THE LAURENT EXTENSION OF A NOETHERIAN INTEGRAL DOMAIN

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ABSTRACT. Let $R[\alpha, \alpha^{-1}]$ be an extension of a Noetherian integral domain $R$ where $\alpha$ is an element of an algebraic field extension over the quotient field of $R$. In the case $\alpha$ is an anti-integral element over $R$ we will give a condition for a prime ideal $p$ of $R$ to be $pR[\alpha, \alpha^{-1}] = R[\alpha, \alpha^{-1}]$. By making use of this we will proceed mainly with the study of flatness and faithful flatness of the extension $R[\alpha, \alpha^{-1}]/R$. Let $\eta_1, \cdots, \eta_d$ be the coefficients of the minimal polynomial of $\alpha$ over the quotient field of $R$. Then we will also investigate the extension $R[\eta_1, \cdots, \eta_d]/R$.

§1. Laurent extensions and ideals $J_{[\alpha]}$, $\gamma$

Let $R$ be a Noetherian integral domain with the quotient field $K$. Let $\alpha$ be an element which is algebraic over $K$ and set $d = [K(\alpha) : K]$. We denote the minimal polynomial of $\alpha$ over $K$ by

$$\phi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d,$$

$$\eta_1, \cdots, \eta_d \in K.$$

Set $I_{\eta_i} = R : R \eta_i$ for $1 \leq i \leq d$ and $I_{[\alpha]} = \cap_{i=1}^d I_{\eta_i}$. We call $I_{[\alpha]}$ the generalized denominator ideal of $\alpha$. Furthermore we will set

$$J_{[\alpha], \eta} = I_{[\alpha]}(\eta_1, \cdots, \eta_d)$$

where $(\eta_1, \cdots, \eta_d)$ is a fractional ideal of $R$ generated by the elements $\eta_1, \cdots, \eta_{d-1}, \eta_d$ and

$$J_{[\alpha], i} = I_{[\alpha]}(1, \eta_1, \cdots, \eta_{i-1}, \eta_{i+1}, \cdots, \eta_d)$$

for $1 \leq i \leq d$. Sometimes we will use the notation $\widetilde{J}_{[\alpha]}$ instead of $J_{[\alpha], \eta}$. Set $J_{[\alpha]} = I_{[\alpha]} + J_{[\alpha], 0} = I_{[\alpha]}(1, \eta_1, \eta_2, \cdots, \eta_d)$.

We call $R[\alpha, \alpha^{-1}]$ the Laurent extension of $\alpha$ over $R$.

Let $R[X]$ be a polynomial ring over $R$ in an indeterminate $X$ and $\pi : R[X] \rightarrow R[\alpha]$ the $R$-algebra homomorphism defined by $\pi(X) = \alpha$. We say that $\alpha$ is an anti-integral element over $R$ of degree $d$ if $\text{Ker}(\pi) = I_{[\alpha]}\phi_\alpha(X)R[X]$. Set

$$\Gamma_{J_{[\alpha]}} = \{ p \in \text{Spec}(R) \mid p + J_{[\alpha]} = R \}$$

and

$$V(\widetilde{J}_{[\alpha]}) = \{ p \in \text{Spec}(R) \mid p \supset \widetilde{J}_{[\alpha]} \}.$$
Our notation is standard and our general reference for unexplained technical terms is H. Matsumura [2]. We will list the following results for later use.

**Lemma 1.1** (M. Kanemitsu and K. Yoshida [1, Theorem 7 (2)]). Assume that \( \alpha \) is an anti-integral element over \( R \) of degree \( d \). Then

\[
\{ p \in \text{Spec}(R) \mid pR[\alpha] = R[\alpha] \} = V(J_{[\alpha]}) \cap \Gamma_{J_{[\alpha]}},
\]

An element \( \gamma \) in \( R[\alpha] \) is said to be an excellent element if there exist elements \( c_0, c_1, \ldots, c_n \in R \) such that

\[
\gamma = c_0 + c_1 \alpha + \cdots + c_n \alpha^n \text{ and } (c_0, c_1, \ldots, c_n)R = R.
\]

**Lemma 1.2** (J. Sato, S. Oda and K. Yoshida [5, Corollary 5]). Assume that \( \alpha \) is an anti-integral element over \( R \) of degree \( d \). Then the following statements are equivalent.

(i) \( R[\alpha] / R \) is a flat extension.
(ii) \( R[\alpha, \alpha^{-1}] / R \) is a flat extension.
(iii) \( \alpha \in \text{rad}(J_{[\alpha]}R[\alpha]) \).
(iv) \( J_{[\alpha]} = R \).
(iv) Every excellent element belongs to \( \text{rad}(J_{[\alpha]}R[\alpha]) \).

Our key result is the following.

**Theorem 1.3.** Let \( R \) be a Noetherian integral domain and \( \alpha \) an anti-integral element over \( R \) of degree \( d \geq 2 \). For a prime ideal \( p \) of \( R \) the following are equivalent to each other.

(i) \( pR[\alpha, \alpha^{-1}] = pR[\alpha, \alpha^{-1}] \).
(ii) \( p + J_{[\alpha]} = R \) and there exists an integer \( i \) \((1 \leq i \leq d)\) such that \( p \supset J_{[\alpha]}, i \).

**Proof:** Set \( A = R[\alpha, \alpha^{-1}] \).

(i) \( \Rightarrow \) (ii). First we will prove that \( p + J_{[\alpha]} = R \). The condition \( pA = A \) implies that \( \alpha \) is in \( \text{rad}(pR[\alpha]) \). Then there exists a natural number \( n \) such that \( \alpha^n = a_0 + a_1 \alpha + \cdots + a_m \alpha^m \) and \( a_0, a_1, \ldots, a_m \in p \) for some \( m \). Let

\[
f(X) = X^n - (a_0 + a_1 X + \cdots + a_m X^m).
\]

Then \( f(X) \) is in \( \text{Ker}(\pi) \). This shows that there exist elements \( b_0, \ldots, b_d \in J_{[\alpha]} \) and \( g(X) \in R[X] \) satisfying

\[
f(X) = (b_0 + b_1 X + \cdots + b_d X^d)g(X).
\]

Hence \( 1 \) is in \( p + J_{[\alpha]} \), and so \( p + J_{[\alpha]} = R \).

Secondly we will show that there exists an integer \( i \) \((1 \leq i \leq d)\) with \( p \supset J_{[\alpha]}, i \). Suppose that \( p \not\supset J_{[\alpha]}, i \) for every \( i \) with \( 1 \leq i \leq d \). We will prove that \( pR[\alpha] \neq R[\alpha] \). If \( pR[\alpha] = R[\alpha] \), then by Lemma 1.1, we get \( p \supset J_{[\alpha]} = J_{[\alpha]} \). This is a contradiction.

We know that

\[
\alpha^d + \eta_1 \alpha^{d-1} + \cdots + \eta_d = 0.
\]
We will prove that there exists an element $c$ of $I_{[\alpha]}$ such that $c \eta_i$ and $c \eta_j$ are not in $p$ for some $i \neq j$ ($1 \leq i, j \leq d$). Note that $J_{[\alpha]}$, $1 = I_{[\alpha]}(1, \eta_2, \ldots, \eta_d)$. Since $J_{[\alpha]} \not\subset p$, there exists an integer $i$ ($2 \leq i \leq d$) such that $a \in I_{[\alpha]}$ and $a \eta_i \not\in p$. If we can take an integer $j$ ($j \neq i$ and $1 \leq j \leq d$) satisfying $a \eta_j \not\in p$, then the assertion is proved. So we may assume that

$$a(\eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_d) \subset p.$$ 

Furthermore, there exists an integer $j$ ($j \neq i$ and $1 \leq j \leq d$) such that $b \in I_{[\alpha]}$ and $b \eta_j \not\in p$. Similarly as above we may assume that

$$b(\eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_d) \subset p.$$ 

Set $c = a + b$. Then $c \eta_i \not\in p$ and $c \eta_j \not\in p$.

From the argument above we see that there are at least two non-zero terms in $c \phi_\alpha(\alpha)$ modulo $p R_p[\alpha]$. Since $J_{[\alpha]} R_p = R_p$, the ideal $I_{[\alpha]} R_p$ is invertible, so $I_{[\alpha]} R_p$ is principal. Hence we may assume that $c$ is a generator of $I_{[\alpha]}$. Let $\pi'$ be the $R-$algebra homomorphism of $R_p[X]$ into $R_p[\alpha]$ defined by $\pi'(X) = \alpha$. Then $\text{Ker}(\pi') = I_{[\alpha]} \phi_\alpha(\alpha) R_p[X] = (c \phi_\alpha(\alpha))$. Hence $R_p[\alpha] \cong R_p[X]/(c \phi_\alpha(X))$. On the other hand

$$R_p[\alpha]/p R_p[\alpha] \cong k(p)[\pi'] \cong k(p)[X]/(c \phi_\alpha(X))$$

where $k(p)$ is the residue field of $p$. Let $Q$ be the prime ideal of $R_p[\alpha]$ which corresponds to the irreducible factor of $c \phi_\alpha(X)$ different from $\overline{X}$. Then $Q$ does not contain $\alpha$ and $Q \supset p R_p[\alpha]$. Set $P = Q \cap R[\alpha]$. Then $P \supset p R[\alpha]$ and $P \not\supset \alpha$. This is absurd from the fact $P \supset \text{rad}(p R[\alpha]) \supset \alpha$.

(ii) $\Rightarrow$ (i). Assume that $p A \neq A$. Then there exists a prime ideal $P$ of $A$ such that $P \supset p A$. By the condition $p + J_{[\alpha]} = R$, there exists elements $b$ of $p$ and $c$ of $J_{[\alpha]}$ such that $b + c = 1$. Since $c$ is in $J_{[\alpha]} = J_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$, we can write

$$c = c_0 + c_1 \eta_1 + \cdots + c_d \eta_d$$

and $c_0, c_1, \ldots, c_d \in I_{[\alpha]}$.

By the condition (ii), there exists an integer $i$ ($1 \leq i \leq d$) such that $p \supset J_{[\alpha], i}$. Multiplying the equality

$$\alpha^d + \eta_1 \alpha^{d-1} + \cdots + \eta_d = 0,$$

by $c_i$, we have

$$c_i \alpha^d + c_i \eta_1 \alpha^{d-1} + \cdots + c_i \eta_d = 0.$$ 

We know that $c_i, c_i \eta_1, \ldots, c_i \eta_d$ other than $c_i \eta_i$ are in $J_{[\alpha], i}$, and so in $P$. Hence $c_i \eta_i \alpha^{d-i}$ is in $P$. Then in the equation

$$\alpha^{d-i} = c_0 \alpha^{d-i} + c_1 \eta_1 \alpha^{d-i} + \cdots + c_d \eta_d \alpha^{d-i},$$

$c_0, c_1 \eta_1, \ldots, c_d \eta_d$ other than $c_i \eta_i$ are in $J_{[\alpha], i}$, and so in $P$. Therefore $\alpha^{d-i}$ is in $P$. Using $c = 1 - b$, we get $\alpha^{d-i} \in P$. If $p \supset J_{[\alpha], a} = J_{[\alpha]}$, then by Lemma 1.1, we know $p R[\alpha] = R[\alpha]$. This claims that $p A = A$. This is absurd. Hence $i \neq d$. This shows that $\alpha$ is in $P$. This contradicts to the fact $\alpha$ is a unit of $A$. □

**Remark 1.4.** (1) By Theorem 1.3, we obtain:
\[
\{ p \in \text{Spec}(R) \mid pR[\alpha, \alpha^{-1}] = R[\alpha, \alpha^{-1}] \} = (\cup_{i=1}^{d} V(J_{[\alpha], i})) \cap \Gamma_{J_{[\alpha]}}.
\]

If \( J_{[\alpha]} = R \), then we have
\[
\{ p \in \text{Spec}(R) \mid pR[\alpha, \alpha^{-1}] = R[\alpha, \alpha^{-1}] \} = \cup_{i=1}^{d} V(J_{[\alpha], i}).
\]

It is a closed set.

(2) In the case \( d = 1 \), we have the following Theorem 1.3'.

**Theorem 1.3'.** Let \( R \) be a Noetherian integral domain and \( \alpha \) an anti-integral element over \( R \) of degree 1. For a prime ideal \( p \) of \( R \), the following are equivalent to each other:

(i) \( pR[\alpha, \alpha^{-1}] = R[\alpha, \alpha^{-1}] \),
(ii) \( p + J_{[\alpha], i} = R \) and there exists an integer \( i (i = 0, 1) \) such that \( p \supset J_{[\alpha], i} \).

**Proof.** We will prove that \( p \supset J_{[\alpha], i} \) for some \( i (i = 0, 1) \) in the proof (i) \( \Rightarrow \) (ii) because the rest of the proof is the same argument as in Theorem 1.3. Assume that \( p \not\supset J_{[\alpha], i} \cap I_{[\alpha]} \). Then there exists an element \( c \) of \( I_{[\alpha]} \) such that \( c \eta \not\in p \) and \( c \eta \not\in I_{[\alpha]} \). The fact \( c \eta \not\in I_{[\alpha]} \) implies \( c \eta^2 \not\in R \). Since \( c \cdot c \eta^2 = (c \eta)^2 \) is not in \( p \), we see that \( c \) is not in \( p \). Hence \( \alpha = -c \eta \in R_p \). Furthermore, \( \alpha^{-1} = -c/c \eta \) is in \( R_p \) because \( c \eta \not\in p \). Therefore \( R_p \supset R[\alpha, \alpha^{-1}] = A \). Thus \( pR_p \supset pA = A \). This is a contradiction. \( \square \)

(3) For another characterization of \( pR[\alpha, \alpha^{-1}] = R[\alpha, \alpha^{-1}] \) in the case \( d = 1 \), see [1, p. 55, Remark].

(4) From now on we will assume \( d \geq 2 \).

**Proposition 1.5.** Let \( R \) be a Noetherian integral domain and \( \alpha \) an anti-integral element over \( R \) of degree \( d \). Let \( A = R[\alpha, \alpha^{-1}] \) and \( \phi \) the contraction mapping of \( \text{Spec}(A) \) into \( \text{Spec}(R) \). Then the following are equivalent.

(i) The contraction mapping \( \phi \) is surjective.
(ii) For every \( i \) with \( 1 \leq i \leq d \), the equality \( \text{rad}(J_{[\alpha], i}) = \text{rad}(J_{[\alpha], i}) \) holds.

**Proof.** (i) \( \Rightarrow \) (ii). Suppose that there exists an integer \( i (1 \leq i \leq d) \) such that \( \text{rad}(J_{[\alpha], i}) \neq \text{rad}(J_{[\alpha], i}) \). By the definitions of \( J_{[\alpha]} \) and \( J_{[\alpha], i} \), we have \( J_{[\alpha], i} \subset J_{[\alpha]} \). Hence \( \text{rad}(J_{[\alpha], i}) \subset \text{rad}(J_{[\alpha]}) \). Then there exists a prime ideal \( p \) of \( R \) such that \( J_{[\alpha], i} \subset p \) and \( J_{[\alpha]} \not\subset p \). This implies that \( pR_p \supset J_{[\alpha], i} \). By Theorem 1.3 to \( A_p = R_p[\alpha, \alpha^{-1}] \), we obtain \( pA_p = A_p \). This shows that \( p \notin \text{Im}(\phi) \). This is a contradiction.

(ii) \( \Rightarrow \) (i). We have only to prove that \( pA_p \neq A_p \) for arbitrary prime ideal \( p \) of \( R \). Assume that \( pA_p = A_p \). Then Theorem 1.3 asserts that \( pR_p + J_{[\alpha], i} R_p = R_p \). Hence \( J_{[\alpha]} R_p = R_p \). Besides, there exists an integer \( i (1 \leq i \leq d) \) such that \( pR_p \supset J_{[\alpha], i} R_p \) by Theorem 1.3. Hence we see that \( \text{rad}(J_{[\alpha]} R_p) \supset \text{rad}(J_{[\alpha], i} R_p) \), hence \( \text{rad}(J_{[\alpha]}) \supset \text{rad}(J_{[\alpha], i}) \). This is absurd. \( \square \)

**Corollary 1.6.** Let \( R \) be a Noetherian integral domain and \( \alpha \) an anti-integral element over \( R \) of degree \( d \). Set \( A = R[\alpha, \alpha^{-1}] \). Then \( A/R \) is a faithfully flat extension if and only if \( J_{[\alpha], i} = R \) for every \( i (1 \leq i \leq d) \).

**Proof.** Let \( \phi \) the contraction mapping of \( \text{Spec}(A) \) into \( \text{Spec}(R) \).
(\(\Rightarrow\)). Since \(A/R\) is faithfully flat, the homomorphism \(\phi\) is surjective by H. Matsumura [2, (4D) Theorem 3]. Then Proposition 1.5 implies that \(\text{rad}(J_{[\alpha]}^i) = \text{rad}(J_{[\alpha]}^i, i)\) for every integer \(i\) \((1 \leq i \leq d)\). Furthermore, by Lemma 1.2, \(J_{[\alpha]} = R\) because \(A/R\) is a flat extension. Hence \(J_{[\alpha]}^i = R\) for every \(i\) \((1 \leq i \leq d)\).

(\(\Leftarrow\)). It is easily verified that \(J_{[\alpha]} = R\) because \(J_{[\alpha]} \supset J_{[\alpha]}^i\). Then \(A/R\) is a flat extension by Lemma 1.2. Moreover, we know that \(\text{rad}(J_{[\alpha]}^i) = R = \text{rad}(J_{[\alpha]}^i, i)\). By Proposition 1.5, the contraction mapping \(\phi\) is surjective. Hence \(A/R\) is a faithfully flat extension by H. Matsumura [2, (4D) Theorem 3]. □

The following holds about \(R[\alpha]\).

**Theorem 1.7.** Let \(R\) be a Noetherian integral domain and \(\alpha\) an anti-integral element over \(R\) of degree \(d\). Set \(B = R[\alpha]\). Then the following are equivalent.

(i) \(J_{[\alpha]} = R\).

(ii) \(J_{[\alpha]}B = B\).

(iii) \(\overline{J_{[\alpha]}B} = B\).

**Proof.** (i) \(\Rightarrow\) (ii) is obvious.

(ii) \(\Rightarrow\) (i). Since \(J_{[\alpha]}B = B\), we know that \(\alpha\) is in \(\text{rad}(J_{[\alpha]}B)\) by Lemma 1.2, we have \(J_{[\alpha]} = R\).

(iii) \(\Rightarrow\) (ii) is clear from the fact \(J_{[\alpha]} \supset \overline{J_{[\alpha]}B}\).

(ii) \(\Rightarrow\) (iii). Let \(p\) be a prime divisor of \(\overline{J_{[\alpha]}B}\). By (ii) \(\Rightarrow\) (i), we get \(J_{[\alpha]} = R\). Therefore Lemma 1.1 implies that \(pB = B\). Hence \(\text{rad}(J_{[\alpha]}B) = B\), and so \(\overline{J_{[\alpha]}B} = B\). □

An analogous result to Theorem 1.7 holds in the case \(R[\alpha, \alpha^{-1}]\).

**Theorem 1.8.** Let \(R\) be a Noetherian integral domain and \(\alpha\) an anti-integral element over \(R\) of degree \(d\). Set \(A = R[\alpha, \alpha^{-1}]\). Then the following are equivalent.

(i) \(J_{[\alpha]} = R\), i.e., the extension \(A/R\) is a flat extension.

(ii) \(J_{[\alpha]}A = A\).

(iii) \(\overline{J_{[\alpha]}A} = A\).

**Proof.** (i) \(\Rightarrow\) (ii) is immediate.

(ii) \(\Rightarrow\) (i). We will prove that \(A/R\) is a flat extension. Let \(P\) be a prime ideal of \(A\) and set \(p = P \cap R\). Then \(p \not\supset J_{[\alpha]}\) because \(J_{[\alpha]}A = A\). Hence \(J_{[\alpha]}R_p = R_p\). By Lemma 1.2 we see that \(A_p = R_p[\alpha, \alpha^{-1}] / R_p\) is a flat extension. Moreover, \(A_p / A_p\) is also a flat extension. Therefore \(A_p / R_p\) is a flat extension. So is \(A/R\).

(iii) \(\Rightarrow\) (ii) is clear from \(J_{[\alpha]} \supset \overline{J_{[\alpha]}A}\).

(ii) \(\Rightarrow\) (iii). Let \(p\) be a prime divisor of \(\overline{J_{[\alpha]}A}\). Then \(J_{[\alpha]} = R\) by (ii) \(\Rightarrow\) (i). Hence Lemma 1.1 shows that \(\alpha R[\alpha] = R[\alpha]\). This means that \(pA = A\). Therefore \(\text{rad}(\overline{J_{[\alpha]}A}) = A\), and so \(\overline{J_{[\alpha]}A} = A\). □

**Remark 1.9.** \(\overline{J_{[\alpha]} = R}\) does not hold necessarily even if \(J_{[\alpha]} = R\).

**Corollary 1.10.** Let \(R\) be a Noetherian integral domain and \(\alpha\) an anti-integral element over \(R\) of degree \(d\). Set \(A = R[\alpha, \alpha^{-1}]\). Then the following are equivalent to the conditions in Theorem 1.8.

(iv) There exists an integer \(i\) \((1 \leq i \leq d)\) such that \(J_{[\alpha]}^i A = A\).
(v) $J_{[\alpha]}$, $\mathfrak{a}A = A$ for every $i$ $(1 \leq i \leq d)$.

**Proof.** (iv) $\implies$ (ii) is clear from $J_{[\alpha]}$, $\mathfrak{a} \subset J_{[\alpha]}$.

(iii) $\implies$ (iv) is immediate from $\overline{J}_{[\alpha]} = J_{[\alpha]}$, $\mathfrak{a}$.

(v) $\implies$ (iv) is obvious.

(i) $\implies$ (v). Let $p$ be a prime divisor of $J_{[\alpha]}$, $i$. By the condition (i) we know that $J_{[\alpha]} = R$. Hence by Lemma 1.1, we get $pR[\alpha] = R[\alpha]$. Hence $pA = A$. This implies that $J_{[\alpha]}$, $\mathfrak{a}A = A$. □

§ 2. Shifting a generator by an element of $A$.

We denote by $U(A)$ the unit group of $A$. We will find a condition for $A$ to coincide with $R[\alpha a, (\alpha a)^{-1}]$.

**Lemma 2.1.** Let $R$ be a Noetherian integral domain with the quotient field $K$. Let $\alpha$ be an element of an algebraic field extension over $K$ and set $A = R[\alpha, \alpha^{-1}]$. If $a$ is an element of $A$ and $A = R[\alpha a, \alpha^{-1}]$, then $a$ is in $U(A)$.

**Proof.** Since $a^{-1} = \alpha(\alpha a)^{-1}$ is in $A$, we know that $a$ is in $U(A)$. □

**Proposition 2.2.** Let $R$ be a Noetherian domain and $\alpha$ an anti-integral element over $R$ of degree $d$. Set $A = R[\alpha, \alpha^{-1}]$. If grade($J_{[\alpha]}$, $i$) $> 1$ for every $i$ $(1 \leq i \leq d)$, then $U(A) \cap R = U(R)$.

**Proof.** It is clear that $U(A) \cap R \supset U(R)$. Assume that

$$U(A) \cap R \supsetneq U(R).$$

Then there exists an element $a$ of $U(A) \cap R$ such that $a \not\in U(R)$. Since $a$ is in $U(A)$, we have $aA = A$. Hence there exists a prime divisor $p$ of $\text{rad}(aR)$ such that $pA = A$. Then by Theorem 1.3, we see that $p \supset J_{[\alpha]}$, $i$ for some $i$ with $1 \leq i \leq d$. By K. Yoshida [6, Proposition 1.10], we obtain depth($R_p) = 1$ because $p$ is a prime divisor of $\text{rad}(aR)$. On the other hand grade($J_{[\alpha]}$, $i$) $> 1$ and $p \supset J_{[\alpha]}$, $i$. This shows that depth($R_p) > 1$, and we reach a contradiction. □

**Theorem 2.3.** Let $R$ be a Noetherian domain and $\alpha$ an anti-integral element over $R$ of degree $d$. Let $a$ be an element of $R$. Set $A = R[\alpha, \alpha^{-1}]$ and assume that grade($J_{[\alpha]}$, $i$) $> 1$ for every $i$ with $1 \leq i \leq d$. Then $A = R[\alpha a, (\alpha a)^{-1}]$ if and only if $a$ is in $U(R)$.

**Proof.** ($\implies$) Lemma 2.1 implies that $a$ is in $U(A)$. Hence $a$ is in $U(A) \cap R$. By Proposition 2.2, we know that $U(A) \cap R = U(R)$. Therefore $a$ is in $U(R)$.

($\impliedby$) Since $a$ is in $U(R)$, we see that $(\alpha a)^{-1}$ is in $A$. Hence

$$R[\alpha a, (\alpha a)^{-1}] \subset A.$$ Note that $\alpha = (\alpha a)a^{-1}$ and $\alpha^{-1} = (\alpha a)^{-1}a$. Then we get $A \subset R[\alpha a, (\alpha a)^{-1}]$. Therefore $A = R[\alpha a, (\alpha a)^{-1}]$. □
§3. Generalized denominator ideals.

We will consider the ring $R[\eta_1, \ldots, \eta_d]$. We can refer to S. Oda and K. Yoshida [3, Corollary 15.2] and S. Oda and K. Yoshida [4, Corollary 1.2] for the condition $I_{[\alpha]} R[\alpha] = R[\alpha]$ and the ring $R[\eta_1, \ldots, \eta_d]$. In this section we will study the ring $R[\eta_1, \ldots, \eta_d]$ in the case the condition $I_{[\alpha]} R[\alpha, \alpha^{-1}] = R[\alpha, \alpha^{-1}]$ holds.

**Lemma 3.1.** Set $C = R[\eta_1, \ldots, \eta_d]$. If $I_{[\alpha]} R[\alpha, \alpha^{-1}] = R[\alpha, \alpha^{-1}]$, then $C \subset R[\alpha, \alpha^{-1}]$.

**Proof.** By definition of $I_{[\alpha]}$, it is easily seen that $\eta_1, \ldots, \eta_d$ are in $I_{[\alpha]}^{-1}$. We know that

$$I_{[\alpha]}^{-1} \subset I_{[\alpha]}^{-1} R[\alpha, \alpha^{-1}] = I_{[\alpha]}^{-1} I_{[\alpha]} R[\alpha, \alpha^{-1}] \subset R[\alpha, \alpha^{-1}]$$

Hence $C \subset R[\alpha, \alpha^{-1}]$. □

**Proposition 3.2.** Let $R$ be a Noetherian integral domain and $\alpha$ an anti-integral element over $R$ of degree $d$. If $I_{[\alpha]} R[\alpha, \alpha^{-1}] = R[\alpha, \alpha^{-1}]$, then $J_{[\alpha]} = R$ and $R[\alpha, \alpha^{-1}]/R$ is a flat extension.

**Proof.** We will show that $R[\alpha, \alpha^{-1}]/R$ is a flat extension. Let $P$ be a prime ideal of $R[\alpha, \alpha^{-1}]$ and set $p = P \cap R$. By the condition $I_{[\alpha]} R[\alpha, \alpha^{-1}] = R[\alpha, \alpha^{-1}]$, we know that $I_{[\alpha]} \nsubseteq p$. So $J_{[\alpha]} \nsubseteq p$ because $I_{[\alpha]} \subset J_{[\alpha]}$. Hence $J_{[\alpha]} R_p = R_p$. Then Lemma 1.2 shows that $R_p[\alpha, \alpha^{-1}]/R_p$ is a flat extension. Hence $R[\alpha, \alpha^{-1}]/R$ is also a flat extension. By Lemma 1.2, we get $J_{[\alpha]} = R$. □

The following is an analogous result to S. Oda and K. Yoshida [3, Theorem 11].

**Theorem 3.3.** Let $R$ be a Noetherian integral domain and $\alpha$ an anti-integral element over $R$ of degree $d$. Set $C = R[\eta_1, \ldots, \eta_d]$. If $I_{[\alpha]} R[\alpha, \alpha^{-1}] = R[\alpha, \alpha^{-1}]$, then the following conditions hold.

1. $C \subset R[\alpha, \alpha^{-1}]$.
2. $I_{[\alpha]} C = C$.
3. $C/R$ is a birational and flat extension.

**Proof.** We have already proved (1) in Lemma 3.1.

(2) Proposition 3.2 says that $J_{[\alpha]} = R$.

Moreover, $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. Then we get $I_{[\alpha]} C = C$.

(3) It is clear that $C/R$ is a birational extension. Let $p$ be a prime ideal of $R$. Then we will show that $pC = C$ or $C \subset R_p$. From this fact it is easily seen that $C/R$ is a flat extension. If $p \supseteq I_{[\alpha]}$, then $I_{[\alpha]} C = C$ means that $pC = C$. If $p \nsubseteq I_{[\alpha]}$, then $\eta_1, \ldots, \eta_d$ are in $R_p$ because $I_{[\alpha]} = \cap_{i=1}^d J_{[\alpha]_i}$. Hence $C \subset R_p$. □

**Theorem 3.4.** Let $R$ be a Noetherian integral domain and $\alpha$ an anti-integral element over $R$ of degree $d$. Assume that $J_{[\alpha]} = R$. Then the following two statements hold.

1. $I_{[\alpha]} R[\alpha] = R[\alpha]$ if and only if $\text{rad}(I_{[\alpha]}) = \text{rad}(J_{[\alpha]_1}, \ldots, J_{[\alpha]_d})$.
2. $I_{[\alpha]} R[\alpha, \alpha^{-1}] = R[\alpha, \alpha^{-1}]$ if and only if $\text{rad}(I_{[\alpha]}) = \text{rad}(J_{[\alpha]_1}, \ldots, J_{[\alpha]_d})$.

**Proof.** (1) ($\Rightarrow$) It is immediate from $\text{rad}(I_{[\alpha]}) \subset \text{rad}(J_{[\alpha]_1}, \ldots, J_{[\alpha]_d})$ that $I_{[\alpha]} \subset J_{[\alpha]_1}$. Let $p$ be a prime divisor of $I_{[\alpha]}$. Then we have $p R[\alpha] = R[\alpha]$ because $I_{[\alpha]} R[\alpha] = R[\alpha]$. By Lemma 1.1, we get $p \supseteq J_{[\alpha]}_1$. Therefore $\text{rad}(J_{[\alpha]_1}, \ldots, J_{[\alpha]_d}) \subset \text{rad}(I_{[\alpha]})$, and so $\text{rad}(I_{[\alpha]}) = (J_{[\alpha]_1}, \ldots, J_{[\alpha]_d})$.

($\Leftarrow$) Let $p$ be a prime divisor of $I_{[\alpha]}$. Then
\[ p \triangleright \text{rad}(I_{[d]}), \, \alpha \triangleright J_{[d]}, \, \alpha. \]

Hence by Lemma 1.1, we obtain \( pR[\alpha] = R[\alpha] \). Therefore \( \text{rad}(I_{[d]}(\alpha))R[\alpha] = R[\alpha] \). This means that \( I_{[d]}(\alpha) \triangleright \alpha \). 

(2) Since \( J_{[d]} \triangleright \alpha \), by Lemma 1.3, the following holds:

\[ pR[\alpha, \alpha^{-1}] = R[\alpha, \alpha^{-1}] \] if and only if there exists an integer \( i \) \((1 \leq i \leq d)\) such that \( p \triangleright J_{[d], i} \).

Note that \( p \triangleright \bigcap_{i=1}^{d} J_{[d], i} \), if and only if there exists an integer \( i \) \((1 \leq i \leq d)\) satisfying \( p \triangleright J_{[d], i} \). By making use of these facts we can prove the assertion (2) in a similar way to the proof of (1). \( \square \)

**Remark 3.5.** In the case \( d = 1 \), Proposition 1.5, Corollary 1.6, 1.10, Proposition 2.2 and Theorem 2.3 hold by rewriting \( i = 0, 1 \) instead of \( i = 1, \cdots, d \). Theorem 3.4 (2) does not hold even if we rewrite \( i = 0, 1 \) because \( I_{[d]} \not\in I_{[d]}\eta_1 \).

**References**


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