ON ATOMS OF BCK-ALGEBRAS

DAJUN SUN

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Abstract. Atoms in BCK-algebras are considered. The notions of the star BCK-algebras and the star part of BCK-algebras are introduced. The properties of some substructures which consist of atoms are investigated. Furthermore, by isomorphic view, there are and only \( n + 1 \) BCK-algebras \( X \) with \( |X| = n + 1 \) and \( |\text{St}(X)| = n \).

1. Introduction By a BCK-algebra we mean an algebra \( (X; *, 0) \) of type \((2, 0)\) satisfying the axioms:

(1) \( ((x * y) * (x * z)) * (z * y) = 0 \),
(2) \( x * (x * y) * y = 0 \),
(3) \( x * x = 0 \),
(4) \( x * y = y * x = 0 \) implies \( x = y \),
(5) \( 0 * x = 0 \)

for any \( x, y \) and \( z \) in \( X \). For any BCK-algebra \( X \), the relation \( \leq \) defined by \( x \leq y \) if and only if \( x * y = 0 \) is a partial order on \( X \) (see [1]).

A BCK-algebra \( X \) has the following properties for any \( x, y, z \in X \):

(6) \( x * 0 = x \),
(7) \( (x * y) * z = (x * z) * y \),
(8) \( x \leq y \) implies that \( x * z \leq y * z \) and \( z * y \leq z * x \).

In a BCK-algebra \( X \), if an element \( a \) satisfying:

(a) \( a \neq 0 \),
(b) \( x \in X \setminus \{0\} \) and \( x \leq a \) imply \( x = a \)

then the element \( a \) is called an atom of \( X \). Since \( 0 \) is the least element of \( X \), it is obvious that an atom of \( X \) is a minimal element of \( X \).

Let \( (X; *, 0) \) be a BCK-algebra. A non-empty subset \( S \) of \( X \) is called a subalgebra if \( x, y \in S \) implies \( x * y \in S \). By an ideal \( I \) of \( X \) we mean \( 0 \in I \) and \( x * y \in I \) imply \( x \in I \). If an ideal \( I \) of \( X \) is also a subalgebra of \( X \), then \( I \) is called a close ideal of \( X \). It has been known that an ideal of a BCK-algebra is a close ideal (see [2]).

2. Star subalgebras of BCK-algebras Let \( X \) be a BCK-algebra. We define

\[ \text{St}_I(X) = \{ a \in X; a = 0 \text{ or } a \text{ is an atom of } X \}. \]
The subset \( S_t(X) \) is called the star part of \( X \).

**Proposition 2.1.** Let \( X \) be a BCK-algebra. If \( a, b \in S_t(X) \) and \( a \neq b \), then \( a \ast b = a \).

**Proof.** In case \( a = 0 \) or \( b = 0 \), the proof is trivial. Assume \( a \neq 0 \) and \( b \neq 0 \), by \( a \ast b \leq a \) and \( a \) is an atom of \( X \), we get \( a \ast b = 0 \) or \( a \ast b = a \). If \( a \ast b = 0 \), then \( a = 0 \) or \( a = b \) since \( b \) is an atom of \( X \). It is a contradiction, hence \( a \ast b = a \). The proof is completed.

By Proposition 2.1 we can immediately get

**Theorem 2.2.** For any BCK-algebra \( X \), \( S_t(X) \) is a subalgebra of \( X \).

Let \( X \) be a BCK-algebra and \( S \) be a subalgebra of \( X \). \( S \) is called a star subalgebra of \( X \) if \( S_t(S) = S \). Particularly \( X \) is called a star BCK-algebra if \( X = S_t(X) \).

**Remark.** \( S_t(X) \) may be not a maximal star subalgebra.

**Example 1.** Let \( X = \{0, \ldots, -n - 1, -n, -n + 1, \ldots, -3, -2, -1\} \) and partial order \( \leq \) as follows \( 0 \leq \cdots \leq -n - 1 \leq -n \leq -n + 1 \leq \cdots \leq -3 \leq -2 \leq -1 \). Define operation \( \ast \) by

\[
x \ast y = \begin{cases} 0, & x \leq y \\ x, & \text{otherwise} \end{cases}
\]

for any \( x, y \) in \( X \). Then \( (X; \ast, 0) \) is a BCK-algebra and \( S_t(X) = \{0\} \). If take the subalgebra \( S = \{0, 1\} \) of \( X \), then \( S_t(S) = S \). In this example, \( S_t(X) \) is not a maximal star subalgebra of \( X \).

**Example 2.** Let \( X = \{0, 1, 2, 3\} \). Take the operation table of \( X \) as follows

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<td>3</td>
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<td>3</td>
<td>3</td>
<td>0</td>
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</table>

Then \( (X; \ast, 0) \) is a BCK-algebra. \( S_t(X) = \{0, 1, 3\} \) is a maximal star subalgebra of \( X \) but not the largest star subalgebra of \( X \) since \( S = \{0, 2\} \) is a star subalgebra of \( X \).

**Theorem 2.3.** Let \( X \) be a BCK-algebra and \( S \) be a subalgebra of \( X \). Then \( S \) is a star subalgebra of \( X \) if and only if for any \( a, b \in S \), \( a \neq b \) implies \( a \ast b = a \).

**Proof.** By Proposition 2.1, the necessity part is obvious. In the sufficiency part, for any \( b \in S \setminus \{0\} \), if there exists \( x_0 \in S \setminus \{0\} \) such that \( x_0 \leq b \), then we have \( x_0 = b \) or \( x_0 \neq b \). If \( x_0 \neq b \), then we get \( x_0 \ast b = 0 \) by \( x_0 \leq b \) and \( x_0 \ast b = x_0 \) by the condition of the Theorem, hence \( x_0 = 0 \). It is contradictory that \( x_0 \in S \setminus \{0\} \). Hence \( x_0 = b \) and \( b \) is an atom of \( S \). The proof is completed.

**Theorem 2.4.** Let \( X \) be a BCK-algebra. Then \( S_t(X) \) is a maximal star subalgebra of \( X \) if and only if for any element \( x \) in \( X \setminus S_t(X) \), there exists an element \( a \) in \( S_t(X) \setminus \{0\} \) such that \( a \leq x \).

**Proof.** Assume \( S \) be a star subalgebra of \( X \) and \( S_t(X) \subseteq S \). If there exists an element \( x_0 \) of \( X \) in \( S \setminus S_t(X) \), then there exists an element \( a \) in \( S_t(X) \setminus \{0\} \subseteq S \) such that \( a \leq x_0 \). Hence the element \( x_0 \) is not an atom of \( S \). It is contradictory that \( S \) is a star subalgebra of \( X \). And the sufficient part is proved. On the other hand, if there exists \( x_0 \) in \( X \setminus S_t(X) \) such that for all \( a \) in \( S_t(X) \setminus \{0\} \), \( a \ast x_0 \neq 0 \), then we have \( a \ast x_0 = a \) by \( a \ast x_0 \leq a \) and \( a \in S_t(X) \setminus \{0\} \). Assume \( x_0 \ast a = b \), we get

\[
(x_0 \ast b) \ast a = (x_0 \ast a) \ast b = b \ast b = 0,
\]
that is \( x_0 \ast b \leq a \), hence \( x_0 \ast b = 0 \) or \( x_0 \ast b = a \). If \( x_0 \ast b = a \), then
\[
a \ast x_0 = (x_0 \ast b) \ast x_0 = (x_0 \ast x_0) \ast b = 0 \ast b = 0.
\]
It is a contradiction. Hence \( x_0 \ast b = 0 \). By \( b \ast x_0 = (x_0 \ast a) \ast x_0 = (x_0 \ast x_0) \ast a = 0 \ast a = 0 \), we have \( x_0 = b \neq x_0 \ast a \). Then we get \( x_0 \ast a = x_0 \) and \( a \ast x_0 = a \) for all \( a \in S_1(X) \). Therefore \( S = S_1(X) \cup \{x_0\} \) is a star subalgebra of \( X \) by Theorem 2.3. It is contradictory that \( S_1(X) \) is a maximal star subalgebra. The proof is completed.

**Corollary 2.5.** For a finite BCK-algebra \( X \), \( S_1(X) \) is a maximal star subalgebra of \( X \).

**Theorem 2.6.** Let \( X \) be a BCK-algebra. \( S_1(X) \) is the largest star subalgebra of \( X \) if and only if \( X = S_1(X) \).

**Proof.** The sufficiency part is obvious. Conversely, for any \( x \in X \setminus \{0\} \), \( S = \{0, x\} \) is a star subalgebra of \( X \), hence \( x \in S_1(X) \) since \( S_1(X) \) is the largest star subalgebra. The proof is completed.

Let \( (X; *_1, 0, \cdot) \), \( (Y; *_2, 0) \) be two BCK-algebras. The set \( X \times Y = \{(x, y); x \in X, y \in Y\} \) about operation \( *: (x_1, y_1) \ast (x_2, y_2) = (x_1 \ast x_2, y_1 \ast y_2) \) becomes a BCK-algebra, and \((0,0)\) is the zero element of \( X \times Y \).

Generally, \( S_1(X \times Y) \neq S_1(X) \times S_1(Y) \), but we have

**Theorem 2.7.** Let \( X, Y \) be two BCK-algebras. Then
\[
S_1(X \times Y) = (S_1(X) \times \{0\}) \cup (\{0\} \times S_1(Y))
\]

**Proof.** It is obvious that \( S_1(X \times Y) \supseteq (S_1(X) \times \{0\}) \cup (\{0\} \times S_1(Y)) \). Furthermore, for any \( (x_0, y_0) \in S_1(X \times Y) \), if \( x_0 \neq 0 \) and \( y_0 \neq 0 \), then we get \( (x_0, 0) \ast (x_0, y_0) = (0, 0) \). It is contradictory that \( (x_0, y_0) \in S_1(X \times Y) \). Hence we get \( x_0 = 0 \) or \( y_0 = 0 \). If \( x_0 = 0 \), then it is easy to prove that \( y_0 \in S_1(Y) \). Similarly, if \( y_0 = 0 \), then \( x_0 \in S_1(X) \). The proof is completed.

**Corollary 2.8.** For any finite BCK-algebra \( X, Y \), we have \( |S_1(X \times Y)| = |S_1(X)| + |S_1(Y)| - 1 \).

**Corollary 2.9.** Let \( X, Y \) be two BCK-algebras. Then \( S_1(X \times Y) = S_1(X) \times S_1(Y) \) if and only if \( S_1(x) = \{0\} \) or \( S_1(y) = \{0\} \).

Let \( X \) be a BCK-algebra. If an atom \( b \) of \( X \) satisfies that \( b \ast x = b \) for any \( x \in X \setminus \{b\} \), then we call \( b \) a strong atom of \( X \). Take the subset of \( X \)

\[
D(X) = \{ b \in S_1(X); \ b \ is \ a \ strong \ atom \ of \ X \ or \ b = 0 \}
\]

we have

**Theorem 2.10.** For any BCK-algebra \( X \), \( D(X) \) is a closed ideal of \( X \).

**Proof.** We need to prove that \( D(X) \) is an ideal of \( X \) only. Assume \( y, x \ast y \in D(X) \), if \( x \ast y = x \), then \( x \in D(X) \). If \( x \ast y \neq x \), then \((x \ast y) \ast x = x \ast y \) by the definition of \( D(X) \). On the other hand,
\[
(x \ast y) \ast x = (x \ast x) \ast y = 0 \ast 0 = 0,
\]
hence we get \( x \ast y = 0 \). By \( y \in D(X) \) and \( x \ast y = 0 \), we have \( x = 0 \) or \( x = y \), hence \( x \in D(X) \).

The proof is completed.

3. **On star BCK-algebras** Suppose \( (X; *_0, 0) \) be a BCK-algebra. For any \( a \in X \), we use \( a^{-1} \) denote the selfmap of defined by \( xa^{-1} = x \ast a \). Let \( M(X) \) denote the set of all finite
product $a^{-1} \cdots b^{-1}$ of selfmaps with $a, \ldots, b \in X$. It is clear that $M(X)$ becomes a commutative monoid under composition of maps and $0^{-1}$ is the identity. We define a relation $\leq_1$ on $M(X)$ as follows:

$$u^{-1} \cdots v^{-1} \leq_1 a^{-1} \cdots b^{-1} \iff (xu^{-1} \cdots v^{-1}) \ast (xa^{-1} \cdots b^{-1}) = 0$$

for any $x \in X$. We call $M(X)$ the adjoint semigroup of $X$ (see [3]). It is obvious that $M(S) = \{u^{-1} \cdots v^{-1}; u, \ldots, v \in S\}$ becomes a subsemigroup of $M(X)$ for any non-empty subset $S$ of $X$.

**Lemma 3.1.** Let $X$ be a BCK-algebra and $\sigma = a_1^{-1} \cdots a_n^{-1} \in M(S_1(X))$. If $S_1(X)$ is an ideal of $X$, then $\text{Ker}\sigma = \{0, a_1, a_2, \ldots, a_n\}$

**Proof.** It is obvious that $\{0, a_1, a_2, \ldots, a_n\} \subseteq \text{Ker}\sigma$ by Section 1. Conversely, if $\sigma = a_1^{-1}$ and $b \in \text{Ker}\sigma$, then $ba_1^{-1} = b \ast a_1 = 0$. We get $b = 0$ or $b = a_1$ by $a_1 \in S_1(X)$, hence $b \in \{0, a_1\}$ and the Lemma holds for $n = 1$. Now we assume the Lemma has already been proved for $n = k$, then we prove the case of $\sigma = a_1^{-1} \cdots a_k^{-1}a_{k+1}^{-1}$. If $b \in \text{Ker}\sigma$, $b \ast c = 0$, then we have $b \in S_k(X)$ by $S_k(X)$ is an ideal of $X$ and $a_1, \ldots, a_{k+1} \in S_k(X)$. Since $(b \ast a_{k+1}) \ast b = 0$ and $b \in S_k(X)$, we get $b \ast a_{k+1} = 0$ or $b \ast a_{k+1} = b$. If $b \ast a_{k+1} = 0$, then $b = a_{k+1} \ast b = 0$ hence $b \in \{0, a_1, \ldots, a_{k+1}\}$. If $b \ast a_{k+1} = b$, then

$$0 = b \ast c = ba_1^{-1} \cdots a_{k+1}^{-1}a_{k+1} = (ba_{k+1})a_1^{-1} \cdots a_{k+1}^{-1} = ba_1^{-1} \cdots a_{k+1}^{-1},$$

we have $b \in \{0, a_1, \ldots, a_k\} \subseteq \{0, a_1, \ldots, a_k, a_{k+1}\}$ by our assumption. The proof is completed.

**Theorem 3.2.** Let $X$ be a BCK-algebra. Then $X$ is a star BCK-algebra if and only if for all $\sigma = a_1^{-1} \cdots a_n^{-1} \in M(X)$, $\text{Ker}\sigma = \{0, a_1, \ldots, a_n\}$.

**Proof.** By Lemma 3.1, the necessity part is obvious. In sufficiency part, for any element $a \in X \setminus \{0\}$, if there exists $x \in X$ such that $x \leq a$, then $x \in \text{Ker}\sigma^{-1} = \{0, a\}$, hence $x = 0$ or $x = a$, and $a$ is an atom of $X$. The proof is completed.

By a positive implicatve BCK-algebra, we mean a BCK-algebra $(X; \ast, 0)$ such that for all $x, y, z \in X$, $(x \ast y) \ast z = (x \ast z) \ast (y \ast z)$. If for all $x, y \in X$, $y \ast (y \ast x) = x \ast (y \ast y)$, then $X$ is said to be a commutative BCK-algebra. It is also noteworthy that $X$ is a positive implicatve BCK-algebra if and only if $(x \ast y) \ast y = x \ast y$ for all $x, y \in X$ (see [4]). By using these results, we have

**Theorem 3.3.** If $X$ is a star BCK-algebra, then the following results hold:

(a) $X$ is a positive implicatve BCK-algebra;

(b) $X$ is a commutative BCK-algebra.

**Proof.** (a) For any $x, y \in X$, if $x \ast y = 0$, it is obvious that $(x \ast y) \ast y = x \ast y$. Assume $x \ast y \neq 0$, then we have $x \ast y = y \ast x \leq x$ and $x$ is an atom of $X$. Hence $(x \ast y) \ast y = x \ast y$.

(b) For any $x, y \in X$, if $x \ast y = 0$ or $y \ast x = 0$, then we get $x = 0$ or $x = y$ or $y = 0$, hence it is obvious that $y \ast (y \ast x) = x \ast (x \ast x)$. Assume $x \ast y \neq 0$ and $y \ast x \neq 0$, then we get $x \ast y = x$ and $y \ast x = y$ by $x \ast y \leq y$ and $y \ast x \leq y$, hence $x \ast (x \ast y) = 0 = y \ast (y \ast x)$. The proof is completed.

4. The count of a class finite BCK-algebras Let $u$ be an element in BCK-algebra $X$. If for any $x \in X$, $u \ast x = 0$ implies $u = x$, then $u$ is called a maximal element of $X$.

**Theorem 4.1.** Let $X$ be a BCK-algebra. If $u \in X \setminus D(X)$ is a maximal element of $X$, then for any $b \in D(X)$, $u \ast b = u$. 
Proof. It is trivial to see the case $b = 0$. If $b \neq 0$, then we have $(u * (u * b)) * b = 0$ by BCK axiom (2), hence $u * (u * b) = 0$ or $b$, since $b$ is an atom of $X$. If $u * (u * b) = b$, then we can get

$$b * u = (u * (u * b)) * u = (u * u) * (u * b) = 0 * (u * b) = 0,$$

it is contradictory that $b$ is a strong atom of $X$. Hence $u * (u * b) = 0$. Therefore $u * b = u$ by $u$ is a maximal element of $X$. The proof is completed.

Lemma 4.2. If $X = \{0, a_1, \cdots, a_n \}$ is a BCK-algebra with $S_t(X) = \{0, a_1, \cdots, a_{n-1} \}$ and $D(X) = \{0, a_1, \cdots, a_i \}$ $(0 \leq i \leq n - 2)$, then for all $a_k \in S_t(X) \setminus D(X)$, $a_k * a_n = 0$.

Proof. By $(a_k * a_n) * a_k = (a_k * a_k) * a_n = 0 * a_n = 0$, we get $a_k * a_n = 0$ or $a_k * a_n = a_k$ since $a_k$ is an atom of $X$. If $a_k * a_n = a_k$, then we have $a_k \in D(X)$ by Proposition 2.1, it is contradictory that $a_k \notin D(X)$. Hence $a_k * a_n = 0$. The proof is completed.

Corollary 4.3. In Lemma 4.2, the element $a_n$ is a maximal element of $X$.

Let $X$ be a BCK-algebra with $|X| = n + 1$, $|S_t(X)| = n$ and $|D(X)| = i + 1$ $(0 \leq i \leq n - 2)$. Assuming $X = \{0, a_1, a_2, \cdots, a_n \}$, $S_t(X) = \{0, a_1, \cdots, a_{n-1} \}$ and $D(X) = \{0, a_1, \cdots, a_i \}$, by above discussing, the operation table of $X$ must be as table one.

In table one, $a_n * a_k = a_n * a_k (i + 1 \leq k \leq n - 1)$. After, we shall give the number of this class BCK-algebras by determining the value of $a_{nk}$ in table one.

Lemma 4.4. Let BCK-algebra $X = \{0, a_1, a_2, \cdots, a_n \}$ with $S_t(X) = \{0, a_1, a_2, \cdots, a_{n-1} \}$ and $D(X) = \{0, a_1, a_2, a_n \}$ If $|S_t(X) \setminus D(X)| \geq 2$, then the following conclusions hold:

(a) For any $a_k \in S_t(X) \setminus D(X)$, $a_n * a_k \neq a_k$;

(b) If there exists $a_k \in S_t(X) \setminus D(X)$ such that $a_n * a_k = a_l$, and $a_l \neq a_n$, then $a_n * a_l = a_k$;

(c) If there exists $a_k, a_l \in S_t(X) \setminus D(X)$ and $a_k \neq a_l$ such that $a_n * a_k = a_l$, $a_n * a_l = a_k$, then for all $a_p \in S_t(X) \setminus D(X) \cup \{a_k, a_l \}$, $a_n * a_p = a_n$.

Proof. (a) If there exists $a_k \in S_t(X) \setminus D(X)$, such that $a_n * a_k = a_k$, then take $a_l \in S_t(X) \setminus D(X)$, $a_l \neq a_k$, we have

$$0 = ((a_l * a_k) * (a_l * a_n)) * (a_n * a_k) \quad (by \ axiom \ (1))$$
$$= (a_l * a_k) * (a_n * a_k) \quad (by \ Proposition \ 2.1)$$
$$= (a_l * 0) * (a_n * a_k) \quad (by \ Lemma \ 4.2)$$
$$= a_l * (a_n * a_k)$$
$$= a_l * a_k \quad (by \ our \ assumption)$$
$$= a_l \quad (by \ Proposition \ 2.1)$$

It is a contradiction. Hence (a) holds.

(b) By BCK axioms (2), we have $0 = (a_n * (a_n * a_k)) * a_k = (a_n * a_l) * a_k$. Hence $a_n * a_l = 0$ or $a_n * a_l = a_k$. If $a_n * a_l = 0$, then we get $a_n = a_l$. By Lemma 4.2. It is contradictory that $a_l \neq a_n$. Therefore $a_n * a_l = a_k$, and the proof of (b) is completed.

(c) If there exists $a_p \in S_t(X) \setminus \{D(X) \cup \{a_k, a_l \} \}$ such that $a_n * a_p = a_q$ and $a_q \neq a_n$, then by BCK axioms (1) we have $0 = ((a_n * a_p) * (a_n * a_k)) * (a_k * a_p) = (a_q * a_l) * a_k$. If $a_q \neq a_l$, then $a_q * a_l = a_q$ by Proposition 2.1. Hence $0 = a_q * a_k$ and $a_q = a_k$, therefore we get

$$0 = (a_n * (a_n * a_p)) * a_p = (a_n * a_q) * a_p = (a_n * a_k) * a_p = a_l * a_p = a_l.$$

It is a contradiction. If $a_q = a_l$, then it is contradictory that

$$0 = (a_n * (a_n * a_p)) * a_p = (a_n * a_q) * a_p = (a_n * a_l) * a_p = a_k * a_p = a_k.$$
Hence (c) holds. And the proof of the Lemma is completed.

**Theorem 4.5.** By isomorphic view, there are total \( n + 1 \) BCK-algebras \( X \) with \(|X| = n + 1 \) and \(|S_t(X)| = n \).

**Proof.** Assuming \( X = \{0, a_1, a_2, \ldots, a_n\} \) with \( S_t(X) = \{0, a_1, a_2, \ldots, a_{n-1}\} \) and \( D(X) = \{0, a_1, a_2, \ldots, a_i\} \) \((0 \leq i \leq n - 2)\), we determine the operation tables of \( X \) according to the order of \( D(X) \).

Case 1. \(|D(X)| = n - 1\), that is \( D(X) = \{0, a_1, a_2, \ldots, a_{n-2}\} \). In this case, we only need to determine the value of \( a_n(n-1) = a_{n-1} * a_{n-1} \) in table one. By \( a_n * a_{n-1} \neq 0 \) and \( a_n * a_{n-1} \leq a_n \), we have \( a_n * a_{n-1} = a_{n-1} \) or \( a_n * a_{n-1} = a_n \). Taking \( a_n(n-1) = a_{n-1} \) and \( a_n(n-1) = a_n \) each, we get two different operation tables—table two and table three.

By BCK-algebra axioms (1)—(5) we can verify that table two and table three indeed give two BCK-algebras. Hence, there are and only two BCK-algebras in Case 1.

Case 2. \(|D(X)| = n - 2\), that is \(|S_t(X) \setminus D(X)| = 2\). In this case, if \( a_n * a_{n-2} = a_n * a_{n-1} = a_n \), then by table one we get the operation table of \( X \) as table four.

If the operation table of \( X \) is different from table four, then we have \( a_n * a_{n-2} = a_{n-1} \) and \( a_n * a_{n-1} = a_{n-2} \) by Lemma 4.4. Hence by table one the operation table must be as table five.

By BCK-algebra axioms (1)—(5) we can verify \( X \) which are given by table four and table five are BCK-algebras. Hence, there are and only two BCK-algebras in Case 2.

Case 3. \(|D(X)| < n - 2\), that is \(|S_t(X) \setminus D(X)| > 2\). In this case, if \( a_n * a_i = a_n \), \( k = i + 1, \ldots, n - 1 \), then by table one we get the operation table of \( X \) as table six.

By BCK-algebra axioms (1)—(5) we can verify \( X \) which is given by table six is BCK-algebra. If the operation table of \( X \) is different from table six, then by Lemma 4.4, there are two elements \( a_k, a_i \in S_t(X) \setminus D(X) \) such that \( a_n * a_k = a_i \) and \( a_n * a_k = a_k \). Assume \( a_k = a_{n-2} \) and \( a_i = a_{n-1} \). We get \( a_n * a_i = a_n, \) \( p = i + 1, \ldots, n - 3 \) by Lemma 4.4. Hence by table one the operation table must be as follows

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>\ldots</th>
<th>( a_i )</th>
<th>( a_{i+1} )</th>
<th>\ldots</th>
<th>( a_{n-3} )</th>
<th>( a_{n-2} )</th>
<th>( a_{n-1} )</th>
<th>( a_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>\ldots</td>
<td>0</td>
<td>0</td>
<td>\ldots</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( a_1 )</td>
<td>( a_1 )</td>
<td>0</td>
<td>( a_1 )</td>
<td>\ldots</td>
<td>( a_1 )</td>
<td>( a_1 )</td>
<td>\ldots</td>
<td>( a_1 )</td>
<td>( a_1 )</td>
<td>( a_1 )</td>
<td>( a_1 )</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>( a_2 )</td>
<td>( a_2 )</td>
<td>0</td>
<td>\ldots</td>
<td>( a_2 )</td>
<td>( a_2 )</td>
<td>\ldots</td>
<td>( a_2 )</td>
<td>( a_2 )</td>
<td>( a_2 )</td>
<td>( a_2 )</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\ldots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\ldots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>( a_i )</td>
<td>( a_i )</td>
<td>( a_i )</td>
<td>( a_i )</td>
<td>\ldots</td>
<td>( a_i )</td>
<td>( a_i )</td>
<td>\ldots</td>
<td>( a_i )</td>
<td>( a_i )</td>
<td>( a_i )</td>
<td>( a_i )</td>
</tr>
<tr>
<td>( a_{i+1} )</td>
<td>( a_{i+1} )</td>
<td>( a_{i+1} )</td>
<td>( a_{i+1} )</td>
<td>\ldots</td>
<td>( a_{i+1} )</td>
<td>( a_{i+1} )</td>
<td>\ldots</td>
<td>( a_{i+1} )</td>
<td>( a_{i+1} )</td>
<td>( a_{i+1} )</td>
<td>( a_{i+1} )</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\ldots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\ldots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>( a_{n-3} )</td>
<td>( a_{n-3} )</td>
<td>( a_{n-3} )</td>
<td>( a_{n-3} )</td>
<td>\ldots</td>
<td>( a_{n-3} )</td>
<td>( a_{n-3} )</td>
<td>\ldots</td>
<td>( a_{n-3} )</td>
<td>( a_{n-3} )</td>
<td>( a_{n-3} )</td>
<td>( a_{n-3} )</td>
</tr>
<tr>
<td>( a_{n-2} )</td>
<td>( a_{n-2} )</td>
<td>( a_{n-2} )</td>
<td>( a_{n-2} )</td>
<td>\ldots</td>
<td>( a_{n-2} )</td>
<td>( a_{n-2} )</td>
<td>\ldots</td>
<td>( a_{n-2} )</td>
<td>( a_{n-2} )</td>
<td>( a_{n-2} )</td>
<td>( a_{n-2} )</td>
</tr>
<tr>
<td>( a_{n-1} )</td>
<td>( a_{n-1} )</td>
<td>( a_{n-1} )</td>
<td>( a_{n-1} )</td>
<td>\ldots</td>
<td>( a_{n-1} )</td>
<td>( a_{n-1} )</td>
<td>\ldots</td>
<td>( a_{n-1} )</td>
<td>( a_{n-1} )</td>
<td>( a_{n-1} )</td>
<td>( a_{n-1} )</td>
</tr>
<tr>
<td>( a_n )</td>
<td>( a_n )</td>
<td>( a_n )</td>
<td>( a_n )</td>
<td>\ldots</td>
<td>( a_n )</td>
<td>( a_n )</td>
<td>\ldots</td>
<td>( a_n )</td>
<td>( a_n )</td>
<td>( a_n )</td>
<td>( a_n )</td>
</tr>
</tbody>
</table>

**Table seven**

But the algebra defined by table seven is not a BCK-algebra, for, we have

\[
((a_{n-3} * a_{n-1}) * (a_{n-3} * a_n)) * (a_n * a_{n-1}) = (a_{n-3} * 0) * a_{n-2} = a_{n-3} \neq 0,
\]

namely, the BCK-algebra axiom (1) does not hold. Hence, there exists and only one BCK-algebra \( X \) with \(|D(X)| = i + 1 < n - 2\) in Case 3 by table six. Since the order of \( D(X) \)
can take 1, 2, \cdots, n - 3, the proof is completed by combinig Case 1, Case 2, Case 3, and the operation tables are given by table two — table six.

$$
\begin{array}{cccccccccc}
* & 0 & a_1 & a_2 & \cdots & a_i & a_{i+1} & \cdots & a_{n-3} & a_{n-2} & a_{n-1} & a_n \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
a_1 & a_1 & 0 & a_1 & \cdots & a_1 & a_1 & \cdots & a_1 & a_1 & a_1 & a_1 \\
a_2 & a_2 & a_2 & 0 & \cdots & a_2 & a_2 & \cdots & a_2 & a_2 & a_2 & a_2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
a_i & a_i & a_i & a_i & \cdots & 0 & a_i & \cdots & a_i & a_i & a_i & a_i \\
a_{i+1} & a_{i+1} & a_{i+1} & a_{i+1} & \cdots & a_{i+1} & 0 & \cdots & a_{i+1} & a_{i+1} & a_{i+1} & a_{i+1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
a_{n-3} & a_{n-3} & a_{n-3} & a_{n-3} & \cdots & a_{n-3} & a_{n-3} & \cdots & 0 & a_{n-3} & a_{n-3} & a_{n-3} \\
a_{n-2} & a_{n-2} & a_{n-2} & a_{n-2} & \cdots & a_{n-2} & a_{n-2} & \cdots & a_{n-2} & 0 & a_{n-2} & a_{n-2} \\
a_{n-1} & a_{n-1} & a_{n-1} & a_{n-1} & \cdots & a_{n-1} & a_{n-1} & \cdots & a_{n-1} & a_{n-1} & 0 & 0 \\
a_n & a_n & a_n & a_n & \cdots & a_n & a_n & \cdots & a_n & a_n & a_n & a_n \\
\end{array}
$$

(table one)

$$
\begin{array}{cccccccccc}
* & 0 & a_1 & a_2 & \cdots & a_i & a_{i+1} & \cdots & a_{n-3} & a_{n-2} & a_{n-1} & a_n \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
a_1 & a_1 & 0 & a_1 & \cdots & a_1 & a_1 & \cdots & a_1 & a_1 & a_1 & a_1 \\
a_2 & a_2 & a_2 & 0 & \cdots & a_2 & a_2 & \cdots & a_2 & a_2 & a_2 & a_2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
a_i & a_i & a_i & a_i & \cdots & 0 & a_i & \cdots & a_i & a_i & a_i & a_i \\
a_{i+1} & a_{i+1} & a_{i+1} & a_{i+1} & \cdots & a_{i+1} & 0 & \cdots & a_{i+1} & a_{i+1} & a_{i+1} & a_{i+1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
a_{n-3} & a_{n-3} & a_{n-3} & a_{n-3} & \cdots & a_{n-3} & a_{n-3} & \cdots & 0 & a_{n-3} & a_{n-3} & a_{n-3} \\
a_{n-2} & a_{n-2} & a_{n-2} & a_{n-2} & \cdots & a_{n-2} & a_{n-2} & \cdots & a_{n-2} & 0 & a_{n-2} & a_{n-2} \\
a_{n-1} & a_{n-1} & a_{n-1} & a_{n-1} & \cdots & a_{n-1} & a_{n-1} & \cdots & a_{n-1} & a_{n-1} & 0 & 0 \\
a_n & a_n & a_n & a_n & \cdots & a_n & a_n & \cdots & a_n & a_n & a_n & a_n \\
\end{array}
$$

(table two)

$$
\begin{array}{cccccccccc}
* & 0 & a_1 & a_2 & \cdots & a_i & a_{i+1} & \cdots & a_{n-3} & a_{n-2} & a_{n-1} & a_n \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
a_1 & a_1 & 0 & a_1 & \cdots & a_1 & a_1 & \cdots & a_1 & a_1 & a_1 & a_1 \\
a_2 & a_2 & a_2 & 0 & \cdots & a_2 & a_2 & \cdots & a_2 & a_2 & a_2 & a_2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
a_i & a_i & a_i & a_i & \cdots & 0 & a_i & \cdots & a_i & a_i & a_i & a_i \\
a_{i+1} & a_{i+1} & a_{i+1} & a_{i+1} & \cdots & a_{i+1} & 0 & \cdots & a_{i+1} & a_{i+1} & a_{i+1} & a_{i+1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
a_{n-3} & a_{n-3} & a_{n-3} & a_{n-3} & \cdots & a_{n-3} & a_{n-3} & \cdots & 0 & a_{n-3} & a_{n-3} & a_{n-3} \\
a_{n-2} & a_{n-2} & a_{n-2} & a_{n-2} & \cdots & a_{n-2} & a_{n-2} & \cdots & a_{n-2} & 0 & a_{n-2} & a_{n-2} \\
a_{n-1} & a_{n-1} & a_{n-1} & a_{n-1} & \cdots & a_{n-1} & a_{n-1} & \cdots & a_{n-1} & a_{n-1} & 0 & 0 \\
a_n & a_n & a_n & a_n & \cdots & a_n & a_n & \cdots & a_n & a_n & a_n & a_n \\
\end{array}
$$

(table three)
\begin{table}[h]
\centering
\begin{tabular}{|c|c c c ... c c c |}
\hline
*        & 0 & a_1 & a_2 & ... & a_i & a_{i+1} & ... & a_{n-3} & a_{n-2} & a_{n-1} & a_n \\
\hline
0        & 0 & 0  & 0  & ... & 0  & 0  & ... & 0  & 0  & 0  & 0  \\
\hline
a_1     & 0 & a_1 & 0  & ... & a_1 & a_1 & ... & a_1 & a_1 & a_1 & a_1 \\
\hline
a_2     & 0 & a_2 & a_2 & 0  & ... & a_2 & a_2 & ... & a_2 & a_2 & a_2 \\
\hline
\vdots  & . & .  & .  & ... & .  & .  & ... & .  & .  & .  & .  \\
\hline
a_i     & a_i & a_i & a_i & ... & a_i & a_i & ... & a_i & a_i & a_i & a_i \\
\hline
a_{i+1} & a_{i+1} & a_{i+1} & a_{i+1} & ... & a_{i+1} & a_{i+1} & ... & a_{i+1} & a_{i+1} & a_{i+1} & a_{i+1} \\
\hline
\vdots  & . & .  & .  & ... & .  & .  & ... & .  & .  & .  & .  \\
\hline
a_{n-3} & a_{n-3} & a_{n-3} & a_{n-3} & ... & a_{n-3} & a_{n-3} & ... & 0  & a_{n-3} & a_{n-3} & a_{n-3} \\
\hline
a_{n-2} & a_{n-2} & a_{n-2} & a_{n-2} & ... & a_{n-2} & a_{n-2} & ... & a_{n-2} & a_{n-2} & a_{n-2} & a_{n-2} \\
\hline
a_{n-1} & a_{n-1} & a_{n-1} & a_{n-1} & ... & a_{n-1} & a_{n-1} & ... & a_{n-1} & a_{n-1} & a_{n-1} & a_{n-1} \\
\hline
a_n     & a_n & a_n & a_n & ... & a_n & a_n & ... & a_n & a_n & a_n & a_n \\
\hline
\end{tabular}
\caption{Table Four}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c c c ... c c c |}
\hline
*        & 0 & a_1 & a_2 & ... & a_i & a_{i+1} & ... & a_{n-3} & a_{n-2} & a_{n-1} & a_n \\
\hline
0        & 0 & 0  & 0  & ... & 0  & 0  & ... & 0  & 0  & 0  & 0  \\
\hline
a_1     & 0 & a_1 & 0  & ... & a_1 & a_1 & ... & a_1 & a_1 & a_1 & a_1 \\
\hline
a_2     & 0 & a_2 & a_2 & 0  & ... & a_2 & a_2 & ... & a_2 & a_2 & a_2 \\
\hline
\vdots  & . & .  & .  & ... & .  & .  & ... & .  & .  & .  & .  \\
\hline
a_i     & a_i & a_i & a_i & ... & a_i & a_i & ... & a_i & a_i & a_i & a_i \\
\hline
a_{i+1} & a_{i+1} & a_{i+1} & a_{i+1} & ... & a_{i+1} & a_{i+1} & ... & a_{i+1} & a_{i+1} & a_{i+1} & a_{i+1} \\
\hline
\vdots  & . & .  & .  & ... & .  & .  & ... & .  & .  & .  & .  \\
\hline
a_{n-3} & a_{n-3} & a_{n-3} & a_{n-3} & ... & a_{n-3} & a_{n-3} & ... & 0  & a_{n-3} & a_{n-3} & a_{n-3} \\
\hline
a_{n-2} & a_{n-2} & a_{n-2} & a_{n-2} & ... & a_{n-2} & a_{n-2} & ... & a_{n-2} & a_{n-2} & a_{n-2} & a_{n-2} \\
\hline
a_{n-1} & a_{n-1} & a_{n-1} & a_{n-1} & ... & a_{n-1} & a_{n-1} & ... & a_{n-1} & a_{n-1} & a_{n-1} & a_{n-1} \\
\hline
a_n     & a_n & a_n & a_n & ... & a_n & a_n & ... & a_n & a_n & a_n & a_n \\
\hline
\end{tabular}
\caption{Table Five}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c c c ... c c c |}
\hline
*        & 0 & a_1 & a_2 & ... & a_i & a_{i+1} & ... & a_{n-3} & a_{n-2} & a_{n-1} & a_n \\
\hline
0        & 0 & 0  & 0  & ... & 0  & 0  & ... & 0  & 0  & 0  & 0  \\
\hline
a_1     & 0 & a_1 & 0  & ... & a_1 & a_1 & ... & a_1 & a_1 & a_1 & a_1 \\
\hline
a_2     & 0 & a_2 & a_2 & 0  & ... & a_2 & a_2 & ... & a_2 & a_2 & a_2 \\
\hline
\vdots  & . & .  & .  & ... & .  & .  & ... & .  & .  & .  & .  \\
\hline
a_i     & a_i & a_i & a_i & ... & a_i & a_i & ... & a_i & a_i & a_i & a_i \\
\hline
a_{i+1} & a_{i+1} & a_{i+1} & a_{i+1} & ... & a_{i+1} & a_{i+1} & ... & a_{i+1} & a_{i+1} & a_{i+1} & a_{i+1} \\
\hline
\vdots  & . & .  & .  & ... & .  & .  & ... & .  & .  & .  & .  \\
\hline
a_{n-3} & a_{n-3} & a_{n-3} & a_{n-3} & ... & a_{n-3} & a_{n-3} & ... & 0  & a_{n-3} & a_{n-3} & a_{n-3} \\
\hline
a_{n-2} & a_{n-2} & a_{n-2} & a_{n-2} & ... & a_{n-2} & a_{n-2} & ... & a_{n-2} & a_{n-2} & a_{n-2} & a_{n-2} \\
\hline
a_{n-1} & a_{n-1} & a_{n-1} & a_{n-1} & ... & a_{n-1} & a_{n-1} & ... & a_{n-1} & a_{n-1} & a_{n-1} & a_{n-1} \\
\hline
a_n     & a_n & a_n & a_n & ... & a_n & a_n & ... & a_n & a_n & a_n & a_n \\
\hline
\end{tabular}
\caption{Table Six}
\end{table}
REFERENCES


Department of Mathematics
Langfang Teacher’s College
Langfang 065000, Hebei, P.R.China