ON LOEB AND WEAKLY LOEB HAUSDORFF SPACES

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It is shown that the statements: Compact $T_3$ spaces are weakly Loeb and the product of weakly Loeb $T_2$ spaces is weakly Loeb are not provable in $ZF^0$, Zermelo-Fraenkel set theory without the axiom of regularity.

1. INTRODUCTION AND DEFINITIONS

Let $(X,T)$ be a topological space. Then $X$ is a Loeb (weakly Loeb) space if there is a choice (multiple choice) function on the family of its non-empty closed subsets. Examples of Loeb spaces are:

1. Compact linearly ordered spaces,
2. spaces with a finite number of open sets,
3. spaces $(X,T)$ with $X$ a well ordered set.

Remark. For any set $X$, the Alexandroff one point compactification, $X(a)$, of the discrete space $X$ is an example of a weakly Loeb space. Indeed, for any non-empty closed set $A$ of $X(a)$ either $a \in A$ and we may choose $a$ from $A$, or $A$ is a finite set and we may choose $A$ itself.

P. Howard and J. Rubin ([5], p. 345) ask whether,

- **Form 115**: The product of weakly Loeb $T_2$ spaces is weakly Loeb

and,

- **Form 116**: Compact $T_2$ spaces are weakly Loeb

are provable in $ZF^0$. In view of the remark above, it follows that showing Form 116 is not provable in $ZF^0$ is not easy. Without AC there are not many compact $T_2$ topologies one can define on an infinite set $X$. The interested reader is invited to define such topologies other than $X(a)$.

The purpose of this paper is to show that neither Form 115 nor Form 116 are provable in $ZF^0$. Before we do this let us list some known results in this area and state the choice principles we are going to use in the sequel.

**Theorem 1.1.** [1] In the basic Fraenkel permutation model (model $N1$ in [5]) the following statements are valid.

(i) Form 115.
(ii) Form 116.
(iii) A compact $T_2$ space $(X,T)$ is weakly Loeb if and only if $X$ is expressible as a well ordered union of compact sets.

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1. The axiom of choice AC, Form 1 in [5]: For every family $A = \{A_i : i \in k\}$ of disjoint non-empty sets there exists a set $C = \{c_i : i \in k\}$ such that $c_i \in A_i$ for all $i \in k$.

2. The axiom of multiple choice MC, Form 67 in [5]: For every family $A = \{A_i : i \in k\}$ of disjoint non-empty sets there exists a set $\mathcal{F} = \{F_i : i \in k\}$ of finite non-empty sets such that $F_i \subseteq A_i$ for all $i \in k$.

3. AC_{fin}, Form 62 in [5]: AC restricted to families of non-empty finite sets.

4. LN, Form 118 in [5]: Every linearly orderable space is normal.

AC and MC are related to the notions of Loeb and weakly Loeb spaces respectively via the following

**Theorem 1.2.** (i) AC if and only if every topological space is Loeb.
(ii) MC if and only if every topological space is weakly Loeb.

**Proof.** (i). It suffices to show $(\rightarrow)$ as the other direction is evident. Let $A = \{A_i : i \in k\}$ be a disjoint family of non-empty sets and let $T$ be the discrete topology on $X = \bigcup A$. By our hypothesis $(X, T)$ is a Loeb space, and so the family $A$ of closed subsets of $X$ has a choice set.

(ii). This can be proved as in (i).

**Remark.** U. Felgner and T. Jech in their joint paper [3] proved that in ZF (Zermelo-Fraenkel set theory minus AC), AC and MC are equivalent. Therefore, in conjunction with the latter theorem we deduce that in ZF the notions of Loeb and weakly Loeb spaces coincide. However, this is not the case in ZF$^0$. In the Second Fraenkel Model, see [4] and Model $\mathcal{A}^2$ of [5], MC is true ([6], p. 135), and so every topological space in $\mathcal{A}^2$ is a weakly Loeb space. But, AC fails in $\mathcal{A}^2$. The family $A = \{\{a_n, b_n\} : n \in \omega\}$, where $A = \bigcup A$ is the set of atoms, has no choice set in $\mathcal{A}^2$ (see [6] or [5]). Thus the space $A(a)$ is not Loeb since the family $A$ of closed sets of $A(a)$ has no choice set in $\mathcal{A}^2$. Hence, in ZF$^0$ weakly Loeb spaces need not be Loeb.

2. **Main results**

In what follows CL($T_2$) will denote the statement compact $T_2$ spaces are Loeb.

**Proposition 1.** [1]. (i) If $(X, T)$ is a Loeb (weakly Loeb) space and $G$ a closed subset of $X$, then $G$ is a Loeb (weakly Loeb) space.

(ii) [1]. If $(X, T)$ is a Loeb (weakly Loeb) space, $(Y, Q)$ a topological space and $f : X \to Y$ a continuous onto function, then $Y$ is a Loeb (weakly Loeb) space.

(iii) [8]. A product of a well orderable family of compact spaces is compact if it is Loeb.

(iv) If $(L, \leq)$ is a conditionally complete linear order (i.e. each non-empty subset with an upper bound has a least upper bound), then $L$ with the order topology is a Loeb space.

**Proof.** (iv). Fix $\ell \in L$. Let $G \neq \emptyset$ be a closed subset of $L$. If $\ell \notin G$ choose $\ell$.

If $\ell \notin G$ and $G \cap [\ell, \infty) \neq \emptyset$ choose the inf of $G \cap [\ell, \infty)$.

If $\ell \notin G$ and $G \cap [\ell, \infty) = \emptyset$ choose the sup of $G \cap (-\infty, \ell]$.

We would like to point out here that Proposition 1(i) is not provable in ZF if we require $G$ to be an open subset of $X$. Indeed we have:

**Theorem 2.1.** (i) MC if and only if open subspaces of weakly Loeb spaces are weakly Loeb.

(ii) AC if and only if open subspaces of weakly Loeb spaces are weakly Loeb + AC_{fin}.

**Proof.** (i) $(\rightarrow)$. This is straightforward.
\(\leftarrow\). Fix \(\mathcal{A} = \{A_i : i \in k\}\) a disjoint family of non-empty sets. As we have remarked in the introduction \(X(a), X = \bigcup \mathcal{A}\), is a weakly Loeb \(T_2\) space. By our hypothesis the open subspace \(\bigcup \mathcal{A}\) is weakly Loeb and consequently the family \(\mathcal{A}\) of closed sets in \(\bigcup \mathcal{A}\) admits a multiple choice function \(f\). This completes the proof \(\leftarrow\) and of (i).

(ii). This follows from (i) and the fact that \(AC = MC + AC_{fin}\).

Next we give some easy topological characterizations of \(AC\) and \(MC\).

**Theorem 2.2.** (i) The following are equivalent:
(a) \(AC\).
(b) Every compact \(T_2\) space has a well ordered dense set.
(c) Every compact \(T_2\) space has a well ordered base.
(ii) The following are equivalent:
(a) \(MC\).
(b) Every compact \(T_2\) space has a dense set \(\mathcal{D}\) which is the union of a well ordered set of finite sets.
(c) Every compact \(T_2\) space has a base \(\mathcal{B}\) which is the union of a well ordered set of finite sets.
(d) Every point in a compact \(T_2\) space has a well ordered neighborhood base.

**Proof.** (i), (ii). The conclusion follows from the fact that for any set \(X, X(a)\) is a compact \(T_2\) space and the well known results:
(A) \([11]\) \(AC\) if and only if every set can be well ordered, and
(B) **Levy’s Lemma** \([7]\), \(MC\) if and only if every set can be written as a well ordered union of finite sets.

**Theorem 2.3.** (i) \(CL(T_2)\) implies \(AC_{fin}\).
(ii) \(AC\) if and only if \(MC + CL(T_2)\).
(iii) \(CL(T_2)\) if and only if every family of non-empty compact \(T_2\) spaces has a choice set.

(See \([9]\) and Form 343 in \([5]\)).

**Proof.** (i). Fix a family \(\mathcal{A} = \{A_i : i \in k\}\) of disjoint non-empty finite sets. Then \(\mathcal{A}\) is a family of closed sets in the compact \(T_2\) space \(X(a), X = \bigcup \mathcal{A}\). Let \(f\) be a choice function for the family of all closed sets of \(X\). Then the restriction of \(f\) to \(\mathcal{A}\) is a choice function of \(\mathcal{A}\).

(ii). This follows from (i) and the fact that \(AC = MC + AC_{fin}\).

(iii) \(\rightarrow\). Fix \(\mathcal{A} = \{(X_i, T_i) : i \in k\}\) be a family of non-empty compact \(T_2\) spaces. Without loss of generality we may assume that \(\mathcal{A}\) is pairwise disjoint. Let \((X = \bigcup_{i \in k} X_i, T)\) be the disjoint topological union of the spaces \(X_i\). That is, \(O \in T\) if and only if \(O \cap X_i \in T_i\) for each \(i \in k\). Let \(Y = X \cup \{\infty\}, \infty \notin X\), be the one-point compactification of \(X\). Then \(Y\) is a compact \(T_2\) space and we may finish the proof as in (i).

\(\leftarrow\). Assume 343 and let \((X, T)\) be a compact \(T_2\) space. As each closed subset of \(X\) is a compact \(T_2\) space (with the subspace topology), 343 implies that the family \(\{F : F\) is closed in \(X\}\) \(\{\emptyset\}\) has a choice set. This completes the proof of (iii) and of the theorem.

In \([2]\) it has been established that:

**Van Douwen’s Lemma.** \(LN\) if and only if every family \(\mathcal{A} = \{A_i : i \in k\}\) of non-empty conditionally complete linear orders has a choice set.

**Theorem 2.4.** Form 115 is not provable in ZF.

**Proof.** We show that Form 115 implies \(LN\). As \(LN\) fails in the Pincus model II (see, \([10]\) and Model \(M_{29}\) of \([5]\)) it will follow that Form 115 fails in \(M_{29}\) and consequently Form
115 is not provable in ZF. To this end, it suffices, in view of van Douwen’s lemma, to show that 115 implies every family $\mathcal{A} = \{A_i : i \in k\}$ of non-empty conditionally complete linear orders has a choice set. Fix such a family $\mathcal{A} = \{A_i : i \in k\}$ and let $X = \prod_{i \in k} X_i$, where $X_i = A_i \cup \{\ast\}$ is the disjoint topological union of $A_i$ taken with the order topology and the discrete space $\{\ast\}$. By proposition 1(iv), it follows that each $X_i$ is a Loeb (hence weakly Loeb) space. Thus, by Form 115 $X$ is weakly Loeb and consequently the family $\{\pi^{-1}(A_i) : i \in k\}$ of non-empty closed sets in $X$ has a multiple choice set $\mathcal{F} = \{F_i : i \in k\}$. It can be readily verified that

$$c = \{c_i = \max(\pi_i(F_i)) : i \in k\}$$

is a choice set of the family $\mathcal{A}$ finishing the proof of the theorem. 

\[\square\]

**Theorem 2.5.** Form 116 is not provable in ZF$^0$.

**Proof.** We first give the description of a permutation model $\mathcal{N}$. The set of atoms $A = \{A_n : n \in \omega^+ = \omega \setminus \{\ast\}\}$, where $A_n = \{a_{nx} : x \in B(0,1/n)\}$ and $B(0,1/n)$ is the set of points on the circle of radius $1/n$ centered at 0. The group of permutations $\mathcal{G}$ is the group of all permutations on $A$ which rotate the $A_n$’s by an angle $\theta_n \in \mathbb{R}$ and supports are finite.

**Claim 1.** The family $\{A_n : n \in \omega^+\}$ does not have a multiple choice function in $\mathcal{N}$.

**Proof of claim 1.** Assume the contrary. Let $f$ be a multiple choice function for the family $\{A_n : n \in \omega^+\}$ and let $E$ be a support of $f$. Since $E$ is a finite set there are only finitely many $n \in \omega^+$ such that $A_n \cap E \neq \emptyset$. Fix an $n$ such that $E \cap A_n \neq \emptyset$ for all $m > n$. Let $m > n$ and let $\pi$ be a permutation on $A$ such that $\pi$ is the identity map on each $A_k, k \neq m$ and $\pi|_{A_m}$ is a rotation of $A_m$ by an angle $\theta \neq 0$ such that $\pi(f(A_m)) \neq f(A_m)$. Clearly $\pi$ fixes $E$ pointwise, therefore $\pi(f) = f$. Thus, $\pi(f(A_m)) = f(A_m)$, a contradiction finishing the proof of claim 1. 

Let $d : A \times A \to \mathbb{R}$ be the function given by:

$$d(a_{nx}, a_{ny}) = \begin{cases} 1 & \text{if } n \neq m \\ \rho(x,y) & \text{if } n = m \end{cases},$$

where $\rho$ is the Euclidean metric.

**Claim 2.** $d$ is a metric on $A$ and for every $n \in \omega^+$, $(A_n, d)$ is compact.

**Proof of claim 2.** It can be readily verified that $d$ is a metric on $A$.

To see that $A_n$ is compact fix $n \in \omega^+$ and let $U$ be an open cover of $A_n$ in $\mathcal{N}$. As each $U \in U$ is expressible as a union of open discs we may assume without loss of generality that each member of $U$ is an open disc. Let $f$ be the bijection $a_{nx} \mapsto x$ used to define $A_n$. $f$ is in $\mathcal{N}$ since every singleton of $A_n$ supports $f$. Then $f(U)$ is an open cover of $B(0,1/n)$ and since $B(0,1/n)$ is compact, $f(U)$ has a finite subcover $V$. Clearly $f^{-1}(V)$ is an open cover of $A_n$ and $f^{-1}(V) \in \mathcal{N}$ (every singleton of $A_n$ is a support for $f^{-1}(V)$). 

Let $A(\ast)$ be the Alexandroff one point compactification of $A$. Clearly $A(\ast)$ is a compact $T_2$ space. We claim that $A(\ast)$ is not weakly Loeb. Assume the contrary and let $f$ be a
multiple choice function on the set of all closed subsets of $A(*)$. Since each $A_n$ is clearly a closed set, it follows that the restriction of $f$ to the family $\{A_n : n \in \omega^+\}$ is a multiple choice function. This contradicts claim 1 and completes the proof of the theorem.

3. Summary

The following diagram summarizes some of the results of the paper.

\[
\begin{array}{ccc}
\text{AC} & \rightarrow & \text{CL}(T_2) \equiv 343 \\
\downarrow & & \downarrow \\
\text{MC} & \rightarrow & \text{AC}_{fin} \\
\downarrow & & \downarrow \\
115 & & 116 \\
\downarrow & & \downarrow \\
\text{LN} & & \\
\end{array}
\]

References


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