PARALINDÉLÖF SUBSPACES IN PRODUCTS OF TWO ORDINALS

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Abstract. Let $\rho$ and $\sigma$ be ordinals with the order topologies. It is known from [KTY] that metacompactness, screenability and weak submetaLindelöfness are equivalent for every subspace of $\rho \times \sigma$. However there are a metacompact subspace of $[\omega_1 + 1] \times [\omega_2 + 1]$ which is not subparacompact, and a subparacompact subspace of $[\omega + 1] \times [\omega_1 + 1]$ which is not paracompact, see [KTY, Example 4.2 and 4.4]. Moreover it is not difficult to show that these examples are not paraLindelöf. So it is natural to ask whether all paraLindelöf subspaces of $\rho \times \sigma$ are paracompact for every ordinals $\rho$ and $\sigma$. In this paper, we will see that paraLindelöf subspaces of $(\rho + 1) \times (\omega_1 \cdot \omega)$ are paracompact for every ordinal $\rho$, where $\omega_1 \cdot \omega$ denotes the ordinal number $\omega_1 + \omega_1 + \cdots (\omega$-times), see [Ku, I Definition 7.19]. Moreover we will show that paraLindelöf subspaces of $(\rho + 1)^2$ are paracompact for every ordinal $\rho \leq \omega_1 \cdot \omega_1$. And we will construct a non-paracompact subspace $X$ of $(\omega_1 \cdot \omega_1) \times (\omega_1 \cdot \omega + 1)$ which can be represented as the locally countable union of clopen paracompact subspaces.

All spaces are assumed to be regular $T_1$. Let $\rho$ and $\sigma$ be ordinals with the order topologies. It is known from [KTY] that metacompactness, screenability and weak submetaLindelöfness are equivalent for every subspace of $\rho \times \sigma$. So at least, such paraLindelöf subspaces are metacompact. However there are a metacompact subspace of $(\omega_1 + 1) \times (\omega_2 + 1)$ which is not subparacompact, and a subparacompact subspace of $(\omega + 1) \times (\omega_1 + 1)$ which is not paracompact, see [KTY, Example 4.2 and 4.4]. Moreover it is not difficult to show that these examples are not paraLindelöf. In this connection, it is known in [KY] that for subspaces $A \subseteq \rho$ and $B \subseteq \sigma$, $A \times B$ is paracompact iff $A$ and $B$ are paracompact. Since, by [Be], paraLindelöf G-space are paracompact, for subspaces $A \subseteq \rho$ and $B \subseteq \sigma$, $\sigma \times \sigma$ is paraLindelöf iff $A \times B$ is paracompact. So it is natural to ask whether all paraLindelöf subspaces of $\rho \times \sigma$ are paracompact for every ordinals $\rho$ and $\sigma$. In this paper, we will see that paraLindelöf subspaces of $(\rho + 1) \times (\omega_1 \cdot \omega)$ are paracompact for every ordinal $\rho$, where $\omega_1 \cdot \omega$ denotes the ordinal number $\omega_1 + \omega_1 + \cdots (\omega$-times), see [Ku, I Definition 7.19]. Moreover we will show that paraLindelöf subspaces of $(\rho + 1)^2$ are paracompact for every ordinal $\rho < \omega_1 \cdot \omega_1$. And we will construct a non-paracompact subspace $X$ of $(\omega_1 \cdot \omega_1) \times (\omega_1 \cdot \omega + 1)$ which can be represented as the locally countable union of clopen paracompact subspaces.

We recall basic definitions and introduce specific notation from [KTY].

In our discussion, for some technical reasons, we always assume $X \subseteq (\rho + 1) \times (\sigma + 1)$ for some suitable large ordinals $\rho$ and $\sigma$. Moreover, in general, the letters $\mu$ and $\nu$ stand for limit ordinals with $\mu \leq \rho$ and $\nu \leq \sigma$. For each $A \subseteq \rho + 1$ and $B \subseteq \sigma + 1$ put

$$X_A = A \times (\sigma + 1) \cap X, \quad X_B = (\rho + 1) \times B \cap X,$$

and

$$X_A^B = X_A \cap X_B.$$
For each $\alpha \leq \rho$ and $\beta \leq \sigma$, put
$$V_\beta(X) = \{\beta \leq \sigma : \langle \alpha, \beta \rangle \in X\},$$
$$H_\beta(X) = \{\alpha \leq \rho : \langle \alpha, \beta \rangle \in X\}.$$  

$c\mu$ denotes the cofinality of the ordinal $\mu$. When $c\mu \geq \omega_1$, a subset $S$ of $\mu$ is called stationary in $\mu$ if it intersects all cub (i.e., closed and unbounded) sets in $\mu$. Moreover for each $A \subseteq \mu$, $\text{Lim}_\mu(A)$ is the set $\{\alpha < \mu : \alpha = \sup(A \cap \alpha)\}$, in other words, the set of all cluster points of $A$ in $\mu$. For convenience, we consider $\sup \emptyset = -1$ and $-1$ is the immediate predecessor of the ordinal $0$. Therefore $\text{Lim}_\mu(A)$ is cub in $\mu$ whenever $A$ is unbounded in $\mu$. We will simply denote $\text{Lim}_\mu(A)$ by $\text{Lim}(A)$ if the situation is clear in its context. In particular, assume $C$ is a cub set in $\mu$ with $c\mu \geq \omega$, then $\text{Lim}(C) \subseteq C$. In this case, we define $\text{Succ}(C) = C \setminus \text{Lim}(C)$, and $p_C(\alpha) = \sup(C \cap \alpha)$ for each $\alpha \in C$. Note that, for each $\alpha < C$, $p_C(\alpha) \in C \cup \{-1\}$, and $p_C(\alpha) < \alpha$ iff $\alpha \in \text{Succ}(C)$. So $p_C(\alpha)$ is the immediate predecessor of $\alpha$ in $C \cup \{-1\}$ whenever $\alpha \in \text{Succ}(C)$. Moreover observe that $C \setminus \text{Lim}(C)$ is the union of the pairwise disjoint collection $\{(p_C(\alpha), \alpha) : \alpha \in \text{Succ}(C)\}$ of open intervals of $\mu$ and that $\mu \setminus \text{Lim}(C)$ is the union of the pairwise disjoint collection $\{(p_C(\alpha), \alpha) : \alpha \in \text{Succ}(C)\}$ of clopen intervals of $\mu$. For short, let denote $\text{Lim} = \text{Lim}(\omega_1)$ and $\text{Succ} = \text{Succ}(\omega_1)$.

Let $\kappa$ be a regular uncountable cardinal and $A \subseteq \kappa$. Assume that a cub set $C_\gamma$ is assigned for each $\gamma \in A$. Then, by the argument of [Ku, II 6.14], the diagonal intersection
$$\Delta_{\gamma \in A} C_\gamma = \{\delta < \kappa : \forall \gamma \in A \cap \delta \in C_\gamma\}$$
is cub in $\kappa$.

A strictly increasing function $M : c\mu + 1 \to \mu + 1$ is said to be a normal function for $\mu$ if $M(\gamma) = \sup\{M(\gamma') : \gamma' < \gamma\}$ for each limit ordinal $\gamma \leq c\mu$ and $M(c\mu) = \mu$. Observe that, if $c\mu \geq \omega_1$, then two normal functions for $\mu$ coincide on a cub set of $c\mu$. Note that a normal function for $\mu$ always exists if $c\mu \geq \omega$. So we always fix a normal function $M$ for each ordinal $\mu$ with $c\mu \geq \omega$. In particular, if $\mu$ is regular, i.e. $c\mu = \mu$, then we can fix the identity map on $\mu + 1$ as the normal function. Then $M$ carries $c\mu + 1$ homeomorphically to the range ran $M$ of $M$ and ran $M$ is closed in $\mu + 1$. Note that for all $S \subseteq \mu$ with $c\mu \geq \omega_1$, $S$ is stationary in $\mu$ if and only if $M^{-1}(S)$ is stationary in $c\mu$. For convenience, we define $M(-1) = -1$.

A space $Y$ is said to be paraLindelöf if every open cover of $Y$ has a locally countable open refinement.

The next lemma follows from [KTY, Lemma 2.2].

**Lemma 1.** Let $Y$ be a subspace of $\rho + 1$ for some ordinal $\rho$. Then the following are equivalent:

(A) $Y$ is paracompact,

(B) $Y$ is paraLindelöf,

(C) For every $\mu \in (\rho + 1) \setminus Y$ with $c\mu \geq \omega_1$, $Y \cap \mu$ is not stationary in $\mu$.

**Lemma 2.** Let $X$ be a paraLindelöf subspace of $(\mu + 1) \times (\nu + 1)$ and let $E$ be a closed subset which is disjoint from $X^{(\nu)}$. If $c\nu \geq \omega_1$, then $E$ and $X^{(\nu)}$ are separated by disjoint open subsets of $X$.

**Proof.** Let $\mathcal{U} = \{X \setminus E\} \cup \{X^{(\delta,\nu)} : \delta < c\nu\}$. Then $\mathcal{U}$ is an open cover of $X$. Since $X$ is paraLindelöf, there is a precise locally countable open refinement $\mathcal{W} = \{W\} \cup \{W(\delta) : \delta < c\nu\}$ of $\mathcal{U}$, where "precise" means $W \subseteq X \setminus E$ and $W(\delta) \subseteq X^{(\delta,\nu)}$ for each $\delta < c\nu$. Let
\[ G = \bigcup_{\delta \in \text{cf} \nu} W(\delta), \text{ then } E \subseteq G. \text{ It suffices to show } X^{\{\nu\}} \subseteq X \setminus \text{Cl} G. \text{ Let } (\alpha, \nu) \in X^{\{\nu\}}. \text{ Take } f(\alpha) < \alpha \text{ and } g(\alpha) < \nu \text{ such that }\]

\[ D_\alpha = \{ \delta < \text{cf} \nu : X^{[f(\alpha), \alpha]}_\{f(\alpha), \alpha\} \cap W(\delta) \neq \emptyset \} \]

is countable. Let \( \nu' = \max\{g(\alpha), \sup D_\alpha \} \). Then \( \nu' < \nu \) and \( X^{[\nu', \nu]} \cap G = \emptyset \). This implies \( (\alpha, \nu') \in X \setminus \text{Cl} G. \qed \]

**Theorem 3.** Let \( X \) be a paraLindelöf subspace of \((\mu + 1) \times (\nu + 1)\). Assume that \( X_{[0, \mu']} \) and \( X^{[0, \nu']} \) are paracompact for each \( \mu' < \mu \) and \( \nu' < \nu \). Then in either case of the following, \( X \) is paracompact.

(a) \( \text{cf} \mu \leq \nu \), moreover either \( \nu \) is regular uncountable or \( \nu < \omega_1 \).

(b) \( \nu < \text{cf} \mu \).

**Proof.** Assume that \( X \) is not paracompact. Then it follows from Lemma 1 that \( \mu \) and \( \nu \) are limit ordinals.

**Claim 1.** \( (\mu, \nu) \notin X \).

**Proof.** Assume \( (\mu, \nu) \in X \). Let \( U \) be an open cover of \( X \). Take \( U \in U \) with \( (\mu, \nu) \in U \), moreover take \( \mu' < \mu \) and \( \nu' < \nu \) such that \( X^{[\mu', \mu']} \subseteq U \). Since, \( X_{[0, \mu']} \cup X^{[0, \nu']} \) is a paracompact clopen subspace, it is not difficult to construct a locally finite open refinement of \( U, \) a contradiction.

Moreover we have \( \text{cf} \mu \geq \omega_1 \) or \( \text{cf} \nu \geq \omega_1 \). Indeed, if \( \text{cf} \mu = \text{cf} \nu = \omega \), then \( X = \bigoplus_{n \in \omega} X_{\{M(n-1), M(n)\}} \cup \bigoplus_{n \in \omega} X_{\{N(n-1), N(n)\}} \) is the union of countably many clopen paracompact subspaces. Therefore \( X \) is paracompact, a contradiction.

We will consider several cases. In all cases, we will derive contradictions. First we consider the case (a).

Case (a). \( \text{cf} \mu \leq \nu \), moreover either \( \nu \) is regular uncountable or \( \nu < \omega_1 \).

In this case, we may assume that \( \text{cf} \mu \leq \nu \) and \( \nu \) is regular uncountable. Because, if \( \text{cf} \mu \leq \nu < \omega_1 \), then \( \text{cf} \mu = \text{cf} \nu = \omega \), a contradiction.

There are three subcases to consider.

Subcase (a-1). \( \text{cf} \mu = \omega \).

Since \( X \) is paraLindelöf, by Lemma 1, we can take a cub set \( D \in \nu \) disjoint from \( V_\mu(X) \). Then \( X_{[\mu]} \) and \( X^{D \cup \{\nu\}} \) are disjoint closed subsets. In particular, \( X_{[\mu]} \) and \( X^{[\nu]} \) are disjoint closed subsets. Applying Lemma 2, take an open set \( G \) of \( X \) such that \( X_{[\mu]} \subseteq G \) and \( \text{Cl} G \cap X^{[\nu]} = \emptyset \). Let \( U = \{ X \setminus \text{Cl} G \} \cup \{X^{[0, \delta]} : \delta < \nu \} \). Then \( U \) is an open cover of \( X \). Since \( X \) is paraLindelöf, there is a precise locally countable open refinement \( \forall = \{ W \} \cup \{ W(\delta) : \delta < \nu \} \) of \( U \). For each \( \beta \in V_\mu(X) \), we can take \( f(\beta) < \mu \) and \( g(\beta) < \beta \) such that \( (g(\beta), \beta) \cap D = \emptyset \), \( H(\beta) = X_{[f(\beta), g(\beta), \beta]} \subseteq G \) and \( \{ \delta < \nu : W(\delta) \cap H(\beta) \neq \emptyset \} \) is countable. Let \( H = \bigcup_{\beta \in V_\mu(X)} H(\beta) \). Then \( X_{[\mu]} \subseteq H \subseteq G \) is obvious.

**Claim 2.** \( S = \{ \delta \in D : X^{[\delta]} \cap \text{Cl} H \neq \emptyset \} \) is not stationary in \( \nu \).

**Proof.** Assume that \( S \) is stationary in \( \nu \). For each \( \delta \in S \), take \( h(\delta) < \mu \) with \( \{h(\delta), \delta\} \in \text{Cl} H \).

Since \( W \) is an open cover, there is \( \psi(\delta) < \nu \) with \( \{h(\delta), \delta\} \in W(\psi(\delta)) \). Note that \( \psi(\delta) > \delta \) because of \( W(\psi(\delta)) \subseteq X^{[0, \psi(\delta)]} \). Since \( (g(\beta), \beta) \cap D = \emptyset \) for each \( \beta \in V_\mu(X) \) and \( W(\psi(\delta)) \) is a neighborhood of \( \{h(\delta), \delta\} \in \text{Cl} H \), we can find \( \beta(\delta) \in V_\mu(X) \) with \( \beta(\delta) < \delta \) such that
$W(\psi(\delta)) \cap H(\beta(\delta)) \neq \emptyset$. For each $\delta \in \nu \setminus S$, define $\psi(\delta) = 0$. Then $D' = \{\delta < \nu : \forall \delta' < \delta(\psi(\delta')) < \delta\}$ is cub in $\nu$. By the PDL(Pressing Down Lemma), there is a stationary set $S' \subseteq S \cap D'$ in $\nu$ and $\beta \in V_\mu(X)$ such that $\beta(\delta) = \beta$ for each $\delta \in S'$. Since $\delta < \psi(\delta)$ for each $\delta \in S'$ and $S' \subseteq D'$, the members of $\{\psi(\delta) : \delta \in S'\}$ are all distinct. Therefore $H(\beta)$ meets uncountably many $W(\psi(\delta))\delta', \delta \in S'$, a contradiction.

Applying Claim 2, take a cub set $E \subseteq D$ in $\nu$ with $E \cap S = \emptyset$. Then, since $H \subseteq G$ and $\text{Cl } G \cap X^{[\nu]} = \emptyset$, $X_{[\mu]}$ and $X_{[\nu]}^{E \cup [\nu]}$ are separated by $H$ and $X \setminus \text{Cl } H$. Since

$$X \setminus H \subseteq X \setminus X_{[\mu]} \subseteq \bigoplus_{n \in \omega} X_{[\mu(n-1), \mu(n)]}$$

and

$$\text{Cl } H \subseteq X \setminus X_{[\nu]}^{E \cup [\nu]} \subseteq \bigoplus_{\delta \in \text{Succ}(E)} X_{[\mu(\delta), \delta]}^{[\nu]}$$

$X = (X \setminus H) \cup \text{Cl } H$ is the union of two paracompact closed subspaces. So $X$ is paracompact, a contradiction.

Subcase (a-2). $\omega_1 \leq \text{cf } \mu < \nu$.

Since $[\mu, \nu) \not\subseteq X$ and $\text{cf } \mu \geq \omega_1$, $H_\mu(X)$ is not stationary in $\mu$, so we can take a cub set $C$ in $\text{cf } \mu$ such that $M(C) \cap H_\mu(X) = \emptyset$. Similarly, for each $\gamma \in C \cup \{\text{cf } \mu\}$, since $V_{M(\gamma)}(X)$ is not stationary in $\nu$, we can take a cub set $D_\gamma$ in $\nu$ disjoint from $V_{M(\gamma)}(X) = \emptyset$. Put $D = \bigcap_{\gamma \in C \cup \{\text{cf } \mu\}} D_\gamma$. Then, since $\text{cf } \mu < \nu = \text{cf } \nu$, $D$ is a cub set in $\nu$ and $X_{M(C) \cup [\mu]}$ and $X^{D \cup [\nu]}$ are disjoint closed subsets. In particular, $X_{M(C) \cup [\mu]}$ and $X^{[\nu]}$ are disjoint closed subsets by Lemma 2, we can take an open subset $G$ such that $X_{M(C) \cup [\mu]} \subseteq G$ and $\text{Cl } G \cap X^{[\nu]} = \emptyset$.

Since $\mathcal{U} = \{X \setminus \text{Cl } G \cup \{X^{\beta, \delta} : \delta < \nu\}$ is an open cover of the paracompact Tikhonov space $X$, there is a precise locally countable open refinement $W = \{W\} \cup \{W(\delta) : \delta \in \nu\}$ of $\mathcal{U}$. For each $\gamma \in C \cup \{\text{cf } \mu\}$ and each $\beta \in V_{M(\gamma)}(X)$, we can take $f(\gamma, \beta) < M(\gamma)$ and $g(\gamma, \beta) < \beta$ such that $(g(\gamma, \beta), \beta) \cap D = \emptyset, H(\gamma, \beta) = X_{[f(\gamma, \beta), M(\gamma)]} \subseteq G$ and $\{\delta : W(\delta) \cap H(\gamma, \beta) \neq \emptyset\}$ is countable. Let

$$H = \bigcup_{\gamma \in C \cup \{\text{cf } \mu\}, \beta \in V_{M(\gamma)}(X)} H(\gamma, \beta).$$

Then $X_{M(C) \cup [\mu]} \subseteq H \subseteq G$ is obvious.

**Claim 3.** $S = \{\delta \in D : X^{[\delta]} \cap \text{Cl } H \neq \emptyset\}$ is not stationary in $\nu$.

**Proof.** Assume that $S$ is stationary in $\nu$. For each $\delta \in S$ take $h(\delta) < \mu$ with $(h(\delta), \delta) \in \text{Cl } H$ and take $\psi(\delta) < \nu$ with $(h(\delta), \delta) \in W(\psi(\delta))$. Note $\psi(\delta) > \delta$ because of $W(\psi(\delta)) \subseteq X^{[\nu]}$. Since $(g(\gamma, \beta), \beta) \cap D = \emptyset$ for each $\gamma \in C \cup \{\text{cf } \mu\}$ and $\beta \in V_{M(\gamma)}(X)$, we can take $\gamma(\delta) \in C \cup \{\text{cf } \mu\}$ and $\beta(\delta) \in V_{M(\gamma(\delta))}(X)$ with $\beta(\delta) < \delta$ such that $W(\psi(\delta)) \cap H(\gamma(\delta), \beta(\delta)) \neq \emptyset$.

As in Claim 2, noting $|C \cup \{\text{cf } \mu\}| = \text{cf } \mu < \nu$, by the PDL, we find a stationary set $S' \subseteq S$, $\beta < \nu$ and $\gamma \in C \cup \{\text{cf } \mu\}$ such that $\beta(\delta) = \beta$ for each $\delta \in S'$, $\gamma(\delta) = \gamma$ and members of $\{\psi(\delta) : \delta \in S'\}$ are all distinct. Then $H(\beta, \gamma)$ meets uncountably many $W(\psi(\delta))$'s, a contradiction.

Applying Claim 3, take a cub set $E \subseteq D$ in $\nu$ with $E \cap S = \emptyset$. Then, since $H \subseteq G$ and $\text{Cl } G \cap X^{[\nu]} = \emptyset$, $X_{M(C) \cup [\mu]}$ and $X_{E \cup [\nu]}$ are separated by $H$ and $X \setminus \text{Cl } H$. Since

$$X \setminus H \subseteq X \setminus X_{M(C) \cup [\mu]} \subseteq \bigoplus_{\gamma \in \text{Succ}(C)} X_{[M(\gamma(C)), M(\gamma)]}$$

and
$$\text{CH} \subset X \setminus X^\mathbb{F}_\nu \subset \bigoplus_{\delta \in \text{Succ}(E)} X^{[\Pi_\nu(\delta, \delta)],}$$

$X \setminus H$ and $\text{CH}$ are paracompact. Therefore $X = (X \setminus H) \cup \text{CH}$ is paracompact, a contradiction.

Subcase (a-3). cf $\mu = \nu$.

Note, in this case, $\omega_1 \leq \text{cf} \mu = \text{cf} \nu = \nu$. First we consider the special case that $X \subset \mu \times (\nu + 1)$. Since $\langle \mu, \nu \rangle \not\in X$, $\Delta(X) = \{ \gamma < \nu : \langle M(\gamma), \gamma \rangle \in X \}$ is homeomorphic to the closed subspace $X \cap \{ \langle M(\gamma), \gamma \rangle : \gamma < \nu \}$ of $X$. $\Delta(X)$ is not stationary in $\nu$. Take a cub set $C$ in $\nu$ such that $C \cap \Delta(X) = \emptyset$.

**Claim 4.** $\mathcal{X} = \{X^{[p_{e}(\gamma), \gamma]}_{\langle M(p_{e}(\gamma)), M(\gamma) \rangle} : \gamma \in \text{Succ}(C) \}$ is a discrete collection of clopen paracompact subspaces.

**Proof.** By the assumption, it suffices to show that $\mathcal{X}$ is discrete. Let $\langle \alpha, \beta \rangle \in X$. If $\alpha \not\in M(C)$, then take $\gamma \in \text{Succ}(C)$ such that $\alpha \in (M(p_{e}(\gamma)), M(\gamma)]$. Then $X^{[p_{e}(\gamma), \gamma]}_{\langle M(p_{e}(\gamma)), M(\gamma) \rangle}$ is neighborhood of $\langle \alpha, \beta \rangle$ which meets at most one member of $\mathcal{X}$. Similarly if $\beta \not\in C \cup \nu$, we can take a neighborhood of $\langle \alpha, \beta \rangle$ which meets at most one member of $\mathcal{X}$. So we may assume $\langle \alpha, \beta \rangle \in X^{\mathbb{F}_\nu(\mu)}$. Take $\gamma(\alpha) \in C$ with $M(\gamma(\alpha)) = \alpha$. Since $C \cap \Delta(X) = \emptyset$, we have $M(\beta) \neq \alpha = M(\gamma(\alpha))$, so $\beta \neq \gamma(\alpha)$. If $\beta < \gamma(\alpha)$, then $X^{[0, \beta]}_{\langle M(\beta), \alpha \rangle}$ is neighborhood of $\langle \alpha, \beta \rangle$ which meets no member of $\mathcal{X}$. If $\beta > \gamma(\alpha)$, then $X^{[\gamma(\alpha), \beta]}_{\langle \alpha, \beta \rangle}$ is neighborhood of $\langle \alpha, \beta \rangle$ which meets no member of $\mathcal{X}$. This completes the proof of Claim 4.

Let

$$Y(0) = \{ \langle \alpha, \beta \rangle \in X : \alpha > M(\beta) \} \setminus (\bigcup \mathcal{X}).$$

Then $Y(0)$ is clopen subspace of $X$. Because $Y(0)$ can be represented as $\{ \langle \alpha, \beta \rangle \in X : \alpha \geq M(\beta) \} \setminus (\bigcup \mathcal{X})$, similarly

$$Y(1) = \{ \langle \alpha, \beta \rangle \in X : \alpha < M(\beta) \} \setminus (\bigcup \mathcal{X})$$

is a clopen subspace of $X$.

**Claim 5.** $Y(0)$ is paracompact.

**Proof.** Note, in this special case, $\mu \not\in H_\delta(X)$ for each $\delta < \nu$. Therefore $H_\delta(X)$ is not stationary in $\mu$ for each $\delta < \nu$. Take a cub set $C_\delta$ in $\text{cf} \mu$ such that $M(C_\delta) \ni H_\delta(X) = \emptyset$. Put

$$C' = C \cap \Delta_{\delta < \nu} C_\delta$$

We shall show $X^{M(C')}_{C'} \cap Y(0) = \emptyset$. Assume on the contrary that $\langle \alpha, \beta \rangle \in X^{M(C')}_{C'} \cap Y(0)$. Take $\gamma(\alpha) \in C'$ with $M(\gamma(\alpha)) = \alpha$. By the definition of $Y(0)$, we have $M(\beta) < \alpha = M(\gamma(\alpha))$, so $\beta < \gamma(\alpha)$. It follows from $\beta < \gamma(\alpha) \in C' \subset \Delta_{\delta < \nu} C_\delta$ that $\gamma(\alpha) \in C_\beta$. So $\alpha = M(\gamma(\alpha)) \in M(C_\beta) \cap H_\beta(X)$, a contradiction. Hence $X^{M(C')}_{C'} \cap Y(0) = \emptyset$. Since $Y(0)$ is clopen and

$$Y(0) \subset X \setminus X^{M(C')}_{C'} \subset \bigoplus_{\gamma \in \text{Succ}(C')} X^{[p_{e}(\gamma), \gamma]}_{\langle M(p_{e}(\gamma)), M(\gamma) \rangle},$$

$Y(0)$ is paracompact.

Since $X = Y(0) \bigoplus (\bigcup \mathcal{X}) \bigoplus Y(1)$ is not paracompact but $\bigcup \mathcal{X}$ and $Y(0)$ are paracompact, $Y(1)$ is not paracompact. By considering $Y(1)$ as $X$, we may now assume that

$$X \subset \{ \langle \alpha, \beta \rangle \in \mu \times (\nu + 1) : \alpha < M(\beta) \}$$
is paralindelöf but not paracompact, moreover $X_{[0, \mu')} \times X^{[0, \nu]}$ are paracompact for each $\mu < \mu'$ and $\nu < \nu$.

Since $X$ is paralindelöf, $H_\mu(X)$ is not stationary in $\mu$. So take a cub set $C'$ in $\text{cf} \mu$ such that $M(C') \cap H_\mu(X) = \emptyset$. Similarly for each $\gamma \in C'$, we can take a cub set $C_\gamma$ in $\nu$ such that $C_\gamma \cap V_{M(\gamma)}(X) = \emptyset$. Put

$$C = C' \cap \Delta_{\gamma \in C'} C_\gamma.$$ 

Claim 6. $X_{M(\gamma)}(x) \cap X^{[\mu, \nu]} = \emptyset$.

Proof. Assume on the contrary that $(\alpha, \beta) \in X_{M(\gamma)}(x) \cap X^{[\mu, \nu]}$. Since $M(C') \cap H_\mu(X) = \emptyset$ and $\alpha \in M(\mu) \subset M(C')$, we have $\beta \neq \nu$ so $\beta \in C \subset C'$. Take $\gamma(\alpha) \in C$ with $M(\gamma(\alpha)) = \alpha$. Since $\alpha < M(\beta)$, we have $\gamma(\alpha) < \beta$. It follows from

$$\gamma(\alpha) < \beta \in C \subset \Delta_{\gamma \in C'} C_\gamma$$

that $\beta \in C_{\gamma(\alpha)}$. So $\beta \in C_{\gamma(\alpha)} \cap V_{\alpha}(X) = C_{\gamma(\alpha)} \cap V_{M(\gamma(\alpha))}$, a contradiction.

By Claim 6, in particular, $X_{M(\gamma)}(x) \cap X^{[\mu, \nu]} = \emptyset$. By Lemma 2, we can find an open subset $G$ such that $X_{M(\gamma)}(x) \subset G$ and $\text{Cl} G \cap X^{[\mu, \nu]} = \emptyset$. Since $U = \{X \setminus \text{Cl} G \cup \{X^{[\mu, \nu]} : \delta < \nu\}$ is an open cover of the paralindelöf space $X$, there is a precise locally countable open refinement $\mathcal{W} = \{W\} \cup \{W(\delta) : \delta < \nu\}$ of $U$. For each $\gamma \in C$ and each $\beta \in V_{M(\gamma)}(X)$, we can find $f(\gamma, \beta) < M(\gamma)$ and $g(\gamma, \beta) < \beta$ such that $(f(\gamma, \beta), \beta) \cap C = \emptyset$, $H(\gamma, \beta) = X_{[f(\gamma, \beta), M(\gamma)]} \subset G$ and $\{\delta < \nu : W(\delta) \cap H(\gamma, \beta) \neq \emptyset\}$ is countable. Let

$$H = \bigcup_{\gamma \in C, \beta \in V_{M(\gamma)}(x)} H(\gamma, \beta).$$

Then $X_{M(\gamma)}(x) \subset H \subset G$ is obvious.

Claim 7. $S = \{\delta \in C : X^{[\delta]} \cap \text{Cl} H \neq \emptyset\}$ is not stationary in $\nu$.

Proof. Assume that $S$ is stationary in $\nu$. For each $\delta \in S$ take $h(\delta) < \mu$ with $\langle h(\delta), \delta \rangle \in \text{Cl} H$ and take $\psi(\delta) \in \nu$ with $\langle h(\delta), \delta \rangle \in W(\psi(\delta))$. As in Claim 3, we can take $\gamma(\delta) \in C$ and $\beta(\delta) \in V_{M(\gamma(\delta))}(x)$ with $\beta(\delta) < \delta$ such that $W(\psi(\delta)) \cap H(\gamma(\delta), \beta(\delta)) \neq \emptyset$. Since $X \subset \{ \langle \alpha, \beta \rangle \in \mu \times (\nu + 1) : \alpha < M(\beta) \}$, we have $M(\gamma(\delta)) < M(\beta(\delta))$). Hence $\gamma(\delta) < \beta(\delta) < \delta$.

Applying the PDL twice, we can find a stationary set $S' \subset S$ in $\nu$, $\gamma \in C$ and $\beta \in V_{M(\gamma)}(x)$ such that $\gamma(\delta) = \gamma$ and $\beta(\delta) = \beta$ for each $\delta \in S'$, moreover members of $\{\psi(\delta) : \delta \in S'\}$ are all distinct. Then $H(\gamma, \beta)$ meets uncontprisingly many $W(\psi(\delta))$’s, a contradiction.

Applying Claim 7, take a cub set $E \subset C$ in $\nu$ with $E \cap S = \emptyset$. Then, by a similar argument of one after Claim 3, we can see that $X$ is paracompact, a contradiction.

Next we consider the general case, that is, $X \subset (\mu + 1) \times (\nu + 1)$. Since $X$ is paralindelöf, $V_\mu(X)$ is not stationary in $\mu$, so we can take a cub set $D$ in $\nu$ such that $D \cap V_\mu(X) = \emptyset$. Then $X^{D \cup [\nu]}$ and $X_{[\mu]}$ are disjoint closed subsets. By Lemma 2, there is an open subset $G$ such that $X^{D \cup [\nu]} \subset G$ and $\text{Cl} G \cap X_{[\mu]} = \emptyset$. Then

$$X \setminus G \subset X \setminus X^{D \cup [\nu]} \subset \bigoplus_{\delta \in \text{Sacc}(D)} X^{[\mu, \delta]} \text{ and}$$

and

$$\text{Cl} G \subset X \setminus X_{[\mu]} \subset \mu \times (\nu + 1).$$
By the special case, \( \text{Cl} G \) is paracompact. Therefore \( X = (X \setminus G) \cup \text{Cl} G \) is paracompact, a contradiction.

Case (b). \( \nu < \text{cf} \mu \).

There are two subcases.

Subcase (b1). \( \omega_1 \leq \text{cf} \nu \).

Since \( \text{cf} \mu > \nu \geq \text{cf} \nu \geq \omega_1 \), we can find a cub set \( D \) in \( \text{cf} \nu \) such that \( N(D) \cap \nu(X) = \emptyset \). Then \( X^N(D) \cup \nu \) and \( X_{\nu} \) are disjoint closed subsets. Applying Lemma 2, take an open set \( G \) such that \( X^N(D) \cup \nu \subset G \) and \( \text{Cl} G \cap X_{\nu} = \emptyset \).

**Claim 8.** \( S = \{ \gamma < \text{cf} \mu : X_{\langle M(\gamma) \rangle} \cap \text{Cl} G \neq \emptyset \} \) is not stationary in \( \text{cf} \mu \).

**Proof.** Assume that \( S \) is stationary in \( \text{cf} \mu \). For each \( \gamma \in S \), fix \( h(\gamma) \leq \nu \) with \( \langle M(\gamma), h(\gamma) \rangle \in \text{Cl} G \). By \( \nu < \text{cf} \mu \) and the PDL, we can find a stationary set \( S' \subset S \) in \( \text{cf} \mu \) and \( \nu' \leq \nu \) such that \( h(\gamma) = \nu' \) for each \( \gamma \in S' \). Then \( M(S') \subset H_{\nu'}(X) \), therefore \( H_{\nu'}(X) \cap \mu \) is stationary in \( \mu \). Since \( X \) is paraLindelöf and \( H_{\nu'}(X) \cap \mu \) is stationary in \( \mu \), we have \( \mu \in H_{\nu'}(X) \). So

\[
\langle \mu, \nu' \rangle \in \text{Cl} \{ \langle M(\gamma), \nu' \rangle : \gamma \in S' \} \cap X_{\nu} \subset \text{Cl} G \cap X_{\nu},
\]

a contradiction.

By Claim 8, we can take a cub set \( C \) in \( \text{cf} \mu \) such that \( C \cap S = \emptyset \). Then we have \( X^N(D) \cup \nu \subset C \) and \( X_{M(C) \cup \nu} \cap \text{Cl} G = \emptyset \). Since

\[
\begin{align*}
X \setminus G \subset X \setminus X_{M(C) \cup \nu} \subset \bigoplus_{\delta \in \text{Succ}(D)} X^{(N[\nu, \nu+\delta), N[\delta])} \quad \text{and} \\
\text{Cl} G \subset X \setminus X_{M(C) \cup \nu} \subset \bigoplus_{\gamma \in \text{Succ}(C)} X_{\langle M(pC(\gamma), M(\gamma)) \rangle},
\end{align*}
\]

\( X = (X \setminus G) \cup \text{Cl} G \) is paracompact, a contradiction.

Subcase (b2). \( \text{cf} \nu = \omega \).

Note that in this case, we have \( \omega_1 \leq \text{cf} \mu \). By Lemma 2, take an open set \( G \) with \( X^\nu \subset G \) and \( \text{Cl} G \cap X_{\nu} = \emptyset \).

By a similar argument of Claim 8, \( S = \{ \gamma < \text{cf} \mu : X_{\langle M(\gamma) \rangle} \cap \text{Cl} G \neq \emptyset \} \) is not stationary in \( \text{cf} \mu \). Then by a similar argument of one after Claim 2, we can show that \( X \) is paracompact, a contradiction. \( \square \)

**Corollary 4.** Let \( X \) be a paraLindelöf subspace of \( (\rho + 1) \times (\nu + 1) \) for a suitably large ordinal \( \rho \) such that \( X^\nu \) is paracompact for each \( \nu' \leq \nu \). If \( \nu \) is regular uncountable or \( \nu < \omega_1 \), then \( X \) is paracompact.

**Proof.** Assume that \( X \) is not paracompact and let

\[
\mu = \min \{ \mu' : X_{[\mu]} \} \text{ is not paracompact } \}.
\]

Then \( X_{[\mu]} \) is not paracompact. First assume \( \nu \) is regular uncountable. Then by Theorem 3, in either cases of \( \text{cf} \mu \leq \nu \) or \( \nu < \text{cf} \mu \), \( X_{[\mu]} \) is paracompact, a contradiction. Next assume \( \nu < \omega_1 \). Similarly by Theorem 3, \( X_{[\mu]} \) is paracompact, a contradiction. \( \square \)

This Corollary immediately yields:
**Corollary 5.** For a suitably large ordinal $\rho$, paraLindelöf subspaces of $(\rho + 1) \times (\omega_1 + 1)$ are paracompact.

**Corollary 6.** For a suitably large ordinal $\rho$, paraLindelöf subspaces of $(\rho + 1) \times (\omega_1 \cdot \omega)$ are paracompact.

*Proof.* Let $X$ be a paraLindelöf subspace of $(\rho + 1) \times (\omega_1 \cdot \omega)$. Put $T_0 = [0, \omega_1]$ and $T_n = (\omega_1 \cdot n, \omega_1 \cdot (n + 1)]$ for each $n \in \omega$ with $n \geq 1$. Since $T_n$ is homeomorphic to $(\omega_1 + 1)$, $X \cap (\rho + 1) \times T_n$ is homeomorphic to a subspace of $(\rho + 1) \times (\omega_1 + 1)$. By Corollary 5,

$$X = \bigoplus_{n \in \omega} [X \cap (\rho + 1) \times T_n]$$

is the free union of paracompact clopen subspaces, so $X$ is paracompact. □

**Corollary 7.** For each $\rho < \omega_1 \cdot \omega_1$, paraLindelöf subspaces of $(\rho + 1)^2$ are paracompact.

*Proof.* Assume that there is a paraLindelöf subspace $X$ of $(\rho + 1)^2$ which is not paracompact for some $\rho < \omega_1 \cdot \omega_1$. Let

$$\mu = \min \{ \mu' \leq \rho : X_{[0, \mu']} \text{ is not paracompact} \} ,$$

$$\nu = \min \{ \nu' \leq \rho : X_{[0, \nu']} \text{ is not paracompact} \} .$$

Since $X_{[0, \mu]}$ is a clopen subspace of $X$, we may assume that $X = X_{[0, \mu]}$. Note that $X$ is not paracompact, $X_{[0, \mu]}$ and $X_{[0, \nu]}$ are paracompact for each $\mu' < \mu$ and $\nu' < \nu$. As in the proof of Theorem 3, we can show that $(\mu, \nu) \notin X$ moreover that $\text{cf} \mu \geq \omega_1$ or $\text{cf} \nu \geq \omega_1$. So we may assume $\nu = \omega_1$. Since $\nu \leq \rho < \omega_1 \cdot \omega_1$, there is a $\xi \in \text{Succ}$ such that $\nu = \omega_1 \cdot \xi$. Let $\xi - 1$ be the immediate predecessor of $\xi$. Since $(\omega_1 \cdot (\xi - 1), \omega_1 \cdot \xi]$ is homeomorphic to $\omega_1 + 1$, by Corollary 6, $X_{[\omega_1 \cdot (\xi - 1), \omega_1 \cdot \xi]}$ is a clopen paracompact subspace. Then $X = X_{[0, \omega_1 \cdot (\xi - 1)]} \cup X_{[\omega_1 \cdot (\xi - 1), \omega_1 \cdot \xi]}$ is the union of two clopen paracompact subspaces, so $X$ is paracompact, a contradiction. □

The reader should observe that, if Theorem 3 also holds for the following remaining case (c):

(c) $\text{cf} \mu \leq \nu$, $\text{cf} \nu < \nu$ and $\nu$ is uncountable.

then all paraLindelöf subspaces of products of two ordinals are paracompact.

However, we have no useful way to extend Corollary 6 for subspaces of $(\rho + 1) \times (\omega_1 \cdot \omega + 1)$ and Corollary 7 for subspaces of $(\omega \cdot \omega_1)^2$. Now we conjecture the following.

**Conjecture 8.** There is a paraLindelöf subspace of $(\omega_1 \cdot \omega_1) \times (\omega_1 \cdot \omega_1 + 1)$ which is not paracompact. Or more generally, there is a paraLindelöf subspace of $(\omega_1 \cdot \omega_1)^2$ which is not paracompact.

The main open problem on paraLindelöf spaces is the following, see [Wa, Problem 39]:

**Problem 9.** Are paraLindelöf spaces countably paracompact?

Now we see:

**Proposition 10.** Let $X$ be a paraLindelöf countably paracompact subspace of $(\mu + 1) \times (\nu + 1)$. Assume that $X_{[0, \mu]}$ and $X_{[0, \nu]}$ are paracompact for each $\mu' < \mu$ and $\nu' < \nu$. Then $X$ is paracompact in the following case:

(c') $\text{cf} \mu \leq \nu$, $\omega = \text{cf} \nu < \nu$ and $\nu$ is uncountable.
\textbf{Proof.} Assume that $X$ is not paracompact, then as usual, we can easily show that $\mu$ and $\nu$ are limit ordinals with $\text{cf} \mu \geq \omega$, and $(\mu, \nu) \notin X$. Take a cub set $C$ in $\text{cf} \mu$ with $M(C) \cap H_\rho(X) = \emptyset$. By Lemma 2, we can also take an open set $G$ with $X^{(\mu)} \subset G \subset X \setminus X_M(C) \cap \mu \setminus [\mu, \nu)$. Let $W = \{W\} \cup \{W(n) : n \in \omega\}$ be a precise locally finite open refinement of $U = \{G\} \cup \{X^{(\alpha)} \cap \mu : n \in \omega\}$. $X_M(C) \cap \mu \subset \bigcup_{n \in \omega} W(n)$ is obvious. By the local finiteness of $W$, $\text{Cl}(\bigcup_{n \in \omega} W(n)) = \bigcup_{n \in \omega} \text{Cl} W(n) \subset X \setminus X^{(\mu)}$. Then in a usual way, we can show that $X$ is paracompact, a contradiction. \hfill \Box

Using a similar argument in Corollary 4 and 5, we see:

\textbf{Corollary 11.} For a suitably large ordinal $\rho$ and for each $\sigma < \omega_1 \cdot \omega_1$, \textit{paraLindelöf} countably paracompact subspaces of $(\rho + 1) \times (\sigma + 1)$ are paracompact. In particular, \textit{paraLindelöf} countably paracompact subspaces of $(\omega_1 \cdot \omega_1) \times (\sigma + 1)$ are paracompact for each $\sigma < \omega_1 \cdot \omega_1$.

Applying this corollary, we can show the positive answer of the conjecture 8 for $(\omega_1 \cdot \omega_1) \times (\omega_1 \cdot \omega + 1)$ yields the negative answer of Problem 9, that is:

\textbf{Corollary 12.} If there is a a \textit{paraLindelöf} subspace $X$ of $(\omega_1 \cdot \omega_1) \times (\omega_1 \cdot \omega + 1)$ which is not paracompact, then $X$ is a \textit{paraLindelöf} space which is not countably paracompact.

However, strangely, we have also no useful way to show that \textit{paraLindelöf} countably paracompact subspaces of $(\omega_1 \cdot \omega_1)^2$ are paracompact. Now we conjecture the following:

\textbf{Conjecture 13.} There is a \textit{paraLindelöf} countably paracompact subspace of $(\omega_1 \cdot \omega_1)^2$ which is not paracompact.

Although the existence of a \textit{paraLindelöf} non-paracompact subspace of $(\omega_1 \cdot \omega_1) \times (\omega_1 \cdot \omega + 1)$ still remains open, now we give some informations about the existence of such a subspace.

From now on, consider the normal function $M$ for $\omega_1 \cdot \omega_1$ by letting $M(\gamma) = \omega_1 \cdot \gamma$ for each $\gamma < \omega_1$.

\textbf{Proposition 14.} Let $X$ be a \textit{paraLindelöf} subspace of $(\omega_1 \cdot \omega_1) \times (\omega_1 \cdot \omega + 1)$. Then for each $\alpha \in H_{\omega_1 \cdot \omega_1}(X)$, there is $g(\alpha) < \omega$ such that $X^{(\omega_1 \cdot g(\alpha) \cdot \omega_1 \cdot \omega)}$ is countable.

\textbf{Proof.} For each $n \leq \omega$, since $H_{\omega_1 \cdot \omega_1}(X)$ is not stationary in $\omega_1 \cdot \omega_1$, take a cub set $C_n$ in $\omega_1$ such that $M(C_n) \cap H_{\omega_1 \cdot \omega_1}(X) = \emptyset$. Set $C = \bigcap_{n \leq \omega} C_n$ and $E = \{\omega_1 \cdot n : n \leq \omega\}$. Then $X_M(C) \cap X$ and $X^{(\mu)}$ are disjoint closed sets of $X$. For each $\gamma < \omega_1$, let $U(\gamma) = X^{(\omega_1 \cdot n + \gamma)} \cap \bigcup_{n \leq \omega} X^{(\omega_1 \cdot n + \gamma)}$. Take a precise locally countable open refinement $W = \{W(\gamma) : \gamma < \omega_1\}$ of $U = \{U(\gamma) : \gamma < \omega_1\} \cup \{X \setminus X_M(C)\}$. Let $\alpha \in H_{\omega_1 \cdot \omega_1}(X)$. Since $W$ is locally countable at $(\alpha, \omega_1 \cdot \omega_1)$, there is $g(\alpha) < \omega$ such that $X^{(\omega_1 \cdot g(\alpha) \cdot \omega_1 \cdot \omega)}$ meets at most countably many $W(\gamma)$'s. Then this $g(\alpha)$ works. \hfill \Box

\textbf{Proposition 15.} Assume that $X$ is a \textit{paraLindelöf} subspace of $(\omega_1 \cdot \omega_1) \times (\omega_1 \cdot \omega + 1)$ such that for some $\alpha_0 < \omega_1 \cdot \omega_1$, $H_{\omega_1 \cdot \omega_1}(X) \cap [\alpha_0, \alpha]$ is \textit{Lindelöf} for every $\alpha < \omega_1 \cdot \omega_1$ with $\alpha_0 \leq \alpha$. Then $X$ is paracompact.

\textbf{Proof.} Note that, by Corollary 6 and 7, $X^{(\omega_1 \cdot \mu)}\omega_1$ and $X^{(\omega_1 \cdot \nu)}\omega_1$ are paracompact for each $\mu < \omega_1 \cdot \omega_1$ and $\nu < \omega_1 \cdot \omega_1$. Since $X^{(\omega_1 \cdot \alpha)}\omega_1$ is paracompact, we may assume $\alpha_0 = 0$, that is, $H_{\omega_1 \cdot \omega_1}(X) \cap [0, \alpha]$ is \textit{Lindelöf} for every $\alpha < \omega_1 \cdot \omega_1$. As in the proof of Proposition 14, define a cub set $C \subset \omega_1$ with $X_M(C) \cap X^{(\mu)} = \emptyset$, where $E = \{\omega_1 \cdot n : n \leq \omega\}$. Let $W = \{W(\gamma) : \gamma < \omega_1\}$ be a precise locally countable open refinement of the open cover $U = \{X^{(\omega_1 \cdot \mu)}\omega_1 : \gamma < \omega_1\}$. For each $\alpha \in H_{\omega_1 \cdot \omega_1}(X)$, take $g(\alpha) < \omega$ and $f(\alpha) < \alpha$ such that
$H(\alpha) = X_{(f(\alpha), \alpha)}^{(\omega_1, \omega_1 \cdot \omega_1)}$ meets at most countably many $W(\gamma)$’s and $(f(\alpha), \alpha) \cap M(C) = \emptyset$.

For each $\gamma \in \text{Succ}(C)$, since $H_{\omega_1, \omega_1}(X) \cap [M(p_C(\gamma)), M(\gamma)] = H_{\omega_1, \omega_1}(X) \cap (M(p_C(\gamma)), M(\gamma))$ is a copen subset of the Lindelöf space $H_{\omega_1, \omega_1}(X) \cap [0, M(\gamma)]$, there is a countable subset $Z(\gamma) \subseteq H_{\omega_1, \omega_1}(X) \cap (M(p_C(\gamma)), M(\gamma))$ such that

$$X_{(M(p_C(\gamma)), M(\gamma))}^{(\omega_1, \omega_1)} \subseteq \bigcup_{\alpha \in Z(\gamma)} H(\alpha) \subseteq X_{(M(p_C(\gamma)), M(\gamma))}^{(\omega_1, \omega_1)}.$$

Of course, $H = \bigcup_{\gamma \in \text{Succ}(C)} (\bigcup_{\alpha \in Z(\gamma)} H(\alpha))$ covers $X_{(\omega_1, \omega_1)}^{(\omega_1, \omega_1)}$. In a usual way, the following Claim shows that $X$ is paracompact.

**Claim.** $S = \{ \delta \in C : X_{(M(\gamma))} \cap \text{Cl} H \neq \emptyset \}$ is not stationary in $\omega_1$.

**Proof.** Assume that $S$ is stationary. For each $\delta \in S$, fix $h(\delta) \leq \omega_1 \cdot \omega_1$ and $\psi(\delta) > \delta$ such that $(M(\delta), h(\delta)) \subseteq \text{Cl} H \cap W(\psi(\delta))$. Moreover once $\bigcup_{\alpha \in Z(\gamma)} H(\alpha) \subseteq X_{(M(p_C(\gamma)), M(\gamma))}^{(\omega_1, \omega_1)}$ for each $\gamma \in \text{Succ}(C)$, we can find $\varphi(\delta) \in \text{Succ}(C) \cap \delta$ with $(\bigcup_{\alpha \in Z(\varphi(\delta))} H(\alpha)) \cap W(\psi(\delta)) \neq \emptyset$.

Applying the PDL, we can find a stationary set $S' \subseteq S$ and $\delta_0 \in \text{Succ}(C)$ such that $\varphi(\delta) = \delta_0$ for each $\delta \in S'$ and the members of $\{\psi(\delta) : \delta \in S'\}$ are all distinct. Then $\bigcup_{\alpha \in Z(\delta_0)} H(\alpha)$ meets uncountably many $W(\psi(\delta))$’s, $\delta \in S'$. Since $Z(\delta_0)$ is countable, we can find $\alpha_0 \in Z(\delta_0)$ such that $H(\alpha_0)$ meets uncountably many $W(\psi(\delta))$’s, a contradiction. □

**Corollary 16.** Assume that there exists a paralindelöf subspace $X$ of $(\omega_1 \cdot \omega_1) \times (\omega_1 \cdot \omega_1 + 1)$ which is not paracompact. Then $I = \{ \alpha \in \omega_1 \cdot \omega_1 \setminus H_{\omega_1, \omega_1}(X) : cf \alpha = \omega_1 \}$ is unbounded in $\omega_1 \cdot \omega_1$.

**Proof.** Assume that $I$ is bounded by some $\alpha_0 < \omega_1 \cdot \omega_1$. The following general fact shows that $H_{\omega_1, \omega_1}(X) \cap [\alpha_0, \alpha]$ is Lindelöf for every $\alpha < \omega_1 \cdot \omega_1$ with $\alpha_0 \leq \alpha$.

**Fact.** If $Z$ is a subspace of $\rho + 1$ for some ordinal $\rho$ such that $cf \beta \leq \omega$ for every $\beta \in (\rho + 1) \setminus Z$, then $Z$ is Lindelöf.

**Proof.** Assume that $Z$ is not Lindelöf and let

$$\mu = \min \{ \mu' \leq \rho : Z \cap [0, \mu'] \text{ is not Lindelöf} \}.$$

Then we see that $\mu \notin Z$, $\mu$ is limit, $Z \cap [0, \mu']$ is Lindelöf for every $\mu' < \mu$ and $Z \cap [0, \mu]$ is not Lindelöf. Since $\mu \notin Z$, we have $cf \mu = \omega$. Then $Z \cap [0, \mu] = \bigcup_{n \in \omega} (Z \cap [0, \mu(n)])$ can be represented as the countable union of Lindelöf subspaces, where $\{\mu(n) : n \in \omega\}$ is a strictly increasing cofinal sequence in $\mu$, so it is Lindelöf, a contradiction. □

Taking account of these informations, the authors have tried to construct a paralindelöf non-paracompact subspace of $(\omega_1 \cdot \omega_1) \times (\omega_1 \cdot \omega_1 + 1)$. But now we present the following by-product of these considerations. Here note that the locally finite union of copen paracompact subspaces are also paracompact.

**Example 17.** There exists a non-paracompact subspace $X$ of $(\omega_1 \cdot \omega_1) \times (\omega_1 \cdot \omega_1 + 1)$, which can be represented as the locally countable union of copen paracompact subspaces, such that $X_{(0, \mu]}$ and $X_{(0, \nu]}$ are paracompact for each $\mu < \omega_1 \cdot \omega_1$ and $\nu < \omega_1 \cdot \omega_1$. In fact, unfortunately, this $X$ is not paraLindelöf.

Our space is defined as follows:

$$X = \left[ \bigcup_{\gamma, \beta \in \text{Succ}} \{\omega_1 \cdot (\gamma - 1) + \beta\} \times \left(\{\omega_1 \cdot \omega_1\} \cup \bigcup_{n \in \omega} \text{Succ}((\omega_1 \cdot n, \omega_1 \cdot n + \beta))\right) \right]$$
\[
\bigcup_{\delta \in \text{Lim}} \{\omega_1 : \delta\} \times \left( \bigcup_{n \in \omega} \{\omega_1 : n + \delta + 1\} \right),
\]

where \(\text{Succ}(\omega_1 : n, \omega_1 : n + \beta)\) denotes the set of all successor ordinals in the open interval \((\omega_1 : n, \omega_1 : n + \beta)\). Observe that \(X_{\omega_1 : n} = \emptyset\) for each \(\gamma \in \text{Succ}\) and \(X^{\omega_1 : n} = \emptyset\) for each \(n \in \omega\).

**Claim 1.** \(X^{\omega_1 : n}\) is paracompact for each \(\nu < \omega_1 : \omega\).

**Proof.** Since \(X^{\omega_1 : n}\) is homeomorphic to \(X^{\omega_1 : n, \omega_1 : (n+1)}\) for each \(n \in \omega\), it suffices to show that \(X^{\omega_1 : n}\) is paracompact. Note that for each \(\eta < \omega_1\), \(H_\eta(X)\) has at most one limit ordinal, so \(X^{\eta}\) is paracompact. Since \(X^{\eta} = \emptyset\) for each \(\eta \in \text{Lim} \cup \{\omega_1\}\), \(X^{\omega_1 : n} = \bigoplus_{\eta \in \text{Succ}} X^{\omega_1 : n}\) is paracompact.

**Claim 2.** \(X_{\omega_1 : n}\) is paracompact for each \(\mu < \omega_1 : \omega_1\).

**Proof.** Note that for each \(\zeta < \omega_1 : \omega_1\), \(V_\zeta(X)\) has at most one limit ordinal \((\omega_1 : \omega_1\) if it has), so \(X_{\zeta}\) is paracompact. Assuming that \(X_{\omega_1 : n}\) is not paracompact for some \(\mu < \omega_1 : \omega_1\), let \(\mu < \omega_1 : \omega_1\) be the such minimal one. Then \(\mu\) is limit and \((\mu, \omega_1 : \omega) \notin X_{\omega_1 : n}\). If cf \(\mu = \omega_1\), then

\[
X_{\omega_1 : n} = \left( \bigoplus_{n \in \omega} X_{\omega_1 : n} \right) \cup \left( \bigoplus_{n \in \omega} X^{\omega_1 : n} \right)
\]

can be represented as the countable union of paracompact clopen subspaces. Therefore \(X_{\omega_1 : n}\) is paracompact, a contradiction. If cf \(\mu = \omega_1\), then \(\mu = \omega_1 : \gamma\) for some \(\gamma \in \text{Succ}\). Since \(X_{\omega_1 : (\omega_1 + \gamma - 1)}\) is paracompact, by the minimality of \(\mu\), \(X_{\omega_1 : n} = X_{\omega_1 : (\omega_1 + \gamma - 1)} \bigoplus X_{\omega_1 : (\omega_1 + \gamma - 1)}\) is paracompact, a contradiction.

**Claim 3.** Let \(V(\gamma) = X_{\omega_1 : (\omega_1 + \gamma - 1)}\) for each \(\gamma \in \text{Succ}\) and \(V(\delta, n) = X_{\omega_1 : n + \delta + 1}\) for each \(\delta \in \text{Lim} ~ n \in \omega\). Then \(V = \{V(\gamma) : \gamma \in \text{Succ}\} \cup \{V(\delta, n) : (\delta, n) \in \text{Succ} \times \omega\}\) is a locally countable open refinement of the open cover \(U = \{X_{\omega_1 : n} : \gamma < \omega_1\}\).

**Proof.** Since other properties are not so hard, we only show that \(V\) is locally countable. Let \(\langle \zeta, \eta \rangle \in X\).

First assume \(\zeta \in (\omega_1 : (\omega_1 + \gamma - 1), \omega_1 : \gamma)\) for some \(\gamma \in \text{Succ}\). Then there is \(\beta \in \text{Succ}\) with \(\zeta = (\omega_1 : \omega_1), \omega_1 : \gamma\). If \(\gamma' \in \text{Succ} \setminus \gamma\), then \(X_{\omega_1 : n} \cap V(\gamma') = \emptyset\). Moreover if \(\delta \in \text{Lim}\) with \(\beta < \delta\), then \(X_{\omega_1 : n} \cap V(\delta, n) = \emptyset\) for each \(n \in \omega\). Therefore \(X_{\omega_1 : n}\) is a neighborhood of \(\langle \zeta, \eta \rangle\) which witnesses the local countability of \(V\) at \(\langle \zeta, \eta \rangle\).

Next assume \(\zeta = (\omega_1 : \gamma, \omega_1 : \beta)\) for some \(\delta \in \text{Lim}\). Then by the construction of \(X\), it is not difficult to show that \(X_{\omega_1 : n}\) is a neighborhood of \(\langle \zeta, \eta \rangle\) which witnesses the local countability of \(V\) at \(\langle \zeta, \eta \rangle\).

Since \(V(\gamma)\)'s and \(V(\delta, n)\)'s are clopen in \(X\), Claim 2 and 3 say that \(X\) can be represented as the locally countable union of clopen paracompact subspaces.

**Claim 4.** \(X\) is not paralindelöf.

**Proof.** Let \(V(\gamma, \beta) = X_{\omega_1 : (\gamma + 1), \beta}\) for each \(\gamma, \beta \in \text{Succ}\), \(V(\delta, n) = X_{\omega_1 : n + \delta + 1}\) for each \(\delta \in \text{Lim}\) and \(n \in \omega\). Assume that there is a precise locally countable open refinement \(\mathcal{W} = \{W(\gamma, \beta) : (\gamma, \beta) \in \text{Succ} \times \text{Succ}\} \cup \{W(\delta, n) : (\delta, n) \in \text{Lim} \times \omega\}\) of the open cover \(\mathcal{V} = \{V(\gamma, \beta) : (\gamma, \beta) \in \text{Succ} \times \text{Succ}\} \cup \{V(\delta, n) : (\delta, n) \in \text{Lim} \times \omega\}\). Let \(\delta \in \text{Lim}\) and \(n \in \omega\). Since \(\{W(\gamma, \beta) : (\gamma, \beta) \in \text{Succ} \times \text{Succ}\}\) is locally countable, we can take \(f(\delta, n) < \delta\) such that

\[
I(\delta, n) = \{\langle \gamma, \beta \rangle \in \text{Succ} \times \text{Succ} : W(\gamma, \beta) \cap X_{\omega_1 : f(\delta, n)} \neq \emptyset\}
\]
is countable. Fix \( n \in \omega \) and moving \( \delta \in \text{Lim} \), by the PDL, we can find a stationary set \( S(n) \subseteq \text{Lim} \) and \( f(n) < \omega \) such that \( f(\delta, n) = f(n) \) for each \( \delta \in S(n) \). Let \( \gamma_0 = \sup \{ f(n) : n \in \omega \} \) and fix \( \gamma \in \text{Succ} \) with \( \gamma_0 < \gamma \). Note \( \gamma_0 \leq \gamma - 1 \). For each \( \beta \in \text{Succ} \), since the unique member of \( \gamma \) which contains the point \( \langle \omega_1 : (\gamma - 1) + \beta, \omega_1 \cdot \omega \rangle \in V(\gamma, \delta) \), \( W(\gamma, \delta) \) also contains the point \( \langle \omega_1 : (\gamma - 1) + \beta, \omega_1 \cdot \omega \rangle \). So there is \( n_0 \in \omega \) and an uncountable subset \( K \subseteq \text{Succ} \) such that \( X_{\langle \omega_1 : (\gamma - 1) + \beta, \omega_1 \cdot \omega \rangle} \subseteq W(\gamma, \beta) \) for each \( \beta \in K \). Fix \( \delta \in S(n_0) \) with \( \gamma < \delta \). Since \( I(\delta, n_0) \) is countable and \( K \) is uncountable, we can find \( \beta \in K \) with \( \delta + 1 < \beta \) and \( \langle \gamma, \beta \rangle \notin I(\delta, n_0) \). By \( \delta + 1 < \beta \), the point \( \langle \omega_1 : (\gamma - 1) + \beta, \omega_1 \cdot n_0 + \delta + 1 \rangle \) belongs to \( X \). Moreover, 

\[
\langle \omega_1 : (\gamma - 1) + \beta, \omega_1 \cdot n_0 + \delta + 1 \rangle \in X_{\langle \omega_1 : (\gamma - 1) + \beta, \omega_1 \cdot n_0 + \delta + 1 \rangle} \cap X_{\langle \omega_1 : \gamma, \omega_1 \cdot \delta \rangle}.
\]

Therefore \( \langle \gamma, \beta \rangle \in I(\delta, n_0) \), a contradiction. \( \square \)

In this connection, note that, as is well known, the space \( \omega_1 \) is the locally countable union of closed paracompact subspaces \( \{ \alpha \}'s, \alpha < \omega_1 \), but \( \omega_1 \) is not paralindelöf. But:

**Proposition 18.** Let \( X \subseteq \rho + 1 \) for some ordinal \( \rho \). Assume that \( X \) is the locally countable union of clopen paracompact subspaces \( \{ \lambda \}'s, \lambda \in \Lambda \). Then \( X \) is paracompact.

**Proof.** Assume that \( X \) is not paracompact. Let 

\[ \mu = \min \{ \mu' \leq \rho : X \cap [0, \mu'] \text{ is not paracompact} \}. \]

Then by the minimality of \( \mu \), \( \mu \) is limit ordinal, \( \mu \notin X \), \( \text{cf} \mu > \omega_1 \) and \( X \cap [0, \mu] \) is stationary in \( \mu \). By identifying \( X = X \cap [0, \mu] \), we may assume that \( X \) is a stationary subset of \( \mu \) and \( X \cap [0, \mu'] \) is paracompact for each \( \mu' < \mu \).

**Claim.** \( X(\lambda) \) is bounded in \( \mu \) for each \( \lambda \in \Lambda \).

**Proof.** Since \( X(\lambda) \) is paracompact, it is not stationary in \( \mu \). So there is a cub set \( C \subseteq \text{Lim}(\text{cf} \mu \rangle \) such that \( X(\lambda) \cap M(C) = \emptyset \), where \( M \) is a normal function for \( \mu \). For each \( \gamma \in C \cap M^{-1}(X) \), fix \( f(\gamma) < \gamma \) such that \( X(\lambda) \cap (M(f(\gamma)), M([\gamma]]) = \emptyset \). Then by the PDL, we find a stationary set \( S \subseteq C \cap M^{-1}(X) \) and \( \gamma_0 < \text{cf} \mu \) such that \( f(\gamma) = \gamma_0 \) for each \( \gamma \in S \). Then \( X(\lambda) \subseteq [0, \gamma_0] \), and so \( X(\lambda) \) is bounded.

Since \( X(\lambda) \)'s cover \( X \) and are open, for each \( \gamma \in M^{-1}(X) \cap \text{Lim}(\text{cf} \mu \rangle \), fix \( f(\gamma) < \gamma \), \( \lambda(\gamma) \in \Lambda \) and \( g(\gamma) < \text{cf} \mu \) such that \( X \cap (M(f(\gamma)), M(\gamma]) \subseteq X(\lambda(\gamma)) \subseteq [0, M(g(\gamma)) \rangle \). By the PDL, we find a stationary set \( S \subseteq M^{-1}(X) \cap \text{Lim}(\text{cf} \mu \rangle \) and \( \gamma_0 < \text{cf} \mu \) such that \( f(\gamma) = \gamma_0 \) for each \( \gamma \in S \). Set \( g(\gamma) = 0 \) for each \( \gamma \in \text{cf} \mu \setminus (M^{-1}(X) \cap \text{Lim}(\text{cf} \mu \rangle \) and \( C = \{ \gamma < \text{cf} \mu : \forall \gamma' < \gamma (g(\gamma') < \gamma) \} \). Then members of \( \{ \lambda(\gamma) : \gamma \in S \cap C \} \) are distinct. Take \( \alpha \in X \) with \( M(\gamma_0) < \alpha \). Then \( \alpha \in X(\lambda(\gamma)) \) for each \( \gamma \in S \cap C \) with \( \alpha < M(\gamma) \). This contradicts the local countability of \( \{ X(\lambda) : \lambda \in \Lambda \} \). \( \square \)

Burke [Bu] proved that submetacompact spaces in which every open cover has a \( \sigma \)-locally countable closed refinement are subparacompact. Using this we can see:

**Proposition 19.** Paralindelöf subspaces of products of two ordinals are subparacompact.

**Proof.** Let \( X \) be a paralindelöf subspace of products of two ordinals. Then by [KTY, Theorem 2.3], it is metacompact. By the regularity of \( X \), every open cover has a \( \sigma \)-locally countable closed refinement. So by the result of [Bu], it is subparacompact. \( \square \)
REFERENCES


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