ON n-FOLD FUZZY BCC-IDEALS

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This paper is dedicated to the memory of Prof. Dr. Mehmet Sapanci

Abstract. Fuzzifications of the notion of n-fold BCC-ideals are considered.

1. Introduction

In 1966, Y. Imai and K. Iséki [6] defined a class of algebras of type (2,0) called *BCK-algebras* which generalizes on one hand the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation, on the other hand the notion of implication algebra [7]. The class of all BCK-algebras is a quasivariety. K. Iséki posed an interesting problem (solved by A. Wroński [10]) whether the class of BCK-algebras is a variety. In connection with this problem, Y. Komori [9] introduced a notion of BCC-algebras, and W. A. Dudek [1, 2] redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Y. Komori. In [4], W. A. Dudek and X. H. Zhang introduced a notion of BCC-ideals in BCC-algebras, and W. A. Dudek and Y. B. Jun [3] established the fuzzification of BCC-ideals in BCC-algebras. Y. B. Jun and W. A. Dudek [8] introduced n-fold BCC-ideals, and investigated some related results. This work is concerned with the fuzzification of the notion of n-fold BCC-ideals. We investigate some related properties, and give a relation between an n-fold fuzzy BCC-ideal and a fuzzy BCK-ideal. We consider the characterization of an n-fold fuzzy BCC-ideal.

2. Preliminaries

By a *BCK-algebra* we mean an algebra $(G, *, 0)$ of type (2,0) satisfying the following conditions:

(I) $((x*y)*(x*z))*(z*y)=0$,
(II) $(x*(x*y))*y=0$,
(III) $x*x=0$,
(IV) $0*x=0$,
(V) $x*y=0$ and $y*x=0$ imply $x=y$,

for all $x, y, z \in G$. We can define a partial ordering "$\leq$" on G by $x \leq y$ if and only if $x*y=0$.

In any BCK-algebra G, the following hold:

(P1) $x*0=x$,
(P2) $x*y \leq x$,
(P3) $(x*y)*z=(x*z)*y$,
(P4) $(x*z)*(y*z) \leq x*y$,
(P5) $x \leq y$ implies $x*z \leq y*z$ and $z*y \leq z*x$.

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Definition 2.1. A non-empty set G with a constant 0 and a binary operation $*$ is called a *BCC-algebra* if for all $x, y, z \in G$ the following conditions hold:

(VI) $((x * y) * (z * y)) * (x * z) = 0$,
(III) $x * x = 0$,
(IV) $0 * x = 0$,
(P1) $x * 0 = x$,
(V) $x * y = 0$ and $y * x = 0$ imply $x = y$.

Any BCK-algebra is a BCC-algebra, but there are BCC-algebras which are not BCK-algebras. Note that a BCC-algebra is a BCK-algebra if and only if it satisfies:

(P3) $(x * y) * z = (x * z) * y$.

On any BCC-algebra (similarly as in the case of BCK-algebras) one can define the natural ordering "$\leq$" by putting

(1) $x \leq y \Longleftrightarrow x * y = 0$.

It is not difficult to verify that this order is partial and 0 is its smallest element. Moreover, in any BCC-algebra G, the following are true:

(2) $(x * y) * (z * y) \leq x * z$,
(P5) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

A non-empty subset S of a BCC/BCK-algebra G is called a *subalgebra* of G if $x * y \in S$ for all $x, y \in S$. A non-empty subset A of a BCK-algebra G is called an *ideal* if

(I1) $0 \in A$,
(I2) $y \in A$ and $x * y \in A$ imply $x \in A$.

In the sequel this ideal will be called a *BCK-ideal* and will be considered also in BCC-algebras. A non-empty subset A of a BCC-algebra G is called a *BCC-ideal* if

(I1) $0 \in A$,
(I3) $y \in A$ and $(x * y) * z \in A$ imply $x * z \in A$.

Proposition 2.2 ([4, Lemma 2.4]). *In a BCC-algebra, any BCK-ideal is a subalgebra.*

We now review some fuzzy logic concepts. A fuzzy set in a set G is a function $\mu : G \to [0, 1]$. For a fuzzy set μ in G and $\alpha \in [0, 1]$ define $U(\mu; t)$ to be the set $U(\mu; \alpha) := \{x \in G \mid \mu(x) \geq \alpha\}$, which is called a *level set* of μ.

A fuzzy set μ in a BCC/BCK-algebra G is called a *fuzzy subalgebra* of G if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in G$. A fuzzy set μ in a BCK-algebra G is called a *fuzzy BCK-ideal* of G if

(FI1) $\mu(0) \geq \mu(x)$,
(FK1) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$,

for all $x, y \in G$. A fuzzy set μ in a BCC-algebra G is called a *fuzzy BCC-ideal* of G if

(FI1) $\mu(0) \geq \mu(x)$,
(FC1) $\mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y)\}$,

for all $x, y, z \in G$.

3. n-fold fuzzy BCC-ideals

For any elements x and y of a BCC-algebra, $x * y^n$ denotes

$$(\cdots((x * y) * y) * \cdots) * y$$

in which y occurs n times.

Definition 3.1 ([8, Definition 3.1])**.** A non-empty subset A of a BCC-algebra G is called an *n-fold BCC-ideal* of G if

(I1) $0 \in A$,

(I4) for every $x, y, z \in G$ there exists a natural number n such that $x * z^n \in A$ whenever $(x * y) * z^n \in A$ and $y \in A$.

For a BCC-algebra G, obviously $\{0\}$ and G itself are n-fold BCC-ideals of G for every positive integer n.

Example 3.2 ([8, Example 3.2])**.** (i) Let $G = \{0, 1, 2, 3, 4, 5\}$ be a BCC-algebra (which is not a BCK-algebra) with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

It is routine to check that $A = \{0, 1, 2, 3, 4\}$ is an n-fold BCC-ideal of G for every positive integer n.

(ii) Consider a proper BCC-algebra $G = \{0, 1, 2, 3, 4\}$ with the Cayley table as follows:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

It is easy to check that $A = \{0, 1, 2, 3\}$ is an n-fold BCC-ideal of G for every positive integer n.

We now consider the fuzzification of n-fold BCC-ideals.

Definition 3.3. A fuzzy set μ in a BCC-algebra G is called an *n-fold fuzzy BCC-ideal* of G if

(FI1) $\mu(0) \geq \mu(x)$ for all $x \in G$,

(FC2) for every $x, y, z \in G$ there exists a positive integer n such that

$$\mu(x * z^n) \geq \min\{\mu((x * y) * z^n), \mu(y)\}.$$

The 1-fold fuzzy BCC-ideal is precisely a fuzzy BCC-ideal.

Example 3.4. Let $G = \{0, 1, 2, 3, 4\}$ be a proper BCC-algebra in Example 3.2(ii). Define a fuzzy set μ in G by $\mu(4) = 0.3$ and $\mu(x) = 0.8$ for all $x \neq 4$. Then μ is an n-fold fuzzy BCC-ideal of G

Lemma 3.5 ([3])**.** *In a BCC-algebra, every fuzzy BCK-ideal is a fuzzy subalgebra.*

Theorem 3.6. *In a BCC-algebra, every n-fold fuzzy BCC-ideal is a fuzzy BCK-ideal.*

Proof. Let μ be an n-fold fuzzy BCC-ideal of a BCC-algebra G. Taking $z = 0$ in (FC2) and using (P1), we get

$$\mu(x) = \mu(x * 0^n) \geq \min\{\mu((x * y) * 0^n), \mu(y)\} = \min\{\mu(x * y), \mu(y)\}$$

for all $x, y \in G$. Hence μ is a fuzzy BCK-ideal of G $\square$

Corollary 3.7. *In a BCC-algebra, every n-fold fuzzy BCC-ideal is a fuzzy subalgebra.*

The following example shows that the converse of Corollary 3.7 may not be true.

Example 3.8. Let $G = \{0, 1, 2, 3, 4\}$ be a proper BCC-algebra in Example 3.2(ii) and let μ be a fuzzy set in G given by

$$\mu(x) := \begin{cases} \alpha_1 & \text{if } x \in \{0, 2, 3\}, \\ \alpha_2 & \text{otherwise}, \end{cases}$$

where $\alpha_1 > \alpha_2$ in $[0, 1]$. It is not difficult to see that μ is a fuzzy subalgebra of G. But μ is not an n-fold fuzzy BCC-ideal of G for every positive integer n because

$$\mu(4 * 0^n) = \mu(4) = \alpha_2 < \alpha_1 = \min\{\mu((4 * 3) * 0^n), \mu(3)\}.$$

Theorem 3.9. *In a BCK-algebra, the notion of n-fold fuzzy BCC-ideals and fuzzy BCK-ideals coincide.*

Proof. Since a BCK-algebra is a BCC-algebra, every n-fold fuzzy BCC-ideal is a fuzzy BCK-ideal (see [Theorem 3.6]). Let μ be a fuzzy BCK-ideal of a BCK-algebra G and let $x, y, z \in G$. Then

$$\begin{aligned} \mu(x * z^n) &\geq \min\{\mu((x * z^n) * y), \mu(y)\} && \text{[by (FK1)]} \\ &= \min\{\mu((x * y) * z^n), \mu(y)\}. && \text{[by (P3)]} \end{aligned}$$

Hence μ is an n-fold fuzzy BCC-ideal of G. $\square$

Proposition 3.10. *Let A be a non-empty subset of a BCC-algebra G, n a positive integer and μ a fuzzy set in G defined by*

$$\mu(x) := \begin{cases} \alpha_1 & \text{if } x \in A, \\ \alpha_2 & \text{otherwise}, \end{cases}$$

where $\alpha_1 > \alpha_2$ in $[0, 1]$. Then μ is an n-fold fuzzy BCC-ideal of G if and only if A is an n-fold BCC-ideal of G. Moreover, $G_\mu = A$, where $G_\mu := \{x \in G \mid \mu(x) = \mu(0)\}$.

Proof. Assume that μ is an n-fold fuzzy BCC-ideal of G. Since $\mu(0) \geq \mu(x)$ for all $x \in G$, we have $\mu(0) = \alpha_1$ and so $0 \in A$. Let $x, y, z \in G$ be such that $(x * y) * z^n \in A$ and $y \in A$. Using (FC2), we know that

$$\mu(x * z^n) \geq \min\{\mu((x * y) * z^n), \mu(y)\} = \alpha_1$$

and thus $\mu(x * z^n) = \alpha_1$. Hence $x * z^n \in A$, and A is an n-fold BCC-ideal of G. Conversely suppose that A is an n-fold BCC-ideal of G. Since $0 \in A$, it follows that $\mu(0) = \alpha_1 \geq \mu(x)$ for all $x \in G$. Let $x, y, z \in G$. If $y \notin A$ or $x * z^n \in A$, then clearly

$$\mu(x * z^n) \geq \min\{\mu((x * y) * z^n), \mu(y)\}.$$

Assume that $y \in A$ and $x * z^n \notin A$. Then by (I4), we have $(x * y) * z^n \notin A$. Therefore

$$\mu(x * z^n) = \alpha_2 = \min\{\mu((x * y) * z^n), \mu(y)\}.$$

Finally we have that $G_\mu = \{x \in G \mid \mu(x) = \mu(0)\} = \{x \in G \mid \mu(x) = \alpha_1\} = A$. This completes the proof. $\square$

Theorem 3.11. *Let μ be a fuzzy set in a BCC-algebra G and let n be a positive integer. Then μ is an n-fold fuzzy BCC-ideal of G if and only if the non-empty level set $U(\mu;\alpha)$ of μ is an n-fold BCC-ideal of G.*

We then call $U(\mu;\alpha)$ the *level n-fold BCC-ideal* of μ.

Proof. Suppose that μ is an n-fold fuzzy BCC-ideal of G and $U(\mu;\alpha) \neq \emptyset$ for any $\alpha \in [0,1]$. Then there exists $x \in U(\mu;\alpha)$ and so $\mu(x) \geq \alpha$. It follows from (FI1) that $\mu(0) \geq \mu(x) \geq \alpha$ so that $0 \in U(\mu;\alpha)$. Let $x, y, z \in G$ be such that $(x * y) * z^n \in U(\mu;\alpha)$ and $y \in U(\mu;\alpha)$. Then by (FC2), we have

$$\mu(x * z^n) \geq \min\{\mu((x * y) * z^n), \mu(y)\} \geq \min\{\alpha, \alpha\} = \alpha,$$

and thus $x * z^n \in U(\mu;\alpha)$. Hence $U(\mu;\alpha)$ is an n-fold BCC-ideal of G. Conversely assume that $U(\mu;\alpha) \neq \emptyset$ is an n-fold BCC-ideal of G for every $\alpha \in [0,1]$. For any $x \in G$, let $\mu(x) = \alpha$. Then $x \in U(\mu;\alpha)$. Since $0 \in U(\mu;\alpha)$, it follows that $\mu(0) \geq \alpha = \mu(x)$ so that $\mu(0) \geq \mu(x)$ for all $x \in G$. Now we only need to show that μ satisfies (FC2). If not, then there exist $a, b, c \in G$ such that

$$\mu(a * c^n) < \min\{\mu((a * b) * c^n), \mu(b)\}.$$

Taking $\alpha_0 = \frac{1}{2}(\mu(a * c^n) + \min\{\mu((a * b) * c^n), \mu(b)\})$, then we have

$$\mu(a * c^n) < \alpha_0 < \min\{\mu((a * b) * c^n), \mu(b)\}.$$

Hence $(a * b) * c^n \in U(\mu;\alpha_0)$ and $b \in U(\mu;\alpha_0)$, but $a * c^n \notin U(\mu;\alpha_0)$, which means that $U(\mu;\alpha_0)$ is not an n-fold BCC-ideal of G. This is a contradiction. Therefore μ is an n-fold fuzzy BCC-ideal of G □

Lemma 3.12. *Let μ be an n-fold fuzzy BCC-ideal of a BCC-algebra G and let $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 > \alpha_2$. Then*

(i) *$U(\mu;\alpha_1) \subseteq U(\mu;\alpha_2)$,*

(ii) *whenever $\alpha_1, \alpha_2 \in \mathrm{Im}(\mu)$, then $U(\mu;\alpha_1) \neq U(\mu;\alpha_2)$,*

(iii) *$U(\mu;\alpha_1) = U(\mu;\alpha_2)$ if and only if there does not exist $x \in G$ such that $\alpha_1 \leq \mu(x) < \alpha_2$.*

Proof. Straightforward. □

Theorem 3.13. *Let μ be an n-fold fuzzy BCC-ideal of a BCC-algebra G with $\mathrm{Im}(\mu) = \{\alpha_i \mid i \in \Lambda\}$ and $\Omega = \{U(\mu;\alpha_i) \mid i \in \Lambda\}$ where Λ is an arbitrary index set. Then*

(i) *there exists a unique $i_0 \in \Lambda$ such that $\alpha_{i_0} \geq \alpha_i$ for all $i \in \Lambda$,*

(ii) *$G_\mu = \bigcap_{i\in\Lambda} U(\mu;\alpha_i) = U(\mu;\alpha_{i_0})$,*

(iii) *$G = \bigcup_{i\in\Lambda} U(\mu;\alpha_i)$,*

(iv) *the members of Ω form a chain,*

(v) *Ω contains all level n-fold BCC-ideals of μ if and only if μ attains its infimum on all n-fold BCC-ideals of G.*

Proof. (i) Since $\mu(0) \in \mathrm{Im}(\mu)$, there exists a unique $i_0 \in \Lambda$ such that $\mu(0) = \alpha_{i_0}$. Hence by (FI1), we get $\mu(x) \leq \mu(0) = \alpha_{i_0}$ for all $x \in G$, and so $\alpha_{i_0} \geq \alpha_i$ for all $i \in \Lambda$.

(ii) We have that

$$\begin{aligned} U(\mu;\alpha_{i_0}) &= \{x \in G \mid \mu(x) \geq \alpha_{i_0}\} \\ &= \{x \in G \mid \mu(x) = \alpha_{i_0}\} \\ &= \{x \in G \mid \mu(x) = \mu(0)\} \\ &= G_\mu. \end{aligned}$$

Note that $U(\mu;\alpha_{i_0}) \subseteq U(\mu;\alpha_i)$ for all $i \in \Lambda$, so that $U(\mu;\alpha_{i_0}) \subseteq \bigcap_{i\in\Lambda} U(\mu;\alpha_i)$. Since $i_0 \in \Lambda$, it follows that

$$G_\mu = U(\mu;\alpha_{i_0}) = \bigcap_{i\in\Lambda} U(\mu;\alpha_i).$$

(iii) For any $x \in G$ we have $\mu(x) \in \mathrm{Im}(\mu)$ and so there exists $i(x) \in \Lambda$ such that $\mu(x) = \alpha_{i(x)}$. This implies $x \in U(\mu;\alpha_{i(x)}) \subseteq \bigcup_{i\in\Lambda} U(\mu;\alpha_i)$. Hence $G = \bigcup_{i\in\Lambda} U(\mu;\alpha_i)$.

(iv) Straightforward.

(v) Suppose that Ω contains all level n-fold BCC-ideals of μ. Let A be an n-fold BCC-ideal of G. If μ is constant on A, then we are done. Assume that μ is not constant on A. For the case $A = G$, let $\beta := \inf\{\alpha_i \mid i \in \Lambda\}$. Then $\beta \leq \alpha_i$ for all $i \in \Lambda$, and so $U(\mu;\alpha_i) \subseteq U(\mu;\beta)$ for all $i \in \Lambda$. Note that $G = U(\mu;0) \in \Omega$ so that there exists $j \in \Lambda$ such that $\alpha_j \in \mathrm{Im}(\mu)$ and $U(\mu;\alpha_j) = G$. Thus $G = U(\mu;\alpha_j) \subseteq U(\mu;\beta)$ and so $U(\mu;\beta) = U(\mu;\alpha_j) = G$ because every level n-fold BCC-ideal of μ is an n-fold BCC-ideal of G. Now we prove $\beta = \alpha_j$. If $\beta < \alpha_j$, then there exists $k \in \Lambda$ such that $\alpha_k \in \mathrm{Im}(\mu)$ and $\beta \leq \alpha_k < \alpha_j$. This implies that $G = U(\mu;\alpha_j) \subsetneq U(\mu;\alpha_k)$, which is impossible. Hence $\beta = \alpha_j$. If $A \neq G$, consider the restriction μ_A of μ to A. By Proposition 3.10, μ_A is an n-fold fuzzy BCC-ideal of G. Let $\Lambda_A = \{i \in \Lambda \mid \mu(y) = \alpha_i \text{ for some } y \in A\}$ and $\Omega_A = \{U(\mu_A;\alpha_i) \mid i \in \Lambda_A\}$. Noticing that Ω_A contains all level n-fold BCC-ideals of μ_A, then there exists $z \in A$ such that $\mu_A(z) = \inf\{\mu_A(x) \mid x \in A\}$, which implies that $\mu(z) = \inf\{\mu(x) \mid x \in A\}$.

Conversely assume that μ attains its infimum on all n-fold BCC-ideals of G. Let $U(\mu;\alpha)$ be a level n-fold BCC-ideal of μ. If $\alpha = \alpha_i$ for some $i \in \Lambda$, then clearly $U(\mu;\alpha) \in \Omega$. Suppose that $\alpha \neq \alpha_i$ for all $i \in \Lambda$. Then there does not exist $x \in G$ such that $\mu(x) = \alpha$. Let $A = \{x \in G \mid \mu(x) > \alpha\}$. Obviously $0 \in A$. Let $x, y, z \in G$ be such that $(x * y) * z^n \in A$ and $y \in A$. Then $\mu((x * y) * z^n) > \alpha$ and $\mu(y) > \alpha$. It follows from (FC2) that

$$\mu(x * z^n) \geq \min\{\mu((x * y) * z^n), \mu(y)\} > \alpha$$

so that $x * z^n \in A$. Hence A is an n-fold BCC-ideal of G. By hypothesis, there exists $y \in A$ such that $\mu(y) = \inf\{\mu(x) \mid x \in A\}$. Now $\mu(y) \in \mathrm{Im}(\mu)$ implies $\mu(y) = \alpha_j$ for some $j \in \Lambda$. Thus we get $\inf\{\mu(x) \mid x \in A\} = \alpha_j > \alpha$. Note that there does not exist $z \in G$ such that $\alpha \leq \mu(z) < \alpha_j$, so from Lemma 3.12(iii) that $U(\mu;\alpha) = U(\mu;\alpha_j) \in \Omega$. This completes the proof. $\square$

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