ON THE BRANCH OF BH-ALGEBRAS

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Abstract. In this paper, we give a normal BH-algebra, and we consider the branch in BH-algebra and investigate some related properties.

1. Introduction

Y. Imai and K. Iséki ([4]) and K. Iséki ([5]) introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In ([3]), Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Y. B. Jun, E. H. Roh and H. S. Kim ([6]) discussed the BH-algebras, which is a generalization of BCH-algebras. Moreover, they introduced the notions of ideal, maximal ideal and translation ideal, and investigated some properties.

In this paper, we give a normal BH-algebra, and we consider the branch in BH-algebra and investigate some related properties. This paper is the some generalization of Chaudhry’s results([1]).

2. Preliminaries

A BH-algebra is a non-empty set $X$ with a constant 0 and a binary operation “$*$” satisfying the following axioms:

1. $x * x = 0$,
2. $x * 0 = x$,
3. $x * y = 0$ and $y * x = 0$ imply $x = y$

for all $x, y$ in $X$.

Example 2.1. (a) Let $X = \{0, 1, 2, 3\}$ be a set with the following Cayley table:

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Then $(X; *, 0)$ is a BH-algebra, but not a BCH-algebra, since $(2 * 3) * 2 = 1 \neq 2 = (2 * 2) * 3$.

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(b) Let \( \mathbb{R} \) be the set of all real numbers and define

\[
x \ast y = \begin{cases} 
0 & \text{if } x = 0, \\
\frac{(x-y)^2}{x} & \text{otherwise},
\end{cases}
\]

for all \( x, y \in \mathbb{R} \), where "−" is the usual subtraction of real numbers. Then \((\mathbb{R}, \ast, 0)\) is a BH-algebra, but not a BCH-algebra.

The relations between BH-algebras and BCH-algebras (also, BCK/BCI-algebras) are as follows:

**Theorem 2.2 ([6]).** Every BCH-algebra is a BH-algebra. Every BH-algebra satisfying the condition \((x \ast y) \ast z = (x \ast z) \ast y\) for all \( x, y, z \in X \), is a BCH-algebra.

**Theorem 2.3 ([6]).** Every BH-algebra satisfying the condition

\((c1) \ ((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0, \ \forall x, y, z \in X;\)

is a BCI-algebra.

**Theorem 2.4 ([6]).** Every BH-algebra satisfying the conditions (c1) and

\[(c2) \ (x \ast y) \ast x = 0, \ \forall x, y \in X,\]

is a BCK-algebra.

A nonempty subset \( S \) of a BH-algebra \( X \) is called a subalgebra if \( x, y \in S \) implies \( x \ast y \in S \). A nonempty subset \( A \) of a BH-algebra \( X \) is called an ideal if \( 0 \in A \) and if \( x \ast y \in A \) imply that \( x \in A \).

### 3. Main Results

Now, we see the following examples.

**Example 3.1.** Let \( X = \{0, 1, 2\} \) be a set with the following Cayley table:

\[
\begin{array}{ccc}
\ast & 0 & 1 & 2 \\
0 & 0 & 2 & 0 \\
1 & 1 & 0 & 2 \\
2 & 2 & 2 & 0 \\
\end{array}
\]

Then \((X; \ast, 0)\) is a BH-algebra, but \( X \) is not satisfied the identity \( 0 \ast (x \ast y) = (0 \ast x) \ast (0 \ast y) \) since \( 0 \ast (1 \ast 2) = 0 \neq 2 = (0 \ast 1) \ast (0 \ast 2) \).

**Example 3.2.** Let \( X = \{0, 1, 2\} \) be a set with the following Cayley table:

\[
\begin{array}{ccc}
\ast & 0 & 1 & 2 \\
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 2 \\
2 & 2 & 2 & 0 \\
\end{array}
\]

Then \((X; \ast, 0)\) is a BH-algebra and \( X \) satisfies the identity \( 0 \ast (x \ast y) = (0 \ast x) \ast (0 \ast y) \).

By Examples 3.1 and 3.2, we will define the following definition.

**Definition 3.1.** A BH-algebra \( X \) is called a BH\(_1\)-algebra if it satisfying the following conditions:

\[(4) \ 0 \ast (x \ast y) = (0 \ast x) \ast (0 \ast y)\]
Definition 3.2. Let $X$ be a BH-algebra. Then the set

$$M(X) = \{x \in X | 0 \ast (0 \ast x) = x\}$$

is called a medial part of $X$ and an element of $M(X)$ is called a medial element of $X$.

Obviously $0 \in M(X)$ and so $M(X)$ is nonempty. In general, $M(X)$ is not a subalgebra of a BH-algebra. But we have the following Theorem.

Theorem 3.1. If $X$ is a BH$_1$-algebra, then $M(X)$ is a subalgebra of $X$.

**Proof.** Clearly $0 \in M(X)$. Let $x, y \in M(X)$. Then we have $0 \ast (0 \ast (x \ast y)) = (0 \ast (0 \ast x)) \ast (0 \ast (0 \ast y)) = x \ast y$. Thus $x \ast y \in M(X)$ and so $M(X)$ is a subalgebra of $X$. □

Theorem 3.2. Let $X$ be a BH$_1$-algebra and let

$$A = \{x \in X | 0 \ast x = 0\}.$$

Then $A$ is an ideal and subalgebra of $X$.

**Proof.** Clearly $0 \in A$. Let $x, y \in X$ be such that $x \ast y \in A$ and $y \in A$. Then $0 \ast (x \ast y) = 0$ and $0 \ast y = 0$. Thus we have $0 \ast x = (0 \ast x) \ast (0 \ast y) = 0 \ast (x \ast y) = 0$, and hence $x \in A$. Therefore $A$ is an ideal of $X$. Obviously, $A$ is a subalgebra of $X$. □

Next, we see the following examples.

**Example 3.3.** Let $X = \{0, 1, 2, 3\}$ be a set with the following Cayley table:

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Then $(X; \ast, 0)$ is a BH-algebra in which satisfies the identity $(x \ast y) \ast x = 0 \ast y$, but not satisfied the identity $(x \ast (x \ast y)) \ast y = 0$ because $(2 \ast (2 \ast 1)) \ast 1 = 3 \neq 0$.

**Example 3.4.** Let $X = \{0, 1, 2\}$ be a set with the following Cayley table:

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Then $(X; \ast, 0)$ is a BH-algebra in which satisfies the identity $(x \ast (x \ast y)) \ast y = 0$, but not satisfied the identity $(x \ast y) \ast x = 0 \ast y$ because $(1 \ast 2) \ast 1 \neq 0 \ast 2$.

**Example 3.5.** Let $X = \{0, 1, 2, 3\}$ be a set with the following Cayley table:

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Then $(X; \ast, 0)$ is a BH-algebra in which satisfies the identities $(x \ast y) \ast x = 0 \ast y$ and $(x \ast (x \ast y)) \ast y = 0$.

By Examples 3.3, 3.4 and 3.5, next conditions (5) and (6) are independent. We give the following definition.
Definition 3.3. A BH-algebra $X$ is said to be normal if it satisfying the following condition: (4) and

(5) $(x \ast y) \ast x = 0 \ast y$,

(6) $(x \ast (x \ast y)) \ast y = 0$.

Example 3.6. Let $X = \{0,1,2,3\}$ be a set with the following Cayley table:

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Then $(X; \ast, 0)$ is a BH-algebra in which satisfies the identities (4), (5) and (6).

Theorem 3.3. Let $X$ be a normal BH-algebra. Then for each $x \in X$, there is a unique $x_m \in M(X)$ such that $x_m \ast x = 0$.

Proof. Let $x \in X$, then $(0 \ast (0 \ast x)) \ast x = 0$ by (6). We take $x_m = 0 \ast (0 \ast x)$, then $x_m \ast x = 0$. To prove that $x_m$ is in $M(X)$. By (5), we have $0 \ast (0 \ast (0 \ast x)) = ((0 \ast (0 \ast x)) \ast x) \ast (0 \ast (0 \ast x)) = 0 \ast x$. Thus $0 \ast (0 \ast x_m) = 0 \ast (0 \ast (0 \ast x)) = 0 \ast (0 \ast x) = x_m$, and so $x_m \in M(X)$. To prove uniqueness we assume that $y_m \in M(X)$ be such that $y_m \ast x = 0$. Then by (5), we get

$0 \ast y_m = (y_m \ast x) \ast y = 0 \ast x$. Thus $0 \ast (0 \ast x) = 0 \ast (0 \ast y_m) = y_m$, and hence $x_m = y_m$. □

Corollary 3.4. Let $X$ be a normal BH-algebra and let $x, y \in X$ be such that $x \ast y = 0$. Then $x_m = y_m$ where $x_m, y_m \in M(X)$.

Remark. Let $X$ be a normal BH-algebra. If $x_m \in M(X)$ and $y \ast x_m = 0$, then $y = x_m$. Thus each medial point of a normal BH-algebra is also minimal point.

Theorem 3.5. Let $X$ be a normal BH-algebra. Then for any $x, y \in X$, we have

$(x \ast y)_m = x_m \ast y_m$.

where $(x \ast y)_m, x_m, y_m \in M(X)$.

Proof. By Theorem 3.1, $M(X)$ is a subalgebra of $X$, we get $x_m \ast y_m \in M(X)$. Then by (4) and (6) we have $(x_m \ast y_m) \ast (x \ast y) = ((0 \ast (0 \ast x)) \ast (0 \ast (0 \ast y))) \ast (x \ast y) = 0 \ast (0 \ast (x \ast y)) \ast (x \ast y) = 0$. By Theorem 3.3, we know that $(x \ast y)_m = x_m \ast y_m$. □

Definition 3.4. Let $X$ be a normal BH-algebra and let $x_m \in M(X)$. The set

$\{x \in X | x_m \ast x = 0\}$

is called the branch of $X$ determined by $x_m$ and is denoted by $V(x_m)$.

Theorem 3.6. Let $X$ be a normal BH-algebra. Then

(i) $X = \bigcup_{x_m \in M(X)} V(x_m)$

(ii) $V(x_m) \cap V(y_m) = \emptyset$ if $x_m \neq y_m$ and $x_m, y_m \in M(X)$.

Proof. (i). Clearly $V(x_m) \subseteq X$ for all $x_m \in M(X)$. Thus $\bigcup_{x_m \in M(X)} V(x_m) \subseteq X$. Let $y \in X$, then there is $y_m \in M(X)$ such that $y_m \ast y = 0$. Thus $y \in V(y_m) \subseteq \bigcup_{x_m \in M(X)} V(x_m)$. Hence $X \subseteq \bigcup_{x_m \in M(X)} V(x_m)$. Therefore $X = \bigcup_{x_m \in M(X)} V(x_m)$.

(ii). Let $z \in V(x_m) \cap V(y_m)$ where $x_m \neq y_m$ in $M(X)$. Then $x_m \ast z = 0$ and $y_m \ast z = 0$. Thus $z$ has two medial points, a contradiction to Theorem 3.3. Hence $V(x_m) \cap V(y_m) = \emptyset$ if $x_m \neq y_m$. □
Theorem 3.7. Let $X$ be a normal BH-algebra. Then

(i) If $x*y \in A$ and $y*x \in A$, then $x, y \in V(x_m)$ for some $x_m \in M(X)$,

(ii) If $x \in V(x_m), y \in V(y_m)$ and $x_m \neq y_m$, then $x*y, y*x \in X-A$.

Proof. (i). Let $x*y \in A$ and $y*x \in A$. If $x \in V(x_m)$ and $y \in V(y_m)$. Then by Theorem 3.5 gives $(x*y)_m = x_m*y_m$ and $(y*x)_m = y_m*x_m$. Since $x*y, y*x \in A = V(0)$, we have $(x*y)_m = 0 = (y*x)_m$. Now uniqueness of medial point gives $x_m*y_m = 0 = y_m*x_m$.

Thus $x_m = y_m$. Hence $x, y \in V(x_m)$ for some $x_m \in M(X)$.

(ii). Let $x \in V(x_m), y \in V(y_m)$ and $x_m \neq y_m$. If $x*y \in A = V(0)$, then by Theorem 3.5, we get $(x*y)_m = x_m*y_m$. Thus $x*y \in V(x_m*y_m)$. Hence $x_m*y_m = 0$. Thus $(x_m*y_m)_m = 0 = x_m$, which gives $0*y_m = 0 = x_m$ and hence $0*(y_m) = 0*(x_m)$. Thus $x_m = y_m$, a contradiction. Hence $x*y \in X-A$. Similarly we can be shown that $y*x \in X-A$. □

Remark. We know that every BCH-algebra satisfies conditions (1) - (6). Thus this note is the generalization of Chaudhry’s results.

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