ALPHA-COMPACTNESS WITH RESPECT TO AN IDEAL

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ABSTRACT. We introduce the concept of ideal α-compact spaces and show properties related to this topic. Also we generalize the concepts given in [3] and [4].

Definition 1. [3]. *Given a set X, a collection I of subsets of X is called an ideal on X if it satisfies the following conditions*:

(1) If $J \in I$ and $J_1 \subset J$, then $J_1 \in I$.
(2) If $J_1 \in I$ and $J_2 \in I$, then $J_1 \cup J_2 \in I$.

If $X \notin I$, then I is said to be a proper ideal.

Definition 2. [2]. *Let (X, Γ) be a topological space, B be a subset of X and α be an operator from Γ to $P(X)$, $\alpha : \Gamma \longrightarrow P(X)$. We say that α is an operator on Γ if*

$U \subset \alpha(U)$ for every $U \in \Gamma$.

We say that the operator α on Γ is stable with respect to B if α induce an operator $\alpha_B : \Gamma/_B \longrightarrow P(B)$ such that $\alpha/_B(U \cap B) = \alpha(U) \cap B$ for every $U \in \Gamma$ where Γ_B is the relative topology on B.

We will denote by (X,Γ,I,α) a nonempty set X, a topology Γ on X, an operator $\alpha, \alpha : \Gamma \longrightarrow P(X)$ and an ideal I on X,. and $cl(U)$ is the closure of U.

Definition 3. *A subset A of a space (X,Γ,I,α) is said to be I- α-compact, if for every open cover $\{U_i, i \in \Lambda\}$ of A , there exists a finite subset $\{i_1, \dots , i_n\}$ of Λ such that $X \backslash U_{j=1}^{n} \alpha(U_{i_j}) \in I$.*

Definition 4. *(X,Γ,I,α) is said to be I-α- compact if X is I-α-compact as a subset.*

Remark 1. If I=$\{\phi\}$, then I-α-compact is equivalent to $\alpha-$compact in the usual sense given in [1]. If I=$\{\phi\}$ and α is the identity operator, then I-α-compact is equivalent to compact in the usual sense . Finally if I is an ideal and α is the identity operator, then I-α-compact is equivalent to I-compact in the sense given in [3].

In this case the definition of I-α-compact generalizes the definition of I-compact, α-compact and compact.

Theorem 1. *Let (X,Γ,I,α) be I-α- compact. If J is an ideal on X with $I \subset J$, then (X,Γ,J,α) is J-α-compact.*

Proof. Let $\{U_i\}_{i\in\Lambda}$ be an open cover of X. Since (X,Γ,I,α) is I-α-compact, there exist a finite subset $\{i_1, \dots , i_n\}$ of Λ, such that $X \backslash U_{j=1}^{n} \alpha(U_{i_j}) \in I$. By hypothesis $I \subset J$, therefore $X \setminus U_{j=1}^{n} \alpha(U_{i_j}) \in J$. In this case (X,Γ,J,α) is J-α-compact.

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Theorem 2. *Let I_F denote the ideal of finite subsets of X, then (X,Γ) is α-compact if and only if (X,Γ,I_F,α) is I_F-α-compact.*

Proof. Let $\{U_i : i \in \Lambda\}$ be an open cover of X, since X is α-compact, there exists a finite collection $\{i_1, \dots, i_n\}$ of Λ such that $X \setminus U_{j=1}^{n} \alpha(U_{i_j}) = \phi$, but $\phi \in I_F$, therefore (X,Γ,I_F,α) is I_F- α-compact.

Now let $\{U_i, i \in \Lambda\}$ be an open cover of X, since $(X,\Gamma,I_F,\alpha\ )$ is I_F-α-compact there exists a finite collection $\{i_i, \dots, i_n\}$ of Λ such that $X \setminus U_{j=1}^{n} \alpha(U_{i_j}) \in I_F$. Take $B = X \setminus U_{j=1}^{n} \alpha(U_{i_j})$, then B is a finite set, we can choose $B = \{x_l, l = 1 \dots k\}$. Now for each $x_l \in B$, let U_l a neighborhood of x_l, in consequence $X = U_{j=1}^{n} \alpha(U_{i_j}) \cup U_{l=1}^{k} \alpha(U_l)$ and obtain that X is α-compact.

Definition 5. *Let (X,Γ) be a topological space and α be an operator associated to Γ. We say that X is α-lindeloff if for every open cover $\{U_i / i \in \Lambda\}$ of X there exists a countable subset $\{i_1, \dots, ..\}$ of Λ, such that $X \subset U_{j=1}^{\infty} \alpha(U_{i_j})$.*

Remark 2. For any operator α, every lindeloff space is an α-lindeloff space.

Theorem 3. . *Let I_c denote the ideal of countable subsets of X. If (X,Γ,I_c,α) is I_c-α-compact, then (X,Γ) is α-Lindeloff.*

Proof. Let $\{U_i : i \in \Lambda\}$ be an open cover of X since $(X,\Gamma,I_c,\alpha\ )$ is I_c-α-compact, then there exists a finite subcolletion$\{i_1, \dots, i_n\}$ of Λ, such that $X \setminus U_{j=1}^{n} \alpha(U_{i_j}) \in I_c$. Let $B= X \setminus U_{j=1}^{n} \alpha(U_{i_j})$.Since B is countable, choose $B=\{X_l : l = 1 \dots k\}$ and $U_l^{x_l}$a neighborhood of x_l. Then $X \subset U_{j=1}^{n} \alpha(U_{i_j}) \cup (U_{l=1}^{\infty} U_l^{x_l}) \subset U_{j=1}^{n} \alpha(U_{i_j}) \cup (U_{l=1}^{\infty} U_l^{x_l})$ In consequence X is α-lindeloff.

For an α-lindeloff space which is not I_c-α-compact, simply consider the real space with the usual topology and define $\alpha : \Gamma \longrightarrow P(X)$ as follows: for each basic element $(a,.b)$, α $((a,b)) = (a\text{-}\epsilon, b + \epsilon)$ where ϵ is an irrational number less than 1, and extend the definition to open sets.

Definition 6. *Let (X, Γ) be a topology space an α be an operator associated to Γ . We say that X is α-QHC if every open cover $\{U_i : i \in \Lambda\}$ of the space contains a finite subcollection $\{U_1, \dots, U_n\}$ such that $X \subset \cup_{i=1}^{n} cl(\alpha(U_i))$.*

Remark 3. If α is the identity operation, the definition of α-QHC is equivalent to QHC given in [3].

Definition 7. *An α-T_2 space which is α-QHC is said to be α-H-closed.*

Theorem 4. *Let (X,Γ,I,α) be a space if (X,Γ,I,α) is I-α-compact and $\Gamma \cap I = \phi$, then (X,Γ) is α-QHC.*

Proof. Let $\{U_i : i \in \Lambda\}$ be an open cover of X, since (X,Γ,I,α) is I-α-compact, there exists a finite subcollection $\{i_1, \dots, i_n\}$ of Λ such that $X \setminus U_{j=1}^{n} \alpha(U_{i_j}) \in I$. Since $\alpha(U_{i_j}) \subset cl(\alpha(U_{i_j}))$.We obtain that $X \setminus cl(\alpha(U_{i_j})) \subset X \setminus \alpha(U_{i_j}) \in I$. But $X \setminus U_{j=1}^{n} cl(\alpha(U_{i_j})) \subseteq X \setminus U_{j=1}^{n}(\alpha(U_{i_j}))$. Therefore $X \setminus U_{j=1}^{n} cl(\alpha(U_{i_j})) \in I$. By hypothesis $\Gamma \cap I = \phi$ therefore $X \setminus U_{j=1}^{n} cl(\alpha(U_{i_j})) = \phi$ so $X \subset U_{j=1}^{n} cl(\alpha(U_{i_j}))$and therefore (X,Γ) is α-QHC.

Definition 8. [5]. *Let (X,Γ) and (Y,Ψ) be two topological spaces and α, β be operators on Γ and Ψ respectively. We say that a function $f : (X,\Gamma) \longrightarrow (Y,\Psi)$ is (α, β)-continuous if for each point $x \in X$ and every Ψ-open neighborhood V of $f(x)$, there exist a Γ-open neighborhood U of x such that $f\ (\alpha(U)) \subset \beta(V)$.*

Let $f : X \longrightarrow Y$ be a function with I an ideal on X. Then $f(I) = \{f(J) : J \in I\}$ is an ideal on Y.

Using this fact, we have the following theorem.

Theorem 5. *Let f:$(X,\Gamma,I,\alpha) \longrightarrow (Y,\Psi,f(I),\beta)$ be a (α,β)-continuous surjection function. If (X,Γ,I,α) is I-α-compact, then $(Y,\Psi,f(I),\beta)$ is $f(I)$-β-compact.*

Proof. Let $\{V_i : i \in \Lambda\}$ be an open cover of Y. For each $x \in X$, there exists $i \in \Lambda$ such that $f(x) \in V_i$. By (α,β)-continuity, there exists an open set U_{x_i} such that $f\ (\alpha(U_{x_i})) \subset \beta(V_i)$. So $\{U_{x_i} : x \in X\}$ is an open cover of X. Since (X,Γ,I,α) is I-α-compact, there exist a finite subcollection $\{i_1, \ldots, i_n\}$ of Λ such that $X \setminus U_{j=1}^{n}\alpha(U_{x_{i_j}}) \in I$. Relabel $U_{x_{i_j}}$ to be U_{i_j}.

$Y \setminus U_{j=1}^{n}\beta(V_{i_j}) \subset Y \setminus U_{j=1}^{n} f(\alpha(U_{i_j})) = f(X) \setminus U_{j=1}^{n} f(\alpha(U_{i_j})) \subseteq f(X \setminus U_{j=1}^{n}\alpha(U_{i_j})) \in f(I)$. Since $f(I)$ is an ideal on Y, $Y \setminus U_{j=1}^{n}\beta(V_{i_j}) \in f(I)$. Thus, $(Y,\Psi,f(I),\beta)$ is $f(I)$-β-compact.

Corollary 1. *Let $f : (X,\Gamma,I,\alpha) \longrightarrow (Y,\Psi,f(I))$ be a (α,id)-continuous surjection. If (X,Γ,I,α) is I-α-compact. Then $(Y,\Psi,f(I))$ is $f(I)$-compact.*

Now considering I an ideal on X and $A \subset X$, we denote the restriction of I to A by $I/_A$=$\{J \cap A : J \in I\}$ it is easily seen that $I/_A$ is an ideal and in fact $I/_A$ =$I \cap I(\{A\})$,where $I(\{A\})$ is the ideal generated by A.

Theorem 6. *Let (X,Γ,I,α) be a space and $A \subset X$ with α be stable on A. Then A is I-α -compact if and only if $(A,\Gamma/_A,I/_A,\alpha/_A)$ is $I/_A$-$\alpha/_A$ -compact.*

Proof. Let $\{U_i \cap A : i \in \Lambda\}$ be a $\Gamma/_A$ -open cover of A , where $U_i \in \Gamma$. Then $\{U_i : i \in \Lambda\}$ is a Γ-open cover of A. By hypothesis A is I-α -compact,then there exists a finite subcollection$\{i_1, \ldots, i_n\}$ of Λ, such that $A \setminus U_{j=1}^{n}\alpha(U_{i_j}) \in I$. Now $A \cap (A \setminus U_{j=1}^{n}\alpha(U_{i_j})) \in I$ $/_A$. Using the fact that α is stable on A, we obtain that $A \setminus U_{j=1}^{n}(\alpha_A(A \cap U_{i_j})) = A \setminus U_{j=1}^{n}(A \cap \alpha(U_{i_j})) = A \cap (A \setminus U_{j=1}^{n}\alpha(U_{i_j})) \in I/_A$. In consequence $(A,\Gamma/_A,I/_A,\alpha/_A)$ is $I/_A$-$\alpha/_A$-compact.

Now suppose that $\left\{\acute{U}_i : i \in \Lambda\right\}$ is Γ-open cover of A, then $\{U_i \cap A : i \in \Lambda\}$ is $\Gamma/_A$ open cover of A. By hypothesis $(A,\Gamma/_A,I/_A,\alpha/_A)$ is $I/_A$-$\alpha/_A$-compact, therefore there exists a finite subcollection $i_1, \ldots, i_n$ of Λ such that $A \setminus U_{j=1}^{n}(\alpha_A(A \cap U_{i_j})) \in I/_A \subset I$. But $A \setminus U_{j=1}^{n}(\alpha(U_{i_j})) \subset A \setminus U_{j=1}^{n}(\alpha(U_{i_j}) \cap A) \in I$. therefore (A,Γ,I,α) is I-α-compact.

Corollary 2. [2] . *Let (X,Γ) be a topological space and $K \subset X$. Let α be an operator associated with Γ and stable with respect to K. Then K is α-compact if and only if K is $\alpha/_K$ -compact.*

Proof. Let $I = \{\phi\}$.

Theorem 7. *Let f:$(X,\Gamma,I,\alpha) \longrightarrow (Y,\ \Psi,f(I),\beta)$ be (α,β) continuous map, A be a subset of X such thatα is stable on A and β is stable on $f(A)$, If A is I- α-compact, then $f(A)$ is $f(I)$-β-compact.*

Proof. The proof follows from theorem 6, theorem 3 and using the fact that $f(I/_A) \subset f(I)/_{f(A)}$.

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