ON FUZZY HYPERK-SUBALGEBRAS OF HYPERK-ALGEBRAS

YOUNG BAE JUN

Received November 19, 1999

ABSTRACT. In this paper, we introduce the concept of fuzzy hyper$K$-subalgebras and investigate some related properties. We state a characterization for a fuzzy hyper$K$-subalgebra in terms of it's level hyper$K$-subalgebras. We also consider the notion of hypernormal fuzzy hyper$K$-subalgebras and study some related results.

1. Introduction

The hyper algebraic structure theory was introduced in 1934 [15] by F. Marty at the 8th congress of Scandinavian Mathematiciens. Since then many researchers have worked on this area. Recently Y. B. Jun et al. [14] applied the hyperstructures to BCK-algebras and introduced the concept of a hyperBCK-algebra which is a generalization of a BCK-algebra. R. A. Borzoei et al.[1] defined the notion of a hyper$K$-algebra. For background and notations we refer to R. A. Borzoei et al. [1]. In this paper, we introduce the concept of fuzzy hyper$K$-subalgebras and investigate some related properties. We state a characterization for a fuzzy hyper$K$-subalgebra in term of it's level hyper$K$-subalgebras. We also consider the notion of hypernormal fuzzy hyper$K$-subalgebra and study some related results.

2. Preliminaries

In [1], the present author together with Professors R. A. Borzoei, A. Hasankhani and M. M. Zahedi established the notion of hyper$I$/hyper$K$-algebras as follows:

By a *hyperI-algebra* we mean a non-empty set $\mathcal{H}$ endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:

(HI1) $(x \circ z) \circ (y \circ z) < x \circ y$,

(HI2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(HI3) $x < x$,

(HI4) $x < y$ and $y < x$ imply $x = y$,

for all $x, y, z \in \mathcal{H}$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq \mathcal{H}$, $A < B$ is defined by $\exists a \in A$ and $\exists b \in B$ such that $a < b$. If a hyper$I$-algebra $(\mathcal{H}, \circ, 0)$ satisfies

(HI5) $0 < x$ for all $x \in \mathcal{H}$,

then $(\mathcal{H}, \circ, 0)$ is called a *hyperK-algebra*. Let $(\mathcal{H}, \circ, 0)$ be a hyper$K$-algebra and let S be a subset of $\mathcal{H}$ containing 0. If S is a hyper$K$-algebra with respect to the hyperoperation "$\circ$" on $\mathcal{H}$, we say that S is a *hyperK-subalgebra* of $\mathcal{H}$.

Let $(\mathcal{H}, \circ, 0)$ be a hyper$K$-algebra. Then for all $x, y, z \in \mathcal{H}$ and for all non-empty subsetes A and B of $\mathcal{H}$ the following hold (see [1, Proposition 3.6]):

(i) $x \circ y < x$,

1991 *Mathematics Subject Classification.* 06F35, 03G25, 20N20.

Key words and phrases. (Fuzzy) hyper$K$-subalgebra, level hyper$K$-subalgebra, weak homo-morphism, hypernormal fuzzy hyper$K$-subalgebra.

This paper is dedicated to the memory of Prof. Dr. Mehmet Sapanci

(ii) $A \circ B < A$,
(iii) $A \circ A < A$,
(iv) $0 \in x \circ (x \circ 0)$,
(v) $x < x \circ 0$,
(vi) $A < A \circ 0$,
(vii) $A < A \circ B$ if $0 \in B$.

We now review some fuzzy logic concepts. A fuzzy set in a set X is a function $\mu : X \to [0,1]$. For a fuzzy set μ in X and $\alpha \in [0,1]$ define $U(\mu;\alpha)$ to be the set $U(\mu;\alpha) := \{x \in X \mid \mu(x) \geq \alpha\}$, which is called a *level set* of μ.

3. Fuzzy hyper$K$-subalgebras

In what follows, $\mathcal{H}$ denotes a hyper$K$-algebra unless otherwise specified.

Definition 3.1. A fuzzy set μ in $\mathcal{H}$ is said to be a *fuzzy hyper$K$-subalgebra* of $\mathcal{H}$ if it satisfies the inequality:

$$\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y)\},$$

for all $x, y \in \mathcal{H}$.

Proposition 3.2. *Let μ be a fuzzy hyper$K$-subalgebra of $\mathcal{H}$. Then $\mu(0) \geq \mu(x)$ for all $x \in \mathcal{H}$.*

Proof. Using (HI3), we see that $0 \in x \circ x$ for all $x \in \mathcal{H}$. Hence

$$\mu(0) \geq \inf_{z \in x \circ x} \mu(z) \geq \min\{\mu(x), \mu(x)\} = \mu(x)$$

for all $x \in \mathcal{H}$. $\square$

Example 3.3. (i) Let $\mathcal{H} = \{0, a, b\}$ in which the hyperoperation "$\circ$" is given by the following table:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

Then $\mathcal{H}$ is a hyper$K$-algebra (see [1, Example 3.3(4)]). Define a fuzzy set $\mu : \mathcal{H} \to [0,1]$ by $\mu(0) = \mu(a) = \alpha_1 > \alpha_2 = \mu(b)$. Then μ is a fuzzy hyper$K$-subalgebra of $\mathcal{H}$. A fuzzy set $\nu : \mathcal{H} \to [0,1]$ defined by $\nu(0) = 0.7$, $\nu(a) = 0.5$ and $\nu(b) = 0.2$ is also a fuzzy hyper$K$-subalgebra of $\mathcal{H}$.

(ii) Consider a hyper$K$-algebra $\mathcal{H} = \{0, x, y\}$ with Cayley table as follows:

$\circ$	0	x	y
0	$\{0\}$	$\{0, x, y\}$	$\{0, x, y\}$
x	$\{x\}$	$\{0, x, y\}$	$\{0, x, y\}$
y	$\{y\}$	$\{x, y\}$	$\{0, x, y\}$

Using [1, Theorem 3.9], we see that $\mathcal{H} \times \mathcal{H}$ is a hyper$K$-algebra. Define a fuzzy set μ in $\mathcal{H} \times \mathcal{H}$ by

$$\mu(a, b) := \begin{cases} \alpha_1 & \text{if } b = 0, \\ \alpha_2 & \text{otherwise,} \end{cases}$$

where $\alpha_1 > \alpha_2$ in $[0,1]$. It is routine to check that μ is a fuzzy hyper$K$-subalgebra of $\mathcal{H} \times \mathcal{H}$.

Lemma 3.4 ([1, Theorem 4.12])**.** *Let S be a non-empty subset of $\mathcal{H}$. Then S is a hyper$K$-subalgebra of $\mathcal{H}$ if and only if $x \circ y \subseteq S$ for all $x, y \in S$.*

Theorem 3.5. *Let μ be a fuzzy set in $\mathcal{H}$. Then μ is a fuzzy hyper$K$-subalgebra of $\mathcal{H}$ if and only if for every $\alpha \in [0,1]$ the non-empty level set $U(\mu;\alpha)$ of μ is a hyper$K$-subalgebra of $\mathcal{H}$.*

We then call $U(\mu;\alpha)$ a *level hyper$K$-subalgebra* of μ.

Proof. Suppose that μ is a fuzzy hyper$K$-subalgebra of $\mathcal{H}$ and let $x, y \in U(\mu;\alpha)$ for $\alpha \in [0,1]$. Let $z \in x \circ y$. Then

$$\mu(z) \geq \inf_{w \in x \circ y} \mu(w) \geq \min\{\mu(x), \mu(y)\} \geq \alpha,$$

and so $z \in U(\mu;\alpha)$. This shows that $x \circ y \subseteq U(\mu;\alpha)$. It follows from Lemma 3.4 that $U(\mu;\alpha)$ is a hyper$K$-subalgebra of $\mathcal{H}$. Conversely let $U(\mu;\alpha)(\neq \emptyset)$ be a hyper$K$-subalgebra of $\mathcal{H}$ for every $\alpha \in [0,1]$. For any $x, y \in \mathcal{H}$, let $\beta = \min\{\mu(x), \mu(y)\}$. Then $x, y \in U(\mu;\beta)$ and hence $x \circ y \subseteq U(\mu;\beta)$. It follows that $\mu(z) \geq \beta$ for all $z \in x \circ y$ so that

$$\inf_{z \in x \circ y} \mu(z) \geq \beta = \min\{\mu(x), \mu(y)\}.$$

This shows that μ is a fuzzy hyper$K$-subalgebra of $\mathcal{H}$. □

Theorem 3.6. *Let S be a non-empty subset of $\mathcal{H}$ and let μ_S be a fuzzy set in $\mathcal{H}$ defined by*

$$\mu_S(x) := \begin{cases} \alpha_1 & \text{if } x \in S, \\ \alpha_2 & \text{otherwise,} \end{cases}$$

for all $x \in \mathcal{H}$ and $\alpha_1 > \alpha_2$ in $[0,1]$. Then

(i) *μ_S is a fuzzy hyper$K$-subalgebra of $\mathcal{H}$ if and only if S is a hyper$K$-subalgebra of $\mathcal{H}$.*
(ii) *$\mathcal{H}_{\mu_S} := \{x \in \mathcal{H} \mid \mu_S(x) = \mu_S(0)\} = S$.*

Proof. Assume that μ_S is a fuzzy hyper$K$-subalgebra of $\mathcal{H}$ and let $x, y \in S$. Then $\mu_S(x) = \alpha_1 = \mu_S(y)$. For any $z \in x \circ y$, we have

$$\mu_S(z) \geq \inf_{w \in x \circ y} \mu_S(w) \geq \min\{\mu_S(x), \mu_S(y)\} = \alpha_1$$

and so $\mu_S(z) = \alpha_1$. Hence $z \in S$, which shows that $x \circ y \subseteq S$. Therefore S is a hyper$K$-subalgebra of $\mathcal{H}$ by Lemma 3.4. Conversely suppose that S is a hyper$K$-subalgebra of $\mathcal{H}$ and let $x, y \in \mathcal{H}$. If $x \notin S$ or $y \notin S$, then clearly

$$\inf_{w \in x \circ y} \mu_S(w) \geq \alpha_2 = \min\{\mu_S(x), \mu_S(y)\}.$$

Assume that $x \in S$ and $y \in S$. Then $x \circ y \subseteq S$, and thus

$$\inf_{z \in x \circ y} \mu_S(z) = \alpha_1 = \min\{\mu_S(x), \mu_S(y)\}.$$

Consequently μ_S is a fuzzy hyper$K$-subalgebra of $\mathcal{H}$.

(ii) We have that

$$\mathcal{H}_{\mu_S} = \{x \in \mathcal{H} \mid \mu_S(x) = \mu_S(0)\} = \{x \in \mathcal{H} \mid \mu_S(x) = \alpha_1\} = S.$$

This completes the proof. □

Corollary 3.7. *Any hyper$K$-subalgebra of $\mathcal{H}$ can be realized as a level hyper$K$-sub-algebra of some fuzzy hyper$K$-subalgebra of $\mathcal{H}$.*

Proposition 3.8. *Let μ be a fuzzy hyper$K$-subalgebra of $\mathcal{H}$ and let $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 > \alpha_2$. Then two level hyper$K$-subalgebras $U(\mu;\alpha_1)$ and $U(\mu;\alpha_2)$ of μ are equal if and only if there does not exists $x \in \mathcal{H}$ such that $\alpha_1 \leq \mu(x) < \alpha_2$.*

Proof. Straightforward. □

Corollary 3.9. *Let $\mathcal{H}$ be a finite hyper$K$-algebra and let μ be a fuzzy hyper$K$-subalgebra of $\mathcal{H}$. Then the level hyper$K$-subalgebras of μ form a chain, i.e.,*

$$U(\mu;\alpha_0) \subseteq U(\mu;\alpha_1) \subseteq \cdots \subseteq U(\mu;\alpha_m) = \mathcal{H}$$

where $\alpha_0 > \alpha_1 > \cdots > \alpha_m$ in $[0,1]$.

Corollary 3.10. *Let μ be a fuzzy hyper$K$-subalgebra of $\mathcal{H}$ with*

$$\mathrm{Im}(\mu) = \{\alpha_1, \alpha_2, \cdots, \alpha_m\}.$$

Then

(i) *the family of hyper$K$-subalgebras $U(\mu;\alpha_i)$, $1 \leq i \leq m$, constitutes all the level hyper$K$-subalgebras of μ.*

(ii) *for any $\alpha_i, \alpha_j \in \mathrm{Im}(\mu)$, $U(\mu;\alpha_i) = U(\mu;\alpha_j)$ implies $\alpha_i = \alpha_j$.*

Theorem 3.11. *If μ is a fuzzy hyper$K$-subalgebra of $\mathcal{H}$, then*

$$\mu(x) := \sup\{\alpha \in [0,1] \mid x \in U(\mu;\alpha)\} \text{ for all } x \in \mathcal{H}.$$

Proof. Let $\beta = \sup\{\alpha \in [0,1] \mid x \in U(\mu;\alpha)\}$ and consider an arbitrary positive number ε. Then there exists $\alpha \in [0,1]$ such that $x \in U(\mu;\alpha)$ and $\beta - \varepsilon < \alpha$. It follows that $\mu(x) \geq \alpha > \beta - \varepsilon$ so that $\mu(x) \geq \beta$ since ε is arbitrary. Now let $\mu(x) = \gamma$. Then $x \in U(\mu;\gamma)$ and so $\gamma \in \{\alpha \in [0,1] \mid x \in U(\mu;\alpha)\}$. Hence

$$\mu(x) = \gamma \leq \sup\{\alpha \in [0,1] \mid x \in U(\mu;\alpha)\} = \beta$$

and therefore $\mu(x) = \beta$, as desired. □

Finally we state the converse of Theorem 3.11.

Theorem 3.12. *Let Λ be a non-empty subset of $[0,1]$ and let $\{S_\alpha \mid \alpha \in \Lambda\}$ be a collection of hyper$K$-subalgebras of $\mathcal{H}$ such that $\mathcal{H} = \bigcup_{\alpha \in \Lambda} S_\alpha$ and for all $\alpha, \beta \in \Lambda$, $\beta > \alpha$ if and only if $S_\beta \subset S_\alpha$. Define a fuzzy set μ in $\mathcal{H}$ by*

$$\mu(x) = \sup\{\alpha \in \Lambda \mid x \in S_\alpha\} \text{ for all } x \in \mathcal{H}.$$

Then μ is a fuzzy hyper$K$-subalgebra of $\mathcal{H}$.

Proof. Using Proposition 3.5, it is sufficient to show that the non-empty level set $U(\mu;\delta)$ of μ is a hyper$K$-subalgebra of $\mathcal{H}$ for every $\delta \in [0,1]$. We should consider two cases as follows:

$$(1)\ \delta = \sup\{\alpha \in \Lambda \mid \alpha < \delta\} \text{ and } (2)\ \delta \neq \sup\{\alpha \in \Lambda \mid \alpha < \delta\}.$$

For the case (1), we have

$$x \in U(\mu;\delta) \Leftrightarrow x \in S_\alpha \text{ for all } \alpha < \delta \Leftrightarrow x \in \bigcap_{\alpha < \delta} S_\alpha,$$

whence $U(\mu;\delta) = \bigcap_{\alpha<\delta} S_\alpha$ which is a hyper$K$-subalgebra of $\mathcal{H}$. Case (2) implies that there exists $\varepsilon > 0$ such that $(\delta-\varepsilon,\delta)\cap\Lambda = \emptyset$. If $x \in \bigcup_{\alpha\geq\delta} S_\alpha$, then $x \in S_\alpha$ for some $\alpha \geq \delta$. It follows that $\mu(x) \geq \alpha \geq \delta$ so that $x \in U(\mu;\delta)$. This proves that $\bigcup_{\alpha\geq\delta} S_\alpha \subset U(\mu;\delta)$. Assume that $x \notin \bigcup_{\alpha\geq\delta} S_\alpha$. Then $x \notin S_\alpha$ for all $\alpha \geq \delta$, which implies that $x \notin S_\alpha$ for all $\alpha > \delta-\varepsilon$, i.e., if $x \in S_\alpha$ then $\alpha \leq \delta - \varepsilon$. Thus $\mu(x) \leq \delta-\varepsilon < \delta$ and so $x \notin U(\mu;\delta)$. Therefore $U(\mu;\alpha) \subset \bigcup_{\alpha\geq\delta} S_\alpha$ and consequently $U(\mu;\alpha) = \bigcup_{\alpha\geq\delta} S_\alpha$ which is a hyper$K$-subalgebra of $\mathcal{H}$, ending the proof. $\square$

Definition 3.13. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be hyper$K$-algebras. A mapping $f : \mathcal{H}_1 \to \mathcal{H}_2$ is called a *weak homomorphism* if

(i) $f(0) = 0$,

(ii) $f(x \circ y) \subseteq f(x) \circ f(y)$ for all $x, y \in \mathcal{H}_1$.

Theorem 3.14. *Let $f : \mathcal{H}_1 \to \mathcal{H}_2$ be a weak homomorphism of hyper$K$-algebras. Then*

(i) *If $x < y$ in $\mathcal{H}_1$, then $f(x) < f(y)$ in $\mathcal{H}_2$.*

(ii) *If μ is a fuzzy hyper$K$-subalgebra of $\mathcal{H}_2$, then μ_f is a fuzzy hyper$K$-subalgebra of $\mathcal{H}_1$ where μ_f is defined by $\mu_f(x) = \mu(f(x))$ for all $x \in \mathcal{H}_1$.*

Proof. (i) If $x < y$ in $\mathcal{H}_1$, then $0 \in x \circ y$ and so $0 = f(0) \in f(x\circ y) \subseteq f(x)\circ f(y)$. Therefore $f(x) < f(y)$.

(ii) For any $x, y \in \mathcal{H}_1$, we have

$$\begin{aligned}
\inf_{z\in x\circ y} \mu_f(z) &= \inf_{z\in x\circ y} \mu(f(z)) \\
&\geq \inf_{f(z)\in f(x\circ y)} \mu(f(z)) \\
&\geq \inf_{f(z)\in f(x)\circ f(y)} \mu(f(z)) \\
&\geq \min\{\mu(f(x)), \mu(f(y))\} \\
&= \min\{\mu_f(x), \mu_f(y)\},
\end{aligned}$$

which shows that μ_f is a fuzzy hyper$K$-subalgebra of $\mathcal{H}_1$. $\square$

We construct a new fuzzy hyper$K$-subalgebra from old. Let $t \geq 0$ be a real number. If $\alpha \in [0,1]$, α^t shall mean the positive root in case $t < 1$. We define $\mu^t : \mathcal{H} \to [0,1]$ by $\mu^t(x) = (\mu(x))^t$ for all $x \in \mathcal{H}$.

Theorem 3.15. *If μ is a fuzzy hyper$K$-subalgebra of $\mathcal{H}$, then so is μ^t and $\mathcal{H}_{\mu^t} = \mathcal{H}_\mu$ for all $t \geq 0$.*

Proof. For any $x, y \in \mathcal{H}$ and $t \geq 0$, we have

$$\begin{aligned}
\inf_{z\in x\circ y} \mu^t(z) &= \inf_{z\in x\circ y} (\mu(z))^t \\
&= (\inf_{z\in x\circ y} \mu(z))^t \\
&\geq (\min\{\mu(x), \mu(y)\})^t \\
&= \min\{(\mu(x))^t, (\mu(y))^t\} \\
&= \min\{\mu^t(x), \mu^t(y)\}.
\end{aligned}$$

Hence μ^t is a fuzzy hyper$K$-subalgebra of $\mathcal{H}$. Now we get

$$\begin{aligned}\mathcal{H}_{\mu^t} &= \{x \in \mathcal{H} \mid \mu^t(x) = \mu^t(0)\} \\ &= \{x \in \mathcal{H} \mid (\mu(x))^t = (\mu(0))^t\} \\ &= \{x \in \mathcal{H} \mid \mu(x) = \mu(0)\} \\ &= \mathcal{H}_\mu.\end{aligned}$$

This completes the proof. □

Definition 3.16. A fuzzy hyper$K$-subalgebra μ of $\mathcal{H}$ is said to be *hypernormal* if there exists $x \in \mathcal{H}$ such that $\mu(x) = 1$.

Example 3.17. By taking $\alpha_1 = 1$, the fuzzy hyper$K$-subalgebra μ mentioned in Example 3.3(i) is hypernormal.

Using Proposition 3.2, we see that if a fuzzy hyper$K$-subalgebra μ of $\mathcal{H}$ is hypernormal, then $\mu(0) = 1$; hence μ is a hypernormal fuzzy hyper$K$-subalgebra of $\mathcal{H}$ if and only if $\mu(0) = 1$.

By using Theorem 3.6, we have the following.

Proposition 3.18. *Let S be a hyper$K$-subalgebra of $\mathcal{H}$ and let μ_S be a fuzzy set in $\mathcal{H}$ defined by*

$$\mu_S(x) := \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in \mathcal{H}$. Then μ_S is a hypernormal fuzzy hyper$K$-subalgebra of $\mathcal{H}$ and $\mathcal{H}_{\mu_S} = S$.

Theorem 3.19. *Let μ be a fuzzy hyper$K$-subalgebra of $\mathcal{H}$ and $\theta : [0, \mu(0)] \to [0, 1]$ be an increasing function. Let μ_θ be a fuzzy set in $\mathcal{H}$ defined by $\mu_\theta(x) = \theta(\mu(x))$ for all $x \in \mathcal{H}$. Then*

(i) *μ_θ is a fuzzy hyper$K$-subalgebra of $\mathcal{H}$.*
(ii) *If $\theta(\mu(0)) = 1$, then μ_θ is hypernormal.*
(iii) *If $\theta(\alpha) \geq \alpha$ for all $\alpha \in [0, \mu(0)]$ then $\mu(x) \leq \mu_\theta(x)$ for all $x \in \mathcal{H}$.*

Proof. (i) Let $x, y \in \mathcal{H}$. Then

$$\begin{aligned}\inf_{z \in x \circ y} \mu_\theta(z) &= \inf_{z \in x \circ y} \theta(\mu(z)) \\ &= \theta(\inf_{z \in x \circ y} \mu(z)) \\ &\geq \theta(\min\{\mu(x), \mu(y)\}) \\ &= \min\{\theta(\mu(x)), \theta(\mu(y))\} \\ &= \min\{\mu_\theta(x), \mu_\theta(y)\}.\end{aligned}$$

Hence μ_θ is a fuzzy hyper$K$-subalgebra of $\mathcal{H}$.

(ii) Straightforward.

(iii) Assume that $\theta(\alpha) \geq \alpha$ for all $\alpha \in [0, \mu(0)]$. Since $\mu(0) \geq \mu(x)$ for all $x \in \mathcal{H}$, we get $\mu_\theta(x) = \theta(\mu(x)) \geq \mu(x)$ for all $x \in \mathcal{H}$. □

Theorem 3.20. *Let μ be a fuzzy hyperK-subalgebra of $\mathcal{H}$ and let μ^+ be a fuzzy set in $\mathcal{H}$ defined by $\mu^+(x) = \mu(x) + 1 - \mu(0)$ for all $x \in \mathcal{H}$. Then μ^+ is a hypernormal fuzzy hyperK-subalgebra of $\mathcal{H}$ containing μ, i.e., $\mu^+(x) \geq \mu(x)$ for all $x \in \mathcal{H}$.*

Proof. For any $x, y \in \mathcal{H}$, we obtain

$$\begin{aligned}\inf_{z\in x\circ y} \mu^+(z) &= \inf_{z\in x\circ y} (\mu(z) + 1 - \mu(0)) \\ &= 1 - \mu(0) + \inf_{z\in x\circ y} \mu(z) \\ &\geq 1 - \mu(0) + \min\{\mu(x), \mu(y)\} \\ &= \min\{\mu(x) + 1 - \mu(0), \mu(y) + 1 - \mu(0)\} \\ &= \min\{\mu^+(x), \mu^+(y)\}.\end{aligned}$$

Therefore μ^+ is a hypernormal fuzzy hyper$K$-subalgebra of $\mathcal{H}$, and clearly $\mu^+(x) \geq \mu(x)$. This completes the proof. □

Theorem 3.21. *Let μ be a fuzzy hyperK-subalgebra of $\mathcal{H}$ and let $\widehat{\mu}$ be a fuzzy set in $\mathcal{H}$ defined by $\widehat{\mu}(x) = \frac{1}{\mu(0)}\mu(x)$ for all $x \in \mathcal{H}$. Then $\widehat{\mu}$ is a hypernormal fuzzy hyperK-subalgebra of $\mathcal{H}$ containing μ.*

Proof. Let $x, y \in \mathcal{H}$. Then

$$\begin{aligned}\inf_{z\in x\circ y} \widehat{\mu}(z) &= \inf_{z\in x\circ y} \frac{1}{\mu(0)}\mu(z) \\ &= \frac{1}{\mu(0)} \inf_{z\in x\circ y} \mu(z) \\ &\geq \frac{1}{\mu(0)} \min\{\mu(x), \mu(y)\} \\ &= \min\{\frac{1}{\mu(0)}\mu(x), \frac{1}{\mu(0)}\mu(y)\} \\ &= \min\{\widehat{\mu}(x), \widehat{\mu}(y)\},\end{aligned}$$

and $\widehat{\mu}(x) = \frac{1}{\mu(0)}\mu(x) \geq \mu(x)$ for all $x \in \mathcal{H}$, ending the proof. □

Corollary 3.22. *If μ is a fuzzy hyperK-subalgebra of $\mathcal{H}$ satisfying $\widehat{\mu}(x) = 0$ for some $x \in \mathcal{H}$, then $\mu(x) = 0$ also.*

Theorem 3.23. *A fuzzy hyperK-subalgebra of $\mathcal{H}$ is hypernormal if and only if $\widehat{\mu} = \mu$.*

Proof. Assume that μ is a hypernormal fuzzy hyper$K$-subalgebra of $\mathcal{H}$ and let $x \in \mathcal{H}$. Then $\widehat{\mu}(x) = \frac{1}{\mu(0)}\mu(x) = \mu(x)$, and hence $\widehat{\mu} = \mu$. □

Theorem 3.24. *If μ is a fuzzy hyperK-subalgebra of $\mathcal{H}$, then $\widehat{\widehat{\mu}} = \widehat{\mu}$.*

Proof. Note that $\widehat{\mu}(0) = \frac{1}{\mu(0)}\mu(0) = 1$. Thus for any $x \in \mathcal{H}$ we have $\widehat{\widehat{\mu}}(x) = \frac{1}{\widehat{\mu}(0)}\widehat{\mu}(x) = \widehat{\mu}(x)$, ending the proof. □

Corollary 3.25. *If μ is a hypernormal fuzzy hyperK-subalgebra of $\mathcal{H}$, then $\widehat{\widehat{\mu}} = \mu$.*

Theorem 3.26. *Let μ be a fuzzy hyperK-subalgebra of $\mathcal{H}$. If there is a fuzzy hyperK-subalgebra ν of $\mathcal{H}$ satisfying $\widehat{\nu} \subseteq \mu$, then μ is hypernormal.*

Proof. Suppose there exists a fuzzy hyper$K$-subalgebra ν of $\mathcal{H}$ such that $\widehat{\nu} \subseteq \mu$. Then $1 = \widehat{\nu}(0) \leq \mu(0)$, whence $\mu(0) = 1$. The proof is complete. □

Theorem 3.27. *Let μ be a non-constant hypernormal fuzzy hyperK-subalgebra of $\mathcal{H}$, which is maximal in the poset of hypernormal fuzzy hyperK-subalgebras under set inclusion. Then μ takes only the values 0 and 1.*

Proof. Note that $\mu(0) = 1$. Let $x \in \mathcal{H}$ be such that $\mu(x) \neq 1$. It is enough to show that $\mu(x) = 0$. Assume that there exists $w \in \mathcal{H}$ such that $0 < \mu(w) < 1$. Define a fuzzy set $\nu : \mathcal{H} \to [0, 1]$ by $\nu(x) = \frac{1}{2}(\mu(x) + \mu(w))$ for all $x \in \mathcal{H}$. Then clearly ν is well-defined, and we have that for all $x, y \in \mathcal{H}$

$$\begin{aligned}\inf_{z\in x\circ y} \nu(z) &= \inf_{z\in x\circ y} \frac{1}{2}(\mu(z) + \mu(w)) \\ &= \frac{1}{2}(\inf_{z\in x\circ y} \mu(z) + \mu(w)) \\ &\geq \frac{1}{2}(\min\{\mu(x), \mu(y)\} + \mu(w)) \\ &= \min\{\frac{1}{2}(\mu(x) + \mu(w)), \frac{1}{2}(\mu(y) + \mu(w))\} \\ &= \min\{\nu(x), \nu(y)\}.\end{aligned}$$

Hence ν is a fuzzy hyper$K$-subalgebra of $\mathcal{H}$. By Theorem 3.20, ν^+ is a hypernormal fuzzy hyper$K$-subalgebra of $\mathcal{H}$ where ν^+ is defined by $\nu^+(x) = \nu(x) + 1 - \nu(0)$ for all $x \in \mathcal{H}$. Note that

$$\begin{aligned}\nu^+(w) &= \nu(w) + 1 - \nu(0) \\ &= \frac{1}{2}(\mu(w) + \mu(w)) + 1 - \frac{1}{2}(\mu(0) + \mu(w)) \\ &= \frac{1}{2}(\mu(w) + 1) > \mu(w)\end{aligned}$$

and $\nu^+(w) < 1 = \nu^+(0)$. It follows that ν^+ is non-constant and μ is not maximal. This is a contradiction, ending the proof. $\square$

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Department of Mathematics Education, Gyeongsang National University, Chinju 660-701, Korea,
E-mail: ybjun@nongae.gsnu.ac.kr