STRUCTURE OF GROUP $C^*$-ALGEBRAS OF THE GENERALIZED DISCONNECTED DIXMIER GROUPS

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ABSTRACT. In this paper we consider the structure of the group $C^*$-algebras of the generalized disconnected Dixmier groups. As an application we estimate the stable rank and connected stable rank of these $C^*$-algebras.

§0. INTRODUCTION

We first recall that the generalized discrete Heisenberg group $H^Z_{2n+1}$ of rank $2n + 1$ consists of all the $(n + 2) \times (n + 2)$ matrices:

$$
(w, v, u) = \begin{pmatrix} 1 & u & w \\ 1_n & v^t & 1 \\ 0 & & 1 \end{pmatrix} w \in Z, u = (u_j), v = (v_j) \in Z^n,
$$

which is isomorphic to the semi-direct product $Z^{n+1} \times \mathbb{Z}^n$ by the above identification. Then we define the generalized disconnected Dixmier group $D^d_{2n}$ to be the semi-direct product $\mathbb{C}^n \rtimes_a H^Z_{2n+1}$ with the action $\alpha$ defined by

$$
\alpha_g(z_1, \ldots, z_n, w_1, \ldots, w_n) = (e^{i\alpha_1}z_1, \ldots, e^{i\alpha_n}z_n, e^{i\beta_1}w_1, \ldots, e^{i\beta_n}w_n)
$$

(cf.[Sd5] for the Dixmier group). Then $D^d_{2n}$ is a complex $(2n)$-dimensional, disconnected solvable (Lie) group. This definition is analogous with that of the discrete Mautner group (cf.[B], [Sd7]). We call $D^d_{2n}$ the disconnected Dixmier group.

The structure of the group $C^*$-algebra of $H^Z_{2n}$ was investigated in terms of continuous fields of $C^*$-algebras (cf.[AP], [Dv]). The stable rank and connected stable rank of this group $C^*$-algebra were estimated by the author [Sd6]. Refer to the reference for some other works about these ranks.

In this paper, we investigate the structure of the group $C^*$-algebras of the generalized disconnected Dixmier groups, and construct their finite composition series such that their subquotients are $C^*$-algebras of continuous fields with fibers noncommutative tori. This result would be useful to analyze structure of group $C^*$-algebras of the more general groups. As an application, we estimate the stable rank and connected stable rank of these group $C^*$-algebras using some results of [Rfl] and [Sd6] mainly.

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**Notation.** For a locally compact group $G$, we denote by $C^*(G)$ the (full) group $C^*$-algebra. For a locally compact Hausdorff space $X$, we let $C_0(X)$ the $C^*$-algebra of all continuous functions on $X$ vanishing at infinity. If $X$ is compact, we denote it by $C(X)$. For a $C^*$-algebra $A$ and a locally compact group $G$, we denote by $A \rtimes_G$ the (full) $C^*$-crossed product with the action $\alpha$ of $G$ on $A$. By $\Gamma(T, \{A_t\}_{t \in T})$, we mean a $C^*$-algebra of continuous fields on the torus $T$ with $A_t$ fibers (cf. [Pd], [Dx]).

For a $C^*$-algebra $A$, we denote by $\text{sr}(A)$, $\text{csr}(A)$ the stable rank and the connected stable rank respectively (cf. [Rfl]).

§1. **GENERALIZED DISCONNECTED DIXMIER GROUPS**

Let $D_{2n}^d = \mathbb{C}^n \rtimes_n H_{2n+1}^Z$ be the generalized, disconnected Dixmier group defined in the introduction. Then the group $C^*$-algebra $C^*(D_{2n}^d)$ is isomorphic to the crossed product $C_0(\mathbb{C}^n) \rtimes_H H_{2n+1}^Z$ via the Fourier transform, where $\hat{\alpha}$ is defined by

$$\hat{\alpha}(z_1, \ldots, z_n, w_1, \ldots, w_n) = (e^{-i\alpha_1}z_1, \ldots, e^{-i\alpha_n}z_n, e^{-i\gamma_1}w_1, \ldots, e^{-i\gamma_n}w_n).$$

Note that each restriction of $\hat{\alpha}$ to each direct summand $\mathbb{C}$ of $\mathbb{C}^n$ is a rotation. Considering the restrictions of $\hat{\alpha}$ to the direct sums $(\mathbb{C} \setminus \{0\})^k$ of $\mathbb{C} \setminus \{0\}$ or $\{0\}$ with $0 \leq k \leq 2n$, we have a composition series $\{I_{2n+1}^{2n+1}\}$ of $C^*(D_{2n}^d)$ such that

$$I_{2n+1}/I_{2n} = C^*(D_{2n}^d)/I_{2n} \cong C^*(H_{2n+1}^Z),$$

$$I_{2n+1-j}/I_{2n-j} \cong (2^n_j)C_0((\mathbb{C} \setminus \{0\})^j) \rtimes_H H_{2n+1}^Z$$

where $(2^n_j)$ means $(2^n_j)$-direct sum, and $(2^n_j)$ means the combination. Moreover, we have that

$$C_0((\mathbb{C} \setminus \{0\})^j) \rtimes_H H_{2n+1}^Z \cong C_0(\mathbb{R}^j) \otimes (C(T^j) \rtimes_H H_{2n+1}^Z).$$

We now set the generators of $C^*(H_{2n+1}^Z)$ corresponding to those of $H_{2n+1}^Z$:

$$\begin{align*}
U_i &\leftrightarrow (0, \ldots, 0, u_i = 1, 0, \ldots, 0), \quad 1 \leq i \leq n, \\
V_i &\leftrightarrow (0, \ldots, 0, v_i = 1, 0, \ldots, 0), \quad 1 \leq i \leq n, \\
W &\leftrightarrow (w = 1, 0, \ldots, 0).
\end{align*}$$

Then

$$C^*(H_{2n+1}^Z) \cong C(T^{n+1}) \rtimes \mathbb{Z}^n \cong C^*(C(W, V_1, \ldots, V_n, U_1, \ldots, U_n))$$

where $C(T^{n+1}) = C^*(C(W, V_1, \ldots, V_n))$. Moreover, we set that

$$C(T^n) \cong C^*(Z_1, \ldots, Z_n, W_1, \ldots, W_n)$$

where $Z_i, W_i (1 \leq i \leq n)$ mean the coordinate functions of $\mathbb{C}^n$. Then for $j = k + l$, we let that for $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq j_1 < \cdots < j_l \leq n$,

$$C(T^j) \cong C^*(Z_{i_1}, \ldots, Z_{i_k}, W_{j_1}, \ldots, W_{j_l}).$$

We first assume that $C(T^j) \cong C^*(Z_{i_1}, \ldots, Z_{i_k})$. Then

$$C(T^j) \rtimes_H H_{2n+1}^Z \cong C^*(C(Z_{i_1}, \ldots, Z_{i_k}, W, V_1, \ldots, V_n, U_1, \ldots, U_n)) \cong C(T^{j+n+1}) \rtimes_H \mathbb{Z}^n$$
where the action \( u \) is defined by
\[
u_{u_{i_{i-1}, \ldots, u_{n}}} (z_{i}, \ldots, z_{i}, w, v_{1}, \ldots, v_{n}) = (e^{iu_{i_{i-1}}} z_{i}, \ldots, e^{iu_{1}} z_{i}, w, w^{u_{i_{i-1}}} v_{1}, \ldots, w^{u_{1}} v_{n})
\]

Therefore, the above crossed product is regarded as a \( C^* \)-algebra of continuous fields:
\[
C(T^{j+1}) \times_u Z^n \cong \Gamma(T, \{ C(T^{j+n}) \times_{u_{\in T}} Z^n \})
\]

where \( C(T^{j+n}) \times_{u_{\in T}} Z^n = C(T^{j} \times \{ w \} \times T^{n}) \times_{u} Z^n \). Then the fiber decomposes into the tensor product:
\[
C(T^{j+n}) \times_{u_{\in T}} Z^n = C(T^{2j} \times T^{n-j}) \times_{u_{\in T}} Z^n
\]
\[
\cong (C(T^{2j}) \times Z^n) \otimes (C(T^{n-j}) \times Z^{n-j})
\]
\[
\cong (\otimes^{j} C(T^{2}) \times_{\theta \otimes w} Z) \otimes (\otimes^{n-j} A_w).
\]

where the action \( \theta \otimes w \) means the product type action by the multi-rotation by the multiplication by \( (e^{2\pi i \theta}, w) \) with \( \theta = \frac{1}{2\pi} \), and \( A_{w} \) is the rotation algebra by the multiplication by \( w \). If the rotation by \( w \) on \( T \) is irrational, the fiber \( C(T^{2}) \times_{\theta \otimes w} Z \) is a simple non-commutative 3-torus (cf.[BKR]). By [EL, 2] it is an inductive limit of direct sums of matrix algebras over \( C(T) \), that is, an AT-algebra. If the rotation by \( w \) is rational, the fiber is non-simple and non-rational. By [L, Corollary 2] it is an inductive limit of direct sums of matrix algebras over a rational rotation algebra. We note that rational rotation algebras are homogeneous \( C^* \)-algebras (cf.[Dx], [Dv]).

We next assume that \( C(T^{j}) \cong C^*(W_{i_{i-1}, \ldots, W_{n}}) \). Then
\[
C(T^{j}) \times_{\theta} H^{Z}_{2n+1} \cong C^*(C^*(W_{i_{i-1}, \ldots, W_{i_{1}, W_{1}, \ldots, W_{n}}}, U_{1}, \ldots, U_{n})
\]
\[
\cong (\otimes_{s=1}^{j} C^*(W_{i_{s}}, V_{i_{s}})) \otimes C(T^{n-j+1}) \times_{u} Z^n
\]
\[
\cong (\otimes_{s=1}^{j} A_{\theta} \otimes C(T^{n-j+1})) \times_{u} Z^n
\]

where the action \( u \) is defined by
\[
u_{u_{1_{1}, \ldots, u_{n}}} (w_{i_{i-1}, \ldots, W_{1}, v_{1}, \ldots, v_{n}) = (w_{i_{1}}, \ldots, w_{i_{1}}, w, w^{u_{1}} v_{1}, \ldots, w^{u_{1}} v_{n})
\]
and \( A_{\theta} \cong C^*(W_{i_{s}}, V_{i_{s}}) \) \( (1 \leq k \leq n) \) is the irrational rotation algebra with \( \theta = \frac{1}{2\pi} \).

Therefore, the above crossed product is regarded as a \( C^* \)-algebra of continuous fields:
\[
((\otimes_{s=1}^{j} A_{\theta}) \otimes C(T^{n-j+1})) \times_{u_{\in T}} Z^n \cong \Gamma(T, \{ ((\otimes_{s=1}^{j} A_{\theta}) \otimes C(T^{n-j+1})) \times_{u_{\in T}} Z^n \})_{w \in T}.
\]

Then the fiber decomposes into the tensor product:
\[
((\otimes_{s=1}^{j} A_{\theta}) \otimes C(T^{n-j})) \times_{u_{\in T}} Z^n \cong (\otimes_{s=1}^{j} (A_{\theta} \times_{\theta \otimes w} Z)) \otimes C(T^{n-j}) \times Z^{n-j}
\]
\[
\cong (\otimes_{s=1}^{j} (A_{\theta} \times_{\theta \otimes w} Z)) \otimes (\otimes^{n-j} A_w).
\]

We note that for \( 1 \leq k \leq n, \)
\[
A_{\theta} \times_{\theta \otimes w} Z \cong C^*(W_{i_{k}}, V_{i_{k}}) \times_{\theta \otimes w} Z \cong C^*(W_{i_{k}}, U_{i_{k}}) \times_{\theta \otimes w} Z \cong C(T^{2}) \times_{\theta \otimes w} Z.
\]

Finally, we assume that \( C(T^{j}) \cong C^*(Z_{i_{i-1}, \ldots, Z_{i_{1}, W_{1}, \ldots, W_{1}}}, U_{1}, \ldots, U_{n}) \) for \( j = k + l \) with \( k, l \geq 1 \). Then
\[
C(T^{j}) \times_{\theta} H^{Z}_{2n+1} \cong C^*(C^*(Z_{i_{i-1}, \ldots, Z_{i_{1}, W_{1}, \ldots, W_{1}}}, W_{1}, \ldots, W_{l}), U_{1}, \ldots, U_{n})
\]
\[
\cong (C(T^{k}) \otimes (\otimes_{s=1}^{l} C^*(W_{i_{s}}, V_{i_{s}})) \otimes C(T^{n-i+1})) \times_{u} Z^n
\]
\[
\cong (C(T^{k}) \otimes (\otimes_{s=1}^{l} A_{\theta}) \otimes (\otimes_{s=1}^{l-k} A_{\theta}) \otimes C(T^{n-i+1-k})) \times_{u} Z^n
\]
where \(0 \leq k_1 \leq k, l\) and

\[ u_{(u_1, \ldots, u_n)}(w_{i}, \ldots, w_{j}, w, v_{1}, \ldots, v_{n}) = (w_{i}, \ldots, w_{j}, w, w_{u_{1}+1}, \ldots, w_{u_{n}+1}) \]

Therefore, putting \(p = n - l + k - k_1\),

\[
(C(T)^k) \otimes ((\otimes_{i=k}^l A_\theta) \otimes (\otimes_{i=k}^l A_\theta) \otimes C(T^{p+1})) \times_u \mathbb{Z}^n
\]

\[ \cong \Gamma(T, \{(\otimes_{i=k_1}^l (C(T) \otimes A_\theta) \times_{\theta \otimes w} Z) \otimes (\otimes_{i=k_1}^l (A_\theta \times_w Z)) \otimes (C(T^p) \times \mathbb{Z}^{n-l})\}_{w \in T}) \]

Then noting \(n - l \geq k - k_1\),

\[
(C(T^p) \times \mathbb{Z}^{n-l}) \cong (\otimes_{i=k_1}^l (C(T^2) \times_{\theta \otimes w} Z)) \otimes (\otimes_{i=k_1}^l (A_\theta \times_w Z)).
\]

Moreover, \(B_{w, \theta} = (C(T) \otimes A_\theta) \times_{\theta \otimes w} Z\) is a simple, noncommutative torus. We note that it contains \((C \otimes A_\theta) \times_{\theta \otimes w} Z \cong A_{\theta} \times_w Z\) as a \(C^*_\text{subalgebra}\), and the Rieffel projections of \(A_{\theta}\) (cf. [Wo]) commute with \(C(T) \otimes C\). Therefore, the cut-down method for \(A_{\theta} \times_w Z\) ([EL1], [Ln]) is extended to the case of the fiber \(B_{w, \theta}\). In fact, if \(w\) is the irrational rotation, any element of \(A_{\theta} \times_w Z\) is approximated by matrix algebras of noncommutative 2-tori, so that any element of \(B_{w, \theta}\) is approximated by matrix algebras of noncommutative 3-tori. If \(w\) is rational, then \(A_{\theta} \times_w Z\) is an inductive limit of direct sums of matrix algebras of a rational rotation algebra. Therefore, in both cases we deduce that any element of \(B_{w, \theta}\) is approximated by AT-algebras or approximately homogeneous \(C^*_\text{algebras}\) with slow dimension growth (cf. [BDR]).

Summing up we obtain that

**Theorem 1.1.** Let \(D_{2n} = \mathbb{C}^{2n} \times_{A_{\theta}} H_{2n+1}^{Z}\) be the generalized disconnected Dixmier group. Then \(C^*(D_{2n}^{d})\) has a finite composition series \(\{I_j\}_{j=1}^{2n+1}\) such that

\[
\]
Compare with the case of the generalized Dixmier groups [Sd5]. In particular, $C^*(D^d_2)$ has the finite composition series $\{I_1, I_2, \ldots, I_n\}$ such that

$$I_k/I_{k-1} = C^*(D^d_2)/I_k \cong C^*(H^Z_2),$$

where $[x]$ means the maximal integer $\leq x$.

Proof. By [Sd6], we have that

$$\text{sr}(C^*(K_{2n+1}^Z)) = n + 1 \geq \text{csr}(C^*(K_{2n+1}^Z)) \geq 2.$$
REFERENCES


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