality $\|f\|_{L^p_{\bar{\varphi}}} \leq \|f\|_{L^p_{\varphi}}$ for $f \in L^p_{\varphi}$.

From this follows immediately

5.3. If $A \in \alpha(L^p_{\varphi}, Y)$, then the restriction $A|_{L^p_{\varphi}}$ belongs to $\alpha(L^p_{\bar{\varphi}}, Y)$ and if $A \in \alpha(L^{\infty}_{\varphi}, Y)$, then $A|_{L^{\infty}_{\varphi}} \in \alpha(L^{\infty}_{\bar{\varphi}}, Y)$.

Now we present two main theorems.

**Theorem 5.4** If the Orlicz space $L^p_{\varphi}$ is non-atomic, then for every linear operator $A \in \alpha(L^p_{\varphi}, Y)$ there exists a linear operator $B \in \alpha(L^p_{\bar{\varphi}}, Y)$ such that $A = B|_{L^p_{\varphi}}$.

**Proof.** On virtue of 4.4 one can assume that in the space $Y$ there exists a neighbourhood base $V_0$ of the point 0 which consists of convex and closed sets. Let $A \in \alpha(L^p_{\varphi}, Y)$ and let us take $V \in V_0$. Then there exists $\delta > 0$ such that the conditions $f \in S$ and $\rho_{\varphi}(f) \leq 2\delta$ imply $A(f) \in V$. Let us take an arbitrary simple function $g$ such that $\rho_{\varphi}(g) \leq \delta$. On virtue of 3.3 there exist simple functions $g_1, \ldots, g_k$ such that $\rho_{\varphi}(g_l) \leq 2\delta$ for $l = 1, \ldots, k$ and $g = 2^{-k}(g_1 + \cdots + g_k)$. Therefore one can write $A(g_l) \in V$ for $l = 1, \ldots, 2^k$ and next on virtue of convexity of the set $V$ there is $A(g) \in V$.

Thus we have proved the following remark:

(R) If $A \in \alpha(L^p_{\varphi}, Y)$, then for every $V \in V_0$ there exists $\delta > 0$ such that the inequality $\rho_{\bar{\varphi}}(g) \leq \delta$ for any simple function $g$ implies $A(g) \in V$.

Now we shall prove the theorem. Let $A \in \alpha(L^p_{\varphi}, Y)$, we take an arbitrary $f \in L^p_{\varphi}$. Then $\rho_{\bar{\varphi}}(\lambda f) < \infty$ for some $\lambda > 0$ and on virtue of 3.2 there exists a sequence of simple functions $(g_n)$ such that $\rho_{\bar{\varphi}}(\lambda(f - g_n)) \to 0$ as $n \to \infty$. Since

$$\rho_{\bar{\varphi}}\left(\frac{1}{2}\lambda(g_n - g_m)\right) \leq \rho_{\bar{\varphi}}(\lambda(f - g_n)) + \rho_{\bar{\varphi}}(\lambda(f - g_m)),$$

so we have $\rho_{\bar{\varphi}}\left(\frac{1}{2}\lambda(g_n - g_m)\right) \to 0$ as $n, m \to \infty$. Whence, by virtue of the remark (R), we get

$$A\left(\frac{1}{2}\lambda(g_n - g_m)\right) = \frac{1}{2}\lambda(A(g_n) - A(g_m)) \to \theta, \text{ as } n, m \to \infty.$$

This shows that the sequence $(A(g_n))$ satisfies the Cauchy condition. Since $Y$ is a sequentially complete space, then the sequence $(A(g_n))$ is convergent, we denote $B(f) = \lim_{n \to \infty} A(g_n)$.

Further, let $(h_n)$ be an arbitrary sequence of simple functions such that $h_n \xrightarrow{\bar{\varphi}} f$. Then $\rho_{\bar{\varphi}}(\lambda_1(f - h_n)) \to 0$ as $n \to \infty$ for some $\lambda_1 > 0$. From this, on virtue of the inequality

$$\rho_{\bar{\varphi}}\left(\frac{1}{2}\lambda_2(g_n - h_n)\right) \leq \rho_{\bar{\varphi}}(\lambda(f - g_n)) + \rho_{\bar{\varphi}}(\lambda_1(f - h_n)),$$

where $\lambda_2 = \inf\{\lambda, \lambda_1\}$, we get $\rho_{\bar{\varphi}}\left(\frac{1}{2}\lambda_2(g_n - h_n)\right) \to 0$ as $n \to \infty$. Whence, in view of the remark (R), it follows that

$$A\left(\frac{1}{2}\lambda_2(g_n - h_n)\right) = \frac{1}{2}\lambda_2(A(g_n) - A(h_n)) \to \theta \text{ as } n \to \infty.$$
Hence the sequence \((A(h_n))\) is convergent and
\[
B(f) = \lim_{n \to \infty} A(g_n) = \lim_{n \to \infty} A(h_n).
\]
This means that the value \(B(f)\) is independent on the choice of a sequence of simple functions \((h_n)\) satisfying \(h_n \overset{w}{\rightharpoonup} f\).

Let us take \(f_1, f_2 \in L_p^\varphi\) and numbers \(a\) and \(b\). Then from 3.2 it follows the existence of sequences of simple functions \((g_{1,n})\) and \((g_{2,n})\) such that \(g_{1,n} \overset{w}{\rightharpoonup} f_1\) and \(g_{2,n} \overset{w}{\rightharpoonup} f_2\). This implies \(a g_{1,n} + b g_{2,n} \overset{w}{\rightharpoonup} a f_1 + b f_2\) and therefore
\[
B(a f_1 + b f_2) = \lim_{n \to \infty} A(a g_{1,n} + b g_{2,n}) = a \lim_{n \to \infty} A(g_{1,n}) + b \lim_{n \to \infty} A(g_{2,n}) = a B(f_1) + b B(f_2).
\]
Next, let us take \(f \in L_p^\varphi\). On virtue of 3.2 there exists a sequence of simple functions \((g_n)\) such that \(g_n \overset{w}{\rightharpoonup} f\). From 5.2 it follows that the convergence \(g_n \overset{w}{\rightharpoonup} f\) holds too. Therefore, in this case, we have
\[
B(f) = \lim_{n \to \infty} A(g_n) = A(f).
\]

Further, let \(V \in \mathcal{V}_0, \delta > 0\) be such that the condition \(\rho_\varphi(g) \leq \delta\) for simple function \(g\)
implies \(A(g) \in V\) and let \(f \in L_p^\varphi\) be such that \(\rho_\varphi(f) \leq \delta\). Then, from 3.2 it follows the existence of a sequence of simple functions \((g_n)\) such that \(\rho_\varphi(g_n) \leq \rho_\varphi(f)\) for \(n = 1, 2, \ldots\)
and \(\rho_\varphi(f - g_n) \to 0\) as \(n \to \infty\). Hence there holds \(B(f) = \lim_{n \to \infty} A(g_n) \in V\).

Thus we have proved the existence of a operator \(B \in \alpha(L_p^\varphi, Y)\) with the property \(B|_{L_p^\varphi} = A\). Such operator is only one. Namely if \(B\) and \(B_1\) are operators with the required properties, then taking an arbitrary \(f \in L_p^\varphi\) and more, on virtue of 3.2 a sequence of simple functions \((g_n)\) such that \(g_n \overset{w}{\rightharpoonup} f\) we see that
\[
B(f) = \lim_{n \to \infty} B(g_n) = \lim_{n \to \infty} A(g_n) = \lim_{n \to \infty} B_1(g_n) = B_1(f).
\]

5.5 If the space \(L^\varphi\) is non-atomic, then for every operator \(A \in \alpha(L^\varphi, Y)\) there exists a unique operator \(B \in \alpha(L^\varphi, Y)\) such that \(A = B|_{L^\varphi}\).

Since the proof is similar to the previous one, we omit it.

5.6 On virtue of 2.2 for the convex \(\varphi\)-functions \(\varphi\) and \(\varphi\) generated by \(\varphi\) the inequality \(\varphi(u) \leq \varphi(2u) \leq \varphi(2u)\) for \(u \geq 0\) holds, therefore we may replace in the statements 5.2, 5.3, 5.4 and 5.5 the function \(\varphi\) by the function \(\varphi\). Moreover, we observe that the results 5.3, 5.4 and 5.5 may be formulated in the form announced in the abstract. Namely,

If \((E, \Sigma, \mu)\) is a non-atomic measure space, then
\[
\alpha(L_p^\varphi, Y) = \alpha(L_p^\varphi, Y) \text{ and } \alpha(L^\varphi, Y) = \alpha(L^\varphi, Y).
\]

The first equality is closely connected with the theorem of W. Orlicz in [7]. The other proof of the second equality one can find in [1](th.2.2 b). Our result is a little bit more general than the one mentioned above, but first of all, our proof is essentially different from that in [7].
In the particularity, from our results one can obtain the following theorem with the first part known from [7].

**Theorem 5.7** If the $\varphi$-function $\varphi$ is such that $\lim_{n \to \infty} \frac{\varphi(u)}{u} = 0$, then only trivial linear operator from the non-atomic Orlicz space $L_{p,\varphi}$ into locally convex space $Y$ is pseudomodular continuous and only trivial linear operator from the non-atomic space of finite elements $L_{p,\varphi}$ into $Y$ is pseudonorm continuous.

Proof. In this case we have $\varphi(u) = 0$ for $u \geq 0$. This implies $\rho_{\varphi}(f) = 0$ for all $f \in S$, and next $L_{p,\varphi} = L_{p,\varphi}^* = S$ and $\|f\|_{\varphi} = 0$ for all $f \in S$. Whence we get $\alpha(L_{p,\varphi}^*, Y) = \alpha(L_{p,\varphi}^*, Y) = \{0\}$, because only the trivial operator satisfies the inequality $A(f) \in V$ for all $f \in S$ and all $V \in \mathcal{V}_0$. Thus, by virtue of 5.6 we obtain $\alpha(L_{p,\varphi}^*, Y) = \alpha(L_{p,\varphi}^*, Y) = \{0\}$.

The other proof of this theorem one can find in [1](th. 2.2 a). \hfill $\Box$

**References**


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