AFFINE-DERIVED 3-DESIGNS

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ABSTRACT. For any affine (or Hadamard) 3-(v, k, λ) design, four 3-designs are constructed. Namely a 3-(4λ + 4, λ + 1, \binom{λ}{2}) design, a 3-(4λ + 4, 2λ + 2, \binom{2λ + 2}{2}) design, a 3-(4λ + 4, λ + 1, \binom{λ}{2}) design and a 3-(4λ + 4, 3λ + 3, 3(\binom{λ}{2} + 2)) design. Moreover necessary conditions for the existence of simple such designs, i.e., with no repeated blocks, are also given.

1. Preliminaries. Let t, u, k, λ be integers such that t ≥ 0, k, λ > 0 and v > k + 1. A t-(v, k, λ) design (or t-design) is an ordered pair of points and blocks (V, B) where

(1) |V| = v, (2) B is a family of k-subsets of V, (3) every t-subset of V is contained in λ members of B.

A design will be called nontrivial if t < k ≤ v − 2. Moreover a t-design is also an s-design for every s ≤ t. We shall denote by λ_s the number of blocks of B containing each s-subset of V. Thus

\( \lambda_s = \lambda \left( \frac{v - s}{t - s} \right) \frac{k - s}{t - s} \)

see [2,7]. Let \( r = \lambda_1 \) and \( b = \lambda_0 = |B| \). Then \( bk = uv \).

An affine 3-design is a 3-(4λ + 4, 2λ + 2, λ) design. It is also called a Hadamard 3-design since it is an extension of a Hadamard 2-(4λ + 3, 2λ + 1, λ) design, see [2,4,7]. Using this the following properties of an affine 3-design (V, B) follow easily, see [2,7]: (i) \( r = 4λ + 3 \), (ii) \( \lambda_2 = 2λ + 1 \), (iii) \( b = 8λ + 6 \), (iv) for each block \( a \in B \), its compliment \( a' = V - a \) is also a block of B, (v) \( |a \cap c| = \lambda + 1 \), for all \( a, c \in B, c \neq a, a' \).

Finally let \( A_1, A_2, A_3 \) be any three subsets of a set A. By the Sieve Principle see [3]

(*) \(|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| \ldots\)

2. Designs Constructions.

Theorem 1. Let (V, B) be an affine 3-(4λ + 4, 2λ + 2, λ) design where \( \lambda > 2 \). If \( B_1 = \{ a - c \mid a, c \in B, c \neq a, a' \} \), then (V, B_1) is a 3-(4λ + 4, 2λ + 2, λ) design.

Proof. Let \( a - c \in B_1 \). Hence

\(|a - c| = |a - c| = |a| - |a \cap c| = 2λ + 2 - 2\lambda - 1 = \lambda + 1, \)

which is independent of \( a, c \). Now let \( \{ P, Q, S \} \) be any set of three points of V contained in \( a - c \in B_1 \). Therefore \( \{ P, Q, S \} \subseteq a \) and \( \{ P, Q, S \} \cap c = \emptyset \). Moreover there are \( \lambda \) choices of \( a \). To find number of choices of \( c \), let \( i(P), i(Q), i(S) \) be the set of blocks of \( B \) containing
\( P, Q \) and \( S \) respectively. By the Sieve Principle (eqn. (s) above) and properties (i,ii), the number of blocks of \( B \) which contains at least one of \( P, Q \) and \( S \) is

\[
|\ell(P) \cup \ell(Q) \cup \ell(S)| = |\ell(P)| + |\ell(Q)| + |\ell(S)| - |\ell(P) \cap \ell(Q)| - |\ell(P) \cap \ell(S)| - |\ell(Q) \cap \ell(S)| + |\ell(P) \cap \ell(Q) \cap \ell(S)|
\]

\[
= 3x - 3\lambda_2 + \lambda.
\]

Using this and properties (i,ii,iii) therefore the number of blocks of \( B \) disjoint from \( \{P, Q, S\} \) is

\[
b = 3x + 3\lambda_2 - \lambda = 8\lambda + 6 - 12\lambda - 9 + 6\lambda + 3 - \lambda = \lambda.
\]

Since \( e \neq a' \), the number of choices of \( e \) is therefore \( \lambda - 1 \). Hence \( \{P, Q, S\} \) is contained in \( \lambda(\lambda - 1) = 2(\lambda^2) \) blocks of \( B_1 \). Therefore \( (V, B_1) \) is a \( 3-(4\lambda + 4, 2\lambda + 2, \lambda) \) design. Since \( \lambda > 2 \), then \( 3 < \lambda + 1 < 4\lambda + 2 \) and hence design is not trivial.

Next we investigate repetition of blocks in \( (V, B_1) \).

**Theorem 2.** Let \( (V, B) \) be an affine \( 3-(4\lambda + 4, 2\lambda + 2, \lambda) \) design and \( a, a', c, c', d, d' \) be distinct blocks of \( B \). Then

1) \( V = a \cup c \cup d \) iff \( |a \cap c \cap d| = \lambda + 1 \)
2) \( |a \cap c \cap d| = \lambda + 1 \) then \( \lambda \) is an odd integer

**Proof.** Suppose \( V = a \cup c \cup d \). As before by Sieve Principle and property (v), we get

\[
4\lambda + 4 = |V| = |a \cup c \cup d| = 3(2\lambda + 2) - 3(\lambda + 1) + |a \cap c \cap d|.
\]

Simplifying it gives \( |a \cup c \cup d| = \lambda + 1 \). Conversely suppose \( |a \cap c \cap d| = \lambda + 1 \). By Sieve Principle and property (v) again

\[
|a \cup c \cup d| = 3(2\lambda + 2) - 3(\lambda + 1) + |a \cap c \cap d| = 3\lambda + 3 + \lambda + 1
\]

\[
= 4\lambda + 4 = |V|.
\]

Hence \( V = a \cup c \cup d \).

2) Obviously \( a \cap c, a \cap d, c \cap d \supseteq a \cap c \cap d \). By property (v), therefore

\[
|a \cap c| = |a \cap d| = |c \cap d| = \lambda + 1 = |a \cap c \cap d|.
\]

Using these relations we get \( a \cap c = a \cap d = c \cap d = a \cap c \cap d \) and hence for any \( f \in B - \{a, a', c, c', d, d'\}, f \cap a \cap c = f \cap a \cap d = f \cap c \cap d = f \cap a \cap c \cap d \). Let

(1) \( |f \cap a \cap c| = |f \cap a \cap d| = |f \cap c \cap d| = |f \cap a \cap c \cap d| = i \).

By part (1) of the theorem \( V = a \cup c \cup d \) and therefore

\[
2\lambda + 2 = |f| = |f \cap (a \cup c \cup d)| = |(f \cap a) \cup (f \cap c) \cup (f \cap d)|
\]

\[
= |f \cap a| + |f \cap c| + |f \cap d| - |f \cap a \cap c| - |f \cap a \cap d| - |f \cap c \cap d| + |f \cap a \cap c \cap d| = 3(\lambda + 1) - 2i,
\]
see eqn. (1). Therefore \( i = \frac{\lambda + 1}{2} \) and hence \( \lambda \) must be an odd integer.

**Theorem 3.** Let \((V, B)\) be an affine \(3-(4\lambda + 4, 2\lambda + 2, \lambda)\) design. If \( \lambda \) is even, then \((V, B_1)\) consists of two copies of a \(3-(4\lambda + 4, \lambda + 1, \binom{\lambda}{2})\) design which has no repeated blocks.

**Proof.** Let \( a - c, d - e \in B_1\) where \( \{a, c\} \neq \{d, e\} \) and such that \( a - c = d - e \neq \emptyset \).

Therefore \( e \cap (a - c) = e \cap (d - e) = \emptyset \), which implies that \( e \cap a \subseteq e \cap c \). Similarly \( e \cap d \subseteq e \cap c \).

Therefore

(1)

\[ e \cap a, e \cap d \subseteq e \cap c \]

Since \( a - c = d - e \neq \emptyset \), therefore \( e \neq a, c' \) and \( e \neq d \). By property (v) this implies that \( |e \cap a|, |e \cap d| = 0 \) or \( \lambda + 1 \). Consequently \( |e \cap c| = \lambda + 1 \) or \( 2\lambda + 2 \). Equivalently this implies that: (i) Either \( e = a' \) or \( |e \cap c| = \lambda + 1 \), (ii) Either \( d = c' \) or \( |e \cap d| = \lambda + 1 \) and (iii) Either \( e = c \) or \( |e \cap a| = \lambda + 1 \). Using the above assumption that \( a - c = d - e \neq \emptyset \), we investigate the various possibilities of (i,ii,iii) in the following cases: (1) \( e = a', d = c' \in B \). Since \( e' \neq a' \) therefore \( e' - a' \in B_1 \). Moreover \( e' - a' = a - c \) and hence \( a - c \in B_1 \) is a repeated block of \( B_1 \).

(2) \( e = a', |e \cap d| = \lambda + 1 \). Since \( a - c \in B_1 \) and \( e' \in B \), therefore \( |e \cap a'| = |e \cap c'| = |e \cap a| = \lambda + 1 \), by property (v). This implies that \( e \cap d = e \cap c \in B \) since \( e \cap d \subseteq e \cap c \), see eqn. (1) above. Therefore \( e \cap d \cap c = e \cap c \) and consequently \( |e \cap d| = |e \cap c| = \lambda + 1 \). Since \( |e \cap a| = |e \cap a'| = |e \cap c| = \lambda + 1 \), from above, therefore \( e, c, d, c', c, e' \) are distinct. By Theorem 2 part 2 therefore \( \lambda \) is an odd integer. (3) \( |e \cap a| = \lambda + 1 \) and \( d = c' \). Since \( d - e \in B_1 \), therefore \( |e \cap a| = |e \cap d| = \lambda + 1 \) and hence \( |e \cap c| = \lambda + 1 = |e \cap a| \). By eqn. (1) \( e \cap a \subseteq e \cap c \). Consequently \( e \cap c = a \cap e \) which implies that \( a \cap c \subseteq e \cap c \). Therefore \( |a \cap c|, |e \cap a|, |e \cap c| \) are \( \lambda + 1 \) and hence \( a, a', c, e', c, e, e' \) are distinct. By Theorem 2 part 2 it follows that \( \lambda \) is an odd integer.

(4) \( |e \cap a| = |e \cap d| = \lambda + 1 \). By eqn. (1) and referring to (iii) of this proof this case splits into two parts as follows: I) \( |e \cap c| = \lambda + 1 \). By eqn. (1) therefore \( e \cap a = c \cap e = d \cap c \) which implies that \( a \cap c \subseteq e \cap c \) and as in case (3) \( \lambda \) must be an odd integer.

II) \( e = c \). Therefore \( d - c = d - e = a - c \in B_1 \). This implies that

(2)

\[ 0 = a' \cap (a - c) = a' \cap (d - e) = a' \cap d - a' \cap c. \]

It also implies that \( d \neq a, a', c \), since we are investigating distinct repeated blocks. By eqn. (2) therefore \( a' \cap d = a' \cap c \) and hence \( a' \cap c \subseteq d \cap a \). Moreover \( |a' \cap d| = |a \cap c| = \lambda + 1 \), and \( |a' \cap e| = |e \cap c| = \lambda + 1 \). Similarly as in case (3) \( \lambda \) must be an odd integer. Since \( \lambda \) is even, it follows by case (1) above that each block of \( B_1 \) is repeated once and therefore \((V, B_1)\) consists of two copies of a \(3-(4\lambda + 4, \lambda + 1, \binom{\lambda}{2})\) design which has no repeated blocks.

**Corollary 1.** If there exists a Hadamard matrix of order \( 4(\lambda + 1) \) for even \( \lambda \geq 4 \), then there exists a simple \(3-(4\lambda + 4, \lambda + 1, \lambda(\lambda - 1)/2)\) design.

**Remark 1.** Corollary 1 is a corollary of Theorems 3 and 6.

**Remark 2.** If there exists a Hadamard matrix of order \( 4n \) for a positive integer \( n \), then there exists an affine \(3-(4n, 2n, n-1)\) design. It was conjectured by Hadamard [6] that there exists a Hadamard matrix of order \( 4n \) for any positive integer \( n \). It is known (cf. Colbourn and Dinitz [5]) that (1) there exists a Hadamard matrix of order \( 4n \) for any positive integer \( n \) less than or equal to 107 and (2) there is no positive integer \( n \) such that the nonexistence of Hadamard matrices of order \( 4n \) is known and (3) there exists a Hadamard matrix of order \( 4n \) for infinitely many even integers \( n \). Hence Corollary 1 gives a new series of simple 3-designs.

**Theorem 4.** Let \((V, B)\) be an affine \(3-(4\lambda + 4, 2\lambda + 2, \lambda)\) design. If \( B_2 = \{(a - c) \cup (c - a): a, c \in B, c \neq a, a'\} \), then \((V, B_2)\) is a \(3-(4\lambda + 4, 2\lambda + 2, 2(\lambda + 1)/2)\) design.
Proof. Let \((a - c) \cup (c - a) \in B_2\). By Sieve Principle and property (v)

\[
\left|(a - c) \cup (c - a)\right| = |a - c| + |c - a| - |a \cap c| = 2(2\lambda + 2) - 2(\lambda + 1) = 2\lambda + 2,
\]

which is independent of \(a\) and \(c\). Next let \(\{P, Q, S\}\) be any set of three points of \(V\) contained in \((a - c) \cup (c - a) \in B_2\). There are two types of such blocks: (1) \(\{P, Q, S\} \subseteq a\), \(\{P, Q, S\} \cap c = \emptyset\). Hence \(\{P, Q, S\} \subseteq a - c \in B_1\). By theorem 1 therefore \(\{P, Q, S\}\) is contained in \(\lambda(\lambda - 1)\) blocks of this type. (2) One block, \(a\) say, contains two points only, \(\{P, Q\}\) say, and the other block \(c\) contains \(S\) only. Since the number of blocks of \(B\) which contain \(\{P, Q, S\}\) is \(\lambda\) and the number of blocks which contain \(\{P, Q\}\) is \(\lambda_2 = 2\lambda + 1\) by property (ii), therefore there are \(\lambda_2 - \lambda = \lambda + 1\) choices of \(a\). Since each of \(\{P, S\}, \{Q, S\}\) is on \(\lambda_2\) blocks of \(B\) and \(\{P, Q, S\}\) is on \(\lambda\) blocks, this implies that \(S\) is on

\[
r = 2\lambda_2 + \lambda = 4\lambda + 3 - 2(2\lambda + 1) + \lambda = \lambda + 1,
\]

blocks of \(B\) which do not contain \(\{P, Q\}\); see properties above. Since \(c \neq a'\), therefore the number of choices of \(c\) is \(\lambda + 1 - 1 = \lambda\). Consequently the number of blocks \((a - c) \cup (c - a)\) such that \(a\) contains \(\{P, Q, S\}\) only and \(c\) contains \(S\) only is \(\lambda(\lambda + 1)\). Since \(\{P, Q\}\) is one of three ways of choosing two points from \(\{P, Q, S\}\), therefore the number of blocks of this type is \(3\lambda(\lambda + 1)\). Combining the two counts in the two types implies that \(\{P, Q, S\}\) is contained in

\[
\lambda(\lambda - 1) + 3\lambda(\lambda + 1) = 4\lambda^2 + 2\lambda = 2\left(\frac{2\lambda + 1}{2}\right)
\]

blocks of \(B_2\). Therefore \((V, B_2)\) is a \(3\left(4\lambda + 4, 2\lambda + 2, \lambda\right)\) design which is nontrivial since \(3 < 2\lambda + 2 < 4\lambda + 4 - 2\).

Next we investigate repetition of blocks in \(B_2\).

Theorem 5. Let \((V, B)\) be an affine \(3\left(4\lambda + 4, 2\lambda + 2, \lambda\right)\) design. If \(\lambda\) is even, then \((V, B_2)\) consists of two copies of a \(3\left(4\lambda + 4, 2\lambda + 2, \left(\frac{2\lambda + 1}{2}\right)\right)\) design which has no repeated blocks.

Proof. Let \((a - c) \cup (c - a), (d - e) \cup (e - d)\) be any two blocks of \(B_2\), where \(\{a, c\} \neq \{d, e\}\) and such that

\[
\begin{align*}
(a - c) \cup (c - a) &= (d - e) \cup (e - d) \\
(a - c) &= a \cap ((a - c) \cup (c - a)) = a \cap ((d - e) \cup (e - d)) = (a \cap (d - e)) \cup (a \cap (e - d)) \\
&= (a \cap d - a \cap e) \cup (a \cap e - a \cap d) = (a \cap d - a \cap d \cap e) \cup (a \cap e - a \cap d \cap e).
\end{align*}
\]

Using this we get

\[
\lambda + 1 = |a \cap c| = |a - c| = |(a \cap d - a \cap d \cap e) \cup (a \cap e - a \cap d \cap e)|
\]

(2)

\[
= |a \cap d| - |a \cap d \cap e| + |a \cap e| - |a \cap d \cap e| = |a \cap d| + |a \cap e| - 2|a \cap d \cap e|.
\]

Since \(\{a, e\} \neq \{d, e\}\) we can assume that \(a \neq d, e\) and hence \(|a \cap d|, |a \cap e| = 0, \lambda + 1\), see property (v). If \(a \cap e| = |a \cap d| = \lambda + 1,\) then substituting in eqn. (2) we get \(|a \cap d \cap e| = \frac{1}{2}(\lambda + 1),\)
contradiction since $\lambda$ is even. Therefore exactly one of $|a \cap c|$ and $|a \cap d|$ is equal to 0. If $|a \cap d| = 0$, then $d = a'$. Substituting in eqn. (1) we get $(a - c) \cup (e - a) = (a' - c) \cup (e - a')$.

Using this equation as we did with eqn. (1) above we get

$$c - a = c \cap ((a - c) \cup (c - a)) = c \cap ((a' - c) \cup (c - a'))$$

$$= (c \cap a' - c \cap a' \cap e) \cup (e \cap c \cap a' \cap e).$$

Similarly and as in eqn. (2) this gives

$$\lambda + 1 = |c - a| = |c \cap a'| + |c \cap e| - 2|c \cap a' \cap e| = \lambda + 1 + |c \cap e| - 2|c \cap a' \cap e|$$

since $c \neq a', a$. Therefore $|c \cap a' \cap e| = \frac{1}{2}|c \cap e|$. Since $|c \cap e| = 0$ or $\lambda + 1$ or $2\lambda + 2$, then $c = e'$ or $c \neq a', e'$ or $c = e$ respectively. Therefore we have three cases to consider: (i) If $c = e'$ then $e = e'$. Moreover $d = a'$, from above. Using this we get

$$(a - c) \cup (e - a) = (a - c') \cup (c \cap a') = (c' - a') \cup (a - a') = (d - e) \cup (e - d).$$

Since $(c' - a') \cup (a - c') \in B_2$, therefore $(a - c) \cup (e - a)$ is a repeated block.(ii) If $c \neq e, e'$, therefore $|c \cap e| = \lambda + 1$. Since $|c \cap a' \cap e| = \frac{1}{2}|c \cap e|$ from above, therefore $|c \cap a' \cap e| = \frac{1}{2}(\lambda + 1)$. Contradiction since $\lambda$ is even. (iii) If $c = e$ then by substituting in eqn. (1) we get $(a - c) \cup (c - a) = (a' - c) \cup (c - a')$, since $d = a'$ from above. Contradiction, since the two sides of this equation are non empty and have no common points. Therefore by case (i) above each block of $B_2$ is repeated once and therefore $(V, B_2)$ consists of two copies of a $3- \lambda(4\lambda + 4, 2\lambda + 2, \frac{(3\lambda + 1)}{2})$ design which has no repeated blocks.

**Corollary 2.** If there exists a Hadamard matrix of order $4(\lambda + 1)$ for even $\lambda \geq 4$, then there exists a simple $3-(4\lambda + 4, 2\lambda + 2, 2(\lambda + 1))$ design.

**Remark 3.** Corollary 2 gives a new series of simple $3$-designs.

**Theorem 6.** Let $(V, B)$ be an affine $3-(4\lambda + 4, 2\lambda + 2, \lambda)$ design. If $B_3 = \{a \cap c : a, c \in B, c \neq a, a'\}$ and $\lambda > 2$, then $(V, B_3)$ is a $3-(4\lambda + 4, \lambda + 1, \frac{(3\lambda + 1)}{2})$ design which has no repeated blocks if $\lambda$ is even.

**Proof.** Let $a \cap c \in B_3$, then by property (v) $|a \cap c| = \lambda + 1$ which is independent of $a$ and $c$. Let $\{P, Q, S\}$ be any set of three points of $V$. Since there are $\lambda$ blocks of $B$ containing $P$, it follows that the number of blocks $a \cap c \in B_3$, which contain it is $\frac{(3\lambda + 1)}{2}$. Therefore $(V, B_3)$ is a $3-(4\lambda + 4, \lambda + 1, \frac{(3\lambda + 1)}{2})$ design which is not trivial since $3 < \lambda + 1 < 4\lambda + 4 - 2$. Let $a \cap c, d \cap e \in B_3$ where $\{a, c\} \neq \{d, e\}$ and such that $a \cap c = d \cap e$. Therefore $a \cap c \cap d = d \cap e$ and $|a \cap c \cap d| = |d \cap e| = \lambda + 1$. Since $\{a', c\}$ and $\{d, e\}$ assume that $d \neq a, c$. Therefore $a, a', c, c', d, d'$ are distinct. By theorem 2, $\lambda$ must be odd. Contradiction since $\lambda$ is even. Therefore $(V, B_3)$ has no repeated blocks.

**Remark 4.** By Corollary 1 and Remarks 1 and 2 above theorem 6 gives a new series of simple $3$-designs.

**Theorem 7.** Let $(V, B)$ be an affine $3-(4\lambda + 4, 2\lambda + 2, \lambda)$ design. If $B_4 = \{a \cup c : a, c \in B, c \neq a, a'\}$, then $(V, B_4)$ is a $3-(4\lambda + 4, 3\lambda + 3, 3(\frac{3\lambda + 2}{2}))$ design which has no repeated blocks if $\lambda$ is even.

**Proof.** Let $a \cup c \in B_4$. Hence $|a \cup c| = |a| + |c| - |a \cap c| = 4\lambda + 4 - \lambda - 1 = 3\lambda + 3$ which is independent of $a$ and $c$. For any $a, c \in B_M(a \cup c)^T = a' \cap c' \in B_4$, see theorem 6 and
property (iv) above. Conversely for any $a \cap c \in B_3$, $(a \cap c)' = a' \cup c' \in B_4$. This implies that $B_4 = \{a': a \in B_3\}$. Let \( \{P, Q, S\} \) be any set of three points of \( V \) and \( i(P), i(Q) \) and \( i(S) \) be the set of blocks of \( B_3 \) containing \( P, Q \) and \( S \) respectively. By the Sieve Principle the number of blocks of \( B_4 \) which contains at least one of \( P, Q \) and \( S \) is equal to

\[
|\{i(P) \cup i(Q) \cup i(S)\}| = 3r - 3\lambda_2 + \left(\frac{\lambda}{2}\right),
\]

where \( r \) and \( \lambda_2 \) here are the blocks of \( B_3 \) which contain each point and each two points of \( V \) respectively. By eqn. (\( \Delta \)) above \( r = \frac{1}{2}(4\lambda + 3)(4\lambda + 2) \) and \( \lambda_2 = \frac{1}{2}\lambda(4\lambda + 2) \). Using the relation \( b \cdot k = v \cdot r \) for the design \( (V, B_3) \) we get \( b(\lambda + 1) = (4\lambda + 4)r \) which implies that \( b = 4r = 2(4\lambda + 3)(4\lambda + 2) \). Using eqn. (1) above the number of blocks of \( B_3 \) disjoint from \( \{P, Q, S\} \) is therefore equal to \( b - 3r + 3\lambda_2 - \left(\frac{\lambda}{2}\right) \). Since \( B_4 = \{a': a \in B_3\} \) therefore \( \{P, Q, S\} \) is contained in \( b - 3r + 3\lambda_2 - \left(\frac{\lambda}{2}\right) \) blocks of \( B_4 \). Substituting for \( b, r \) and \( \lambda_2 \) this number is equal to

\[
b - 3r + 3\lambda_2 - \left(\frac{\lambda}{2}\right) = 2(4\lambda + 3)(4\lambda + 2) - \frac{3}{2}(4\lambda + 3)(4\lambda + 2) \\
+ \frac{3}{2}\lambda(4\lambda + 2) - \frac{1}{2}\lambda(\lambda - 1) \\
= 3\left(\frac{3\lambda + 2}{2}\right).
\]

Hence \( (V, B_4) \) is a 3-(4\( \lambda + 4 \), 3\( \lambda + 3 \), 3(\( \lambda + 2 \)) \) design. Since \( \lambda \) is even, therefore \( B_3 \) has no repeated blocks by theorem 6 and hence \( B_4 \) has not repeated blocks as \( B_4 = \{a': a \in B_3\} \).

**Remark 5.** Using the points and blocks of an affine 3-(4\( \lambda + 4 \), 4\( \lambda + 2 \), \( \lambda \)) Cameron has constructed a 2-design see [4].

**Remark 6.** Although we have focused on affine 3-designs only, yet our methods may apply for any affine design. In fact it applies to any symmetric design.

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**References**


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