CERTAIN REDUCED FREE PRODUCTS WITH AMALGAMATION OF C^*–ALGEBRAS

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Abstract. It is proved that some of the reduced free products of matrix algebras with amalgamation over finite dimensional commutative C^*-algebras can be identified, up to stable isomorphism, with the reduced free product C^*-algebras obtained from M_n, $C(\mathbb{T})$ and $\mathcal{O}_k$.

§1. Preliminaries

We shall examine some examples in the reduced free product of two matrix algebras with amalgamation over a finite dimensional commutative C^*-algebra. The notion of reduced free product with amalgamation of C^*-algebras was introduced by D.Voiculescu ([14]). Let $(A_i)_{i\in I}$ be unital C^*-algebras and let B be a unital C^*-algebra with unital embedding $\iota_i : B \hookrightarrow A_i$ for each $i \in I$. In addition, for each $i \in I$, let $E_i : A_i \to B$ be a projection of norm one which satisfies the condition that $x \in A_i$ is equal to 0 whenever $E_i(y^*x^*xy) = 0$ for all $y \in A_i$. In this setting, the reduced free product of $(A_i, E_i)_{i\in I}$ with amalgamation over B is the unique unital C^*-algebra A with unital embeddings $\tilde{\iota_i} : A_i \hookrightarrow A$ and a projection of norm one $E : A \to B$ satisfying the following property:

(i) $\tilde{\iota_i}(\iota_i(b)) = \tilde{\iota_j}(\iota_j(b))$ for all $b \in B$ and $i, j \in I$, (So B can be naturally identified with a C^*-subalgebra of A.)
(ii) $E \circ \tilde{\iota_i} = E_i$ for each $i \in I$,
(iii) $(A_i)_{i\in I}$ is *free* in (A, E), that is, the set
$$\{\tilde{\iota_{i_1}}(x_1)\cdots\tilde{\iota_{i_n}}(x_n) \mid x_l \in \ker E_{i_l}, i_l \neq i_{l+1} (1 \le l \le n-1), n \in \mathbb{N}\}$$
is contained in $\ker E$,
(iv) $A = C^*(\cup_{i\in I}\tilde{\iota_i}(A_i))$,
(v) $x \in A$ is equal to 0 whenever $E(y^*x^*xy) = 0$ for all $y \in A$.

We shall denote the reduced free product with amalgamation by
$$(A, E) = \underset{B}{*}(A_i, E_i)$$
and E is called the free product projection of norm one. In particular, for a family $\{(A_i, \varphi_i) \mid i \in I\}$ of a unital C^*-algebra A_i with a state φ_i whose GNS-representation is faithful, the C^*-algebra $(A, \varphi) = \underset{\mathbb{C}}{*}(A_i, \varphi_i)$ is called the reduced free product of $(A_i, \varphi_i)_{i\in I}$, and φ is called the free product state.

In this article, we study some examples of reduced free product C^*-algebras with amalgamation defined as follows:

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Definition 1.1. Let $p, q_1, ..., q_m, r, s_1, ..., s_m \in \mathbb{N}$, and put $q_0 = 0$, $s_0 = 0$. Let $\{e_{ij}\}_{1\le i,j\le p}$, $\{f_{ij}\}_{1\le i,j\le q_1+\cdots+q_m}$, $\{g_{ij}\}_{1\le i,j\le r}$ and $\{h_{ij}\}_{1\le i,j\le s_1+\cdots+s_m}$ be systems of matrix units for the matrix algebras M_p, $M_{q_1+\cdots+q_m}$, M_r and $M_{s_1+\cdots+s_m}$, respectively.

We shall define the reduced free product C^*-algebra with amalgamation

$$(M_p \otimes M_{q_1+\cdots+q_m}, E_1) \underset{\mathbb{C}^m}{*} (M_r \otimes M_{s_1+\cdots+s_m}, E_2)$$

as follows:

(i) the unital embedding

$$i_1 : \mathbb{C}^m \hookrightarrow M_p \otimes M_{q_1+\cdots+q_m}$$

is given by

$$i_1((\lambda_l)_{l=1}^m) = \sum_{l=1}^m \lambda_l (1 \otimes \sum_{k=q_0+\cdots+q_{l-1}+1}^{q_0+\cdots+q_l} f_{kk}),$$

and $i_2 : \mathbb{C}^m \hookrightarrow M_r \otimes M_{s_1+\cdots+s_m}$ is defined analogously to i_1 by replacing p, q_* with r, s_*, respectively.

(ii) the projection of norm one

$$E_1 : M_p \otimes M_{q_1+\cdots+q_m} \to \mathbb{C}^m$$

is given by

$$E_1(\sum_{i,j,k,l} \lambda_{ijkl} e_{ij} \otimes f_{kl}) = (\frac{1}{pq_l} \sum_{i=1}^p \sum_{k=q_0+\cdots+q_{l-1}+1}^{q_0+\cdots+q_l} \lambda_{iikk})_{l=1}^m,$$

and $E_2 : M_r \otimes M_{s_1+\cdots+s_m} \to \mathbb{C}^m$ is defined analogously to E_1 by replacing p, q_* with r, s_*, respectively.

Hereafter, the notation

$$(M_p \otimes M_{q_1+\cdots+q_m}, E_1) \underset{\mathbb{C}^m}{*} (M_r \otimes M_{s_1+\cdots+s_m}, E_2)$$

means the reduced free product C^*-algebra with amalgamation defined as in Definition 1.1.

In §2, it is proved that, under the settings that

(1) $q_1 = 1, s_1 = \cdots = s_m = 1$

or

(2) $p = r = 1$ (so $M_p \cong M_r \cong \mathbb{C}$), $m = 2, q_1 = s_1 = 1$ and $q_2 = s_2$,

the C^*-algebra $(A, E) = (M_p \otimes M_{q_1+\cdots+q_m}, E_1) \underset{\mathbb{C}^m}{*} (M_r \otimes M_{s_1+\cdots+s_m}, E_2)$ is stably isomorphic to the reduced free product C^*-algebra obtained from well-known C^*-algebras such as M_n, $C(\mathbb{T})$, and the Cuntz algebra, $\mathcal{O}_k$.

In §3, we shall show that, for given $n \in \mathbb{N}$, two C^*-algebras A_1, A_2 which satisfy $A_1 \ncong A_2$ and $A_1 \otimes M_n \cong A_2 \otimes M_n$ appear in the C^*-algebra $(A, E) = (M_n \otimes M_{n+1}, E_1) \underset{\mathbb{C}^2}{*} (M_2, E_2)$. Moreover, it is proved that the C^*-algebras we exhibit here have the additional property that they are infinite and non-nuclear.

We use the following notations in the rest of sections.

Notation 1.2. Let A be a C^*-algebra and let $X_1, ..., X_n$ be subsets of A. We shall denote the set $\{x_1 \cdots x_s \mid x_i \in X_{l_i}, l_1 \neq l_2 \neq \cdots \neq l_s, s \in \mathbb{N}\}$ by $W(X_1, ..., X_n)$.

§2. Observation of the C^*-algebras defined as in Definition 1.1

2.1 Statement of main result.

First, we shall describe a basic property of reduced free product C^*-algebras with amalgamation, which will be used in this section.

Lemma 2.1.1. *Let $(A, E) = \underset{B}{*}(A_i, E_i)$ be the reduced free product with amalgamation of C^*-algebras. Suppose that E_i is faithful on A_i for each $i \in I$, and suppose that B has a faithful state φ. Then the free product projection of norm one E is faithful on A.*

Proof. This lemma is essentially proved in [6]. We can verify the desired faithfulness similarly to [6], replacing *states* in [6] with *projection of norm one* here. $\square$

Our main results are the followings.

Proposition 2.1.2. *For the C^*-algebra*

$$(A, E) = (M_p \otimes M_{q_1 + \cdots + q_m}, E_1) \underset{\mathbb{C}^m}{*} (M_r \otimes M_{s_1 + \cdots + s_m}, E_2),$$

suppose that

$$q_1 = 1, s_1 = \cdots = s_m = 1.$$

Then, for suitable states φ_l on $\mathcal{O}_{q_l} (2 \leq l \leq m)$, we have

$$A \cong ((M_p, \frac{1}{p}Tr) \underset{\mathbb{C}}{*} (M_r, \frac{1}{r}Tr) \underset{\mathbb{C}}{*} (\mathcal{O}_{q_2}, \varphi_2) \underset{\mathbb{C}}{*} \cdots \underset{\mathbb{C}}{*} (\mathcal{O}_{q_m}, \varphi_m)) \otimes M_m,$$

where $\mathcal{O}_1$ means $C(\mathbb{T})$.

In particular, A is non-nuclear when $m \geq 3$.

Proposition 2.1.3. *Let $(A, E) = (M_{1+n}, E_1) \underset{\mathbb{C}^2}{*} (M_{1+n}, E_2)$. Then*

$$A \otimes M_n \cong ((M_n, \frac{1}{n}Tr) \underset{\mathbb{C}}{*} (C(\mathbb{T}), \tau)) \otimes M_{1+n},$$

where τ is a tracial state on $C(\mathbb{T})$ given by

$$\tau(f) = \int_0^1 f(e^{2\pi i t})dt \quad \text{for } f \in C(\mathbb{T}).$$

In particular, if $n \geq 3$, we have that A is simple, non-nuclear.

2.2 Proof of Proposition 2.1.2.

Let $(A, E) = (M_p \otimes M_{q_1 + \cdots + q_m}, E_1) \underset{\mathbb{C}^m}{*} (M_r \otimes M_{s_1 + \cdots + s_m}, E_2)$ be as in Proposition 2.1.2, that is, $q_1 = 1, s_1 = \cdots = s_m = 1$. Put $\{e_{ij}\}_{1 \leq i,j \leq p}$, $\{f_{ij}\}_{1 \leq i,j \leq q_1 + \cdots + q_m}$, $\{g_{ij}\}_{1 \leq i,j \leq r}$ and $\{h_{ij}\}_{1 \leq i,j \leq s_1 + \cdots + s_m}$ be systems of matrix units for the matrix algebras M_p, $M_{q_1 + \cdots + q_m}$, M_r and $M_{s_1 + \cdots + s_m}$, respectively.

Lemma 2.2.1. *In the C^*-algebra A, we have the following identification:*

$$1 \otimes f_{11} = 1 \otimes h_{11}, \tag{2.2.1}$$
$$1 \otimes \sum_{j=q_1+\cdots+q_l+1}^{q_1+\cdots+q_{l+1}} f_{jj} = 1 \otimes h_{l+1,l+1} \quad (1 \le l \le m-1).$$

Proof. These relations arise from the unital embeddings $\mathbb{C}^m \hookrightarrow M_p \otimes M_{q_1+\cdots+q_m}$ and $\mathbb{C}^m \hookrightarrow M_r \otimes M_{s_1+\cdots+s_m}$ which we start with. $\square$

Proof of Proposition 2.1.2. We remark that

$$(1 \otimes h_{11})A(1 \otimes h_{11}) \otimes M_m \cong A. \tag{2.2.2}$$

In fact, the map

$$\Phi : (1 \otimes h_{11})A(1 \otimes h_{11}) \otimes M_m \ni ((x_{ij})_{1 \le i,j \le m}) \mapsto \sum_{1 \le i,j \le m} (1 \otimes h_{i1})x_{ij}(1 \otimes h_{1j}) \in A$$

is a *-isomorphism.

Now we shall examine the C^*-algebra $(1 \otimes h_{11})A(1 \otimes h_{11})$. Consider the C^*-subalgebras $A_1, A_2, B_1, ..., B_{m-1}$ of A and a state φ of $(1 \otimes h_{11})A(1 \otimes h_{11}) \otimes M_m$ defined as follows:

$$A_1 = C^*(\{e_{1j} \otimes f_{11} \mid 2 \le j \le p\}), \quad A_2 = C^*(\{g_{1j} \otimes h_{11} \mid 2 \le j \le r\}),$$
$$B_l = C^*(\{S_{l,k} \mid 1 \le k \le q_{l+1}\}) \quad (1 \le l \le m-1)$$
$$\text{and} \quad \varphi = \psi \circ E,$$

where $S_{l,k} = (1 \otimes h_{1,l+1})(1 \otimes f_{(q_1+\cdots+q_l)+k,1})$ and $\psi : \mathbb{C}^m \ni (\lambda_l)_{l=1}^m \mapsto \lambda_1 \in \mathbb{C}$.

From Lemma 2.2.1, it follows that

$$A_1, A_2, B_1, ..., B_{m-1} \subset (1 \otimes h_{11})A(1 \otimes h_{11}) \tag{2.2.3}$$

and

$$A_1 \cong M_p, A_2 \cong M_r, B_l \cong \mathcal{O}_{q_{l+1}} \ (1 \le l \le m-1). \tag{2.2.4}$$

In the case that $q_{l+1} \ge 2$, we can show, by a straightforward computation using Lemma 2.2.1, that the generators of B_l satisfy the relation $S_{l,k}^* S_{l,k} = 1 \otimes h_{11}$ and $\sum_{k=1}^{q_{l+1}} S_{l,k} S_{l,k}^* = 1 \otimes h_{11}$. Therefore $B_l \cong \mathcal{O}_{q_{l+1}}$. Similarly, in the case that $q_{l+1} = 1$, it is easy to see that the one generator $S_{l,1}$ is a unitary with $\varphi(S_{l,1}^n) = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$. So $B_l \cong C(\mathbb{T})$.

In addition, by Lemma 2.2.1, it immediately follows that

$$(1 \otimes h_{11})A(1 \otimes h_{11}) = C^*(A_1 \cup A_2 \cup B_1 \cup \cdots \cup B_{m-1}). \tag{2.2.5}$$

Furthermore, by Lemma 2.1.1, the state φ is faithful on $(1 \otimes h_{11})A(1 \otimes h_{11})$. Trivially, the faithfulness of φ implies the following (2.2.6).

$$\begin{aligned} &\text{If } x \in (1 \otimes h_{11})A(1 \otimes h_{11}) \text{ satisfies } \varphi(y^*x^*xy) = 0 \\ &\text{for all } y \in (1 \otimes h_{11})A(1 \otimes h_{11}), \text{ then } x = 0. \end{aligned} \tag{2.2.6}$$

We shall verify the freeness of $\{A_1, A_2, B_1, ..., B_{m-1}\}$ in $((1\otimes h_{11})A(1\otimes h_{11}), \varphi)$. Consider the following subsets of $(1\otimes h_{11})A(1\otimes h_{11})$:

$$\begin{aligned} W_{A_1} &= \{(e_{ij} - \frac{\delta_{ij}}{p}1)\otimes f_{11} \mid 1 \le i, j \le p\}, \\ W_{A_2} &= \{(g_{ij} - \frac{\delta_{ij}}{r}1)\otimes h_{11} \mid 1 \le i, j \le r\}, \\ W_{B_l} &= (1\otimes h_{11})W(\{1\otimes h_{1,l+1}, 1\otimes h_{l+1,1}\}, Y_l)(1\otimes h_{11}), \end{aligned}$$

where

$$Y_l = \left\{ 1\otimes (f_{ij} - \frac{\delta_{ij}}{q_{l+1}} \sum_{k=q_1+\cdots+q_l+1}^{q_1+\cdots+q_{l+1}} f_{kk}) \;\middle|\; \begin{array}{r} (i,j) \in \{1, q_1+\cdots+q_l+1, ..., \\ q_1+\cdots+q_l+q_{l+1}\}^2 \\ \setminus \{(1,1)\} \end{array} \right\}.$$

Then we have that

$$A_1 \cap \ker\varphi \subset \mathrm{span} W_{A_1}, \quad A_2 \cap \ker\varphi \subset \mathrm{span} W_{A_2},$$

$$B_l \cap \ker\varphi \subset \text{ the norm closure of } (\mathrm{span} W_{B_l}) \quad (1 \le l \le m-1).$$

Therefore, to verify the desired freeness, it suffices to show that

$$W(W_{A_1}, W_{A_2}, W_{B_1}, ..., W_{B_{m-1}}) \subset W(\ker E_1, \ker E_2) (\subset \ker E \subset \ker\varphi).$$

We define $X_1, X_2, V_{A_1}, V_{A_2}, V_{B_l}$ $(l = 1, \ldots, m-1)$ as follows:

$$\begin{aligned} X_1 &= \left\{ x\otimes y \;\middle|\; \begin{array}{l} x = e_{ij} - \frac{\delta_{ij}}{p}1 \ (1\le i,j\le p) \text{ or } 1 \\ 1\otimes y \in \cup_{i=1}^{m-1} Y_l \end{array} \right\}, \\ X_2 &= \left\{ x\otimes y \;\middle|\; \begin{array}{l} x = g_{ij} - \frac{\delta_{ij}}{r}1 \ (1\le i,j\le r) \text{ or } 1 \\ y = h_{ij} \ (1\le i,j\le m,\ i\ne j) \text{ or } h_{11} \end{array} \right\}, \\ V_{A_1} &= \left\{ w \in W(X_1, X_2) \;\middle|\; \begin{array}{l} w \text{ ends with } (e_{ij} - \frac{\delta_{ij}}{p}1)\otimes f_{k1} \\ (1\le i,j\le p,\ 1\le k\le q_1+\cdots+q_m) \end{array} \right\}, \\ V_{A_2} &= \left\{ w \in W(X_1, X_2) \;\middle|\; \begin{array}{l} w \text{ ends with } (g_{ij} - \frac{\delta_{ij}}{r}1)\otimes h_{k1} \\ (1\le i,j\le r,\ 1\le k\le m) \end{array} \right\}, \\ V_{B_l} &= \left\{ w \in W(X_1, X_2) \;\middle|\; \begin{array}{l} w \text{ ends with } 1\otimes h_{l+1,1} \text{ or} \\ 1\otimes f_{q_1+\cdots q_l+k,1} \ (1\le k\le q_{l+1}) \end{array} \right\}. \end{aligned}$$

By simple inspections, we can show that

$$W_{A_1} \subset V_{A_1}, W_{A_2} \subset V_{A_2}, W_{B_l} \subset V_{B_1} \ (1 \le l \le m-1) \tag{2.2.7}$$

and

$$V_S W_T \subset V_T, \text{ if } S \ne T,\ (S,T) \in \{A_1, A_2, B_1, ..., B_{m-1}\}^2. \tag{2.2.8}$$

It is clear that (2.2.7) and (2.2.8) imply

$$W(W_{A_1}, W_{A_2}, W_{B_1}, ..., W_{B_{m-1}}) \subset W(X_1, X_2). \tag{2.2.9}$$

On the other hand, from the definitions of X_1 and X_2, we get the inclusion

$$W(X_1, X_2) \subset W(\ker E_1, \ker E_2)(\subset \ker E \subset \ker \varphi). \tag{2.2.10}$$

Then, by (2.2.9) and (2.2.10), we can conclude that

$$\{A_1, A_2, B_1, ..., B_{m-1}\} \text{ is free in } ((1 \otimes h_{11})A(1 \otimes h_{11}), \varphi). \tag{2.2.11}$$

As a consequence of (2.2.3), (2.2.5), (2.2.6) and (2.2.11), we get

$$((1 \otimes h_{11})A(1 \otimes h_{11}), \varphi) \cong (A_1, \varphi|_{A_1}) \underset{\mathbb{C}}{*} (A_2, \varphi|_{A_2}) \underset{\mathbb{C}}{*} (B_1, \varphi|_{B_1}) \underset{\mathbb{C}}{*} \cdots \underset{\mathbb{C}}{*} (B_{m-1}, \varphi|_{B_{m-1}}). \tag{2.2.12}$$

Then, combining (2.2.2), (2.2.4) and (2.2.12), we can get the desired *-isomorphism.

Now we shall show that A is non-nuclear when $m \geq 3$. We only treat the case that $q_2 \geq 2, q_3 \geq 2$. The proof for the other cases are almost similar. Note that there is a projection of norm one from A onto $((\mathcal{O}_{q_2}, \varphi_2) \underset{\mathbb{C}}{*} (\mathcal{O}_{q_3}, \varphi_3)) \otimes M_m$. In addition, using the *-isomorphism

$$(\mathcal{O}_{q_2}, \varphi_2) \underset{\mathbb{C}}{*} (\mathcal{O}_{q_3}, \varphi_3) \cong (\mathcal{O}_{q_2} \otimes M_{q_2}, \varphi_2 \otimes \frac{1}{q_2} Tr) \underset{\mathbb{C}}{*} (\mathcal{O}_{q_3} \otimes M_{q_3}, \varphi_3 \otimes \frac{1}{q_3} Tr),$$

we can naturally construct a projection of norm one from the C^*-algebra $(\mathcal{O}_{q_2}, \varphi_2) \underset{\mathbb{C}}{*} (\mathcal{O}_{q_3}, \varphi_3)$ onto $(M_{q_2}, \frac{1}{q_2} Tr) \underset{\mathbb{C}}{*} (M_{q_3}, \frac{1}{q_3} Tr)$. Then, since $(M_{q_2}, \frac{1}{q_2} Tr) \underset{\mathbb{C}}{*} (M_{q_3}, \frac{1}{q_3} Tr)$ is non-nuclear (see [4],[5]), it follows that A is non-nuclear. $\square$

Remark 2.2.2. In [9], K. Dykema proved that every reduced free product of exact C^*-algebras with amalgamation is exact. Therefore, the C^*-algebra A in Proposition 2.1.2 is exact.

2.3 Proof of Proposition 2.1.3.

Let $(A, E) = (M_{1+n}, E_1) \underset{\mathbb{C}^2}{*} (M_{1+n}, E_2)$ be as in Proposition 2.1.3, and put $\{e_{ij}\}_{1 \leq i,j \leq 1+n}$, $\{f_{ij}\}_{1 \leq i,j \leq 1+n}$ be systems of matrix units of M_{1+n}, M_{1+n}, respectively.

Lemma 2.3.1. *The C^*-algebra A has the relation*

$$\begin{aligned} e_{11} &= f_{11}, \\ e_{22} + \cdots + e_{1+n,1+n} &= f_{22} + \cdots + f_{1+n,1+n}. \end{aligned} \tag{2.3.1}$$

Proof. The relation (2.3.1) arises from the embeddings $i_1 : \mathbb{C}^2 \hookrightarrow M_{1+n}$ and $i_2 : \mathbb{C}^2 \hookrightarrow M_{1+n}$ which we start with. $\square$

Proof of Proposition 2.1.3. We remark that

$$\begin{aligned} &(e_{11} A e_{11}) \otimes M_{1+n} \cong A \text{ and} \\ &(e_{11} A e_{11}) \otimes M_n \cong (e_{22} + \cdots + e_{1+n,1+n}) A (e_{22} + \cdots + e_{1+n,1+n}). \end{aligned} \tag{2.3.2}$$

We shall examine the C^*-algebra $(e_{22}+\cdots+e_{1+n,1+n})A(e_{22}+\cdots+e_{1+n,1+n})$. Consider the C^*-subalgebras A_1, A_2 of A and a state φ on $(e_{22}+\cdots+e_{1+n,1+n})A(e_{22}+\cdots+e_{1+n,1+n})$ defined as follows:

$$A_1 = C^*(\{e_{2j} \mid 3 \le j \le 1+n\}), \quad A_2 = C^*(\sum_{i=2}^{1+n} e_{i1}f_{1i})$$

and

$$\varphi = \psi \circ E,$$

where $\psi : \mathbb{C}^2 \ni (\lambda_1, \lambda_2) \mapsto \lambda_2 \in \mathbb{C}$.

From Lemma 2.3.1, it follows that

$$A_1, A_2 \subset (e_{22}+\cdots+e_{1+n,1+n})A(e_{22}+\cdots+e_{1+n,1+n}) \tag{2.3.3}$$

and

$$A_1 \cong M_n, \quad A_2 \cong C(\mathbb{T}). \tag{2.3.4}$$

By a straightforward computation using Lemma 2.3.1, we can show that the one generator $u = \sum_{i=2}^{1+n} e_{i1}f_{1i}$ is a unitary with $\varphi(u^n) = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$. Therefore, $A_2 \cong C(\mathbb{T})$.

In addition, by Lemma 2.3.1, it immediately follows that

$$(e_{22}+\cdots+e_{1+n,1+n})A(e_{22}+\cdots+e_{1+n,1+n}) = C^*(A_1 \cup A_2). \tag{2.3.5}$$

Furthermore, by Lemma 2.1.1, the state φ is faithful on the C^*-algebra

$$(e_{22}+\cdots+e_{1+n,1+n})A(e_{22}+\cdots+e_{1+n,1+n}).$$

Trivially, the faithfulness of φ implies the following (2.3.6).

(2.3.6) If $x \in (e_{22}+\cdots+e_{1+n,1+n})A(e_{22}+\cdots+e_{1+n,1+n})$ satisfies $\varphi(y^*x^*xy) = 0$ for all $y \in (e_{22}+\cdots+e_{1+n,1+n})A(e_{22}+\cdots+e_{1+n,1+n})$, then $x = 0$.

Now we shall verify the freeness of $\{A_1, A_2\}$ in $((e_{22}+\cdots+e_{1+n,1+n})A(e_{22}+\cdots+e_{1+n,1+n}), \varphi)$. Consider the following subsets of $(e_{22}+\cdots+e_{1+n,1+n})A(e_{22}+\cdots+e_{1+n,1+n})$.

$$W_{A_1} = \{e_{ij} - E(e_{ij}) \mid 2 \le i, j \le 1+n\},$$
$$W_{A_2} = \{(\sum_{i=2}^{1+n} e_{i1}f_{1i})^m \mid m \in \mathbb{Z} \setminus \{0\}\}.$$

Then we have that

$$A_l \cap \ker\varphi \subset \operatorname{span} W_{A_l} \quad (l = 1, 2).$$

Therefore, to verify the desired freeness, it suffices to show that

$$W(W_{A_1}, W_{A_2}) \subset \operatorname{span} W(\ker E_1, \ker E_2) (\subset \ker E \subset \ker\varphi).$$

Let $W_{e,f}$, $W_{f,e}$, W_e and W_f be as follows:

$$\begin{aligned}
W_{e,f} &= \{(\sum_{i=2}^{1+n} e_{i1}f_{1i})^m \mid m \in \mathbb{N}\} \cup \{e_{i1}f_{1j} \mid 2 \le i,j \le 1+n,\ i \neq j\},\\
W_{f,e} &= \{(\sum_{i=2}^{1+n} f_{i1}e_{1i})^m \mid m \in \mathbb{N}\} \cup \{f_{i1}e_{1j} \mid 2 \le i,j \le 1+n,\ i \neq j\},\\
W_f &= \{f_{ij} - E(f_{ij}) \mid 2 \le i,j \le 1+n\},\\
W_e &= \{e_{ij} - E(e_{ij}) \mid 2 \le i,j \le 1+n\}.
\end{aligned}$$

We define V a set of all elements $x_1 \cdots x_m$, where $m \in \mathbb{N}$, $x_j \in W_{e,f} \cup W_{f,e} \cup W_e \cup W_f$ $(1 \le j \le m)$ which satisfies the following condition (*):

(*) for $2 \le j \le m-1$ and $(s,t) = (e,f)$ or (f,e),

$$\begin{aligned}
&x_j \in W_{s,t} \text{ if and only if } x_{j-1} \in W_s \text{ and } x_{j+1} \in W_t, \text{ and}\\
&x_j \in W_s \text{ if and only if } x_{j-1} \in W_{s,t} \cup W_t \text{ and } x_{j+1} \in W_{t,s} \cup W_t.
\end{aligned}$$

Moreover, define V_{A_1}, V_{A_2} by

$$V_{A_1} = \left\{ w \in V \;\middle|\; \begin{array}{l} w \text{ ends with } e_{ij} - E(e_{ij})\ (2 \le i,j \le 1+n) \text{ or} \\ f_{i1}e_{1j}\ (2 \le i,j \le 1+n,\ i \neq j) \end{array} \right\},$$

$$V_{A_2} = \left\{ w \in V \;\middle|\; \begin{array}{l} w \text{ ends with } (\sum_{k=2}^{1+n} e_{k1}f_{1k})^m\ (m \in \mathbb{Z} \setminus \{0\}) \text{ or} \\ f_{ij} - E(f_{ij})\ (2 \le i,j \le 1+n) \text{ or} \\ e_{i1}f_{1j}\ (2 \le i,j \le 1+n,\ i \neq j) \end{array} \right\}.$$

By simple inspections, we can show that

$$W_{A_1} \subset V_{A_1}, W_{A_2} \subset V_{A_2} \tag{2.3.7}$$

and

$$V_S W_T \subset V_T, \text{ if } S \neq T,\ (S,T) \in \{A_1, A_2\}^2. \tag{2.3.8}$$

It is clear that (2.3.7) and (2.3.8) imply

$$W(W_{A_1}, W_{A_2}) \subset V. \tag{2.3.9}$$

On the other hand, from the definition of V, we get the inclusion

$$V \subset \operatorname{span} W(\ker E_1, \ker E_2) (\subset \ker E \subset \ker \varphi). \tag{2.3.10}$$

Then, by (2.3.9) and (2.3.10), we can conclude that

$$\{A_1, A_2\} \text{ is free in } ((e_{22} + \cdots + e_{1+n,1+n})A(e_{22} + \cdots + e_{1+n,1+n}), \varphi). \tag{2.3.11}$$

As a consequence of (2.3.3), (2.3.5), (2.3.6) and (2.3.11), we get

$$((e_{22} + \cdots + e_{1+n,1+n})A(e_{22} + \cdots + e_{1+n,1+n}), \varphi) \cong (A_1, \varphi|_{A_1}) \underset{\mathbb{C}}{*} (A_2, \varphi|_{A_2}). \tag{2.3.12}$$

Then, combining (2.3.2), (2.3.4) and (2.3.12) we can easily get the desired *-isomorphism.

Since the C^*-algebra $(M_n, \frac{1}{n}Tr) \underset{\mathbb{C}}{*} (C(\mathbb{T}), \tau)$ is non-nuclear (see [4],[5]), it is trivial that A is non-nuclear.

Now we shall investigate the simplicity of A. By [2, Proposition 3.1], the C^*-algebra $(M_n, \frac{1}{n}Tr) \underset{\mathbb{C}}{*} (C(\mathbb{T}), \tau)$ is simple. Therefore, it immediately follows that A is simple. $\square$

Remark 2.3.2. As stated in Remark 2.2.2, it is known that the C^*-algebra A in Proposition 2.1.3 is exact.

Remark 2.3.3. In [13], K. McClanahan gives a sufficient condition for simplicity of reduced free product C^*-algebras with amalgamation. But the C^*-algebra A in Proposition 2.1.3 does not satisfy McClanahan's condition. Therefore, we cannot determine whether A is simple or not from his condition.

§3. Related topics

Let $n \in \mathbb{N}$. Here we shall give a pair of C^*-algebras A, B such that $A \not\cong B$ and $A \otimes M_n \cong B \otimes M_n$.

Proposition 3.1. *Let $(A, E) = (M_n \otimes M_{n+1}, E_1) \underset{\mathbb{C}^2}{*} (M_2, E_2)$ be as in Definition 1.1, and let $\{e_{ij}\}_{1 \le i,j \le n}$, $\{f_{ij}\}_{1 \le i,j \le n}$ and $\{g_{ij}\}_{1 \le i,j \le 2}$ be systems of matrix units of M_n, M_{n+1} and M_2, respectively. Then, for C^*-algebras $A_1 = (e_{11} \otimes f_{11})A(e_{11} \otimes f_{11})$, $A_2 = A_1 \otimes M_n$, we have $A_1 \not\cong A_2$ and $A_1 \otimes M_n \cong A_2 \otimes M_n$.*

In the proof of Proposition 3.1, the following lemma, proved by E. Germain, is used.

Lemma 3.2 ([10], [11]). *Given unital C^*-algebras A_1 and A_2 with states φ_1 and, respectively, φ_2, whose GNS-representations are faithful, let $(A, \varphi) = (A_1, \varphi_1) \underset{\mathbb{C}}{*} (A_2, \varphi_2)$ be the corresponding reduced free product. Suppose that A_1 and A_2 are nuclear. Then there is an exact sequence of K-groups,*

$$\begin{array}{ccccc}
\mathbb{Z} \cong K_0(\mathbb{C}) & \xrightarrow{(K_0(i_1), -K_0(i_2))} & K_0(A_1) \oplus K_0(A_2) & \xrightarrow{K_0(j_1)+K_0(j_2)} & K_0(A) \\
\uparrow & & & & \downarrow \\
K_1(A) & \xleftarrow{K_1(j_1)+K_1(j_2)} & K_1(A_1) \oplus K_1(A_2) & \xleftarrow{(K_1(i_1), -K_1(i_2))} & K_1(\mathbb{C}) = 0,
\end{array}$$

*where $i_k : \mathbb{C} \to A_k$ is the unital *-homomorphism and where $j_k : A_k \to A$ is the unital embedding arising from the construction of the reduced free product (A, φ).*

Proof of Proposition 3.1. First, we shall show that $A_1 \otimes M_n \cong A_2 \otimes M_n$. Note that there is a relation in A given by

$$1 \otimes f_{11} = g_{11} \text{ and } 1 \otimes (f_{22} + \cdots + f_{n+1,n+1}) = g_{22}.$$

Since the map

$$A_1 \otimes M_n \ni (x_{ij})_{1 \le i,j \le n} \mapsto \sum_{1 \le i,j \le n} (e_{i1} \otimes f_{11}) x_{ij} (e_{1j} \otimes f_{11}) \in (1 \otimes f_{11}) A (1 \otimes f_{11})$$

is a *-isomorphism, we get $A_1 \otimes M_n \cong g_{11}Ag_{11}$. Similarly, we can construct a *-isomorphism between $A_2 \otimes M_n$ and $g_{11}Ag_{11}$ by

$$\begin{aligned} A_2 \otimes M_n &\cong A_1 \otimes M_n \otimes M_n \cong (1 \otimes f_{11})A(1 \otimes f_{11}) \otimes M_n \\ &\cong (1 \otimes (f_{22} + \cdots + f_{n+1,n+1}))A(1 \otimes (f_{22} + \cdots + f_{n+1,n+1})) \\ &\cong g_{22}Ag_{22} \cong g_{11}Ag_{11}. \end{aligned}$$

Therefore, $A_1 \otimes M_n \cong A_2 \otimes M_n$.

Next, we prove that $A_1 \not\cong A_2$ by investigating $K_0(A_1)$ and $K_0(A_2)$. Define $\iota_1 : A_1 \to A_2 = A_1 \otimes M_n$ and $\iota_2 : A_2 \to (1 \otimes f_{11})A(1 \otimes f_{11})$ by

$$\iota_1(x) = \mathrm{diag}(x, 0, ..., 0) \quad \text{ for } x \in A_1,$$

$$\iota_2((x_{ij})_{1 \le i,j \le n}) = \sum_{1 \le i,j \le n} (e_{i1} \otimes f_{11})x_{ij}(e_{1j} \otimes f_{11})$$

for $(x_{ij})_{1 \le i,j \le n} \in A_2$, and $\iota_3 : (1 \otimes f_{11})A(1 \otimes f_{11}) \to (M_n, \tau_n) * (\mathcal{O}_n, \varphi)$ be the *-isomorphism constructed in Proposition 2.1.2.

Then it is clear that

$$[\iota_3 \circ \iota_2 \circ \iota_1](1_{A_1}) = j_1(\mathrm{diag}(1, 0, ..., 0)) \text{ and } [\iota_3 \circ \iota_2](1_{A_2}) = j_1(\mathrm{diag}(1, ..., 1)), \tag{3.1}$$

where $j_1 : M_n \hookrightarrow (M_n, \tau_n) \underset{\mathbb{C}}{*} (\mathcal{O}_n, \varphi)$ be the unital embedding arising from the construction of the reduced free product.

According to Lemma 3.2, we can get the following exact sequence of K-groups

$$K_0(\mathbb{C}) \xrightarrow{(K_0(i_1), -K_0(i_2))} K_0(M_n) \oplus K_0(\mathcal{O}_n) \xrightarrow{K_0(j_1)+K_0(j_2)} K_0(B) \longrightarrow 0, \tag{3.2}$$

where $B = (M_n, \tau_n) \underset{\mathbb{C}}{*} (\mathcal{O}_n, \varphi)$.

Through the group isomorphisms $K_0(M_n) \cong \mathbb{Z}$ ($[1] \mapsto n$), $K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$ ($[1] \mapsto [1]$) and $K_0(\iota_3 \circ \iota_2) : K_0(A_2) \cong K_0(B)$, we can obtain the following exact sequence from (3.2).

$$\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}/(n-1)\mathbb{Z} \longrightarrow K_0(A_2) \longrightarrow 0. \tag{3.3}$$

So, from (3.1) and (3.3), it follows that

$$\exists \varphi_2 : K_0(A_2) \cong \mathbb{Z}/(n^2 - n)\mathbb{Z} \ \text{ such that } \varphi_2([1]) = [n]. \tag{3.4}$$

On the other hand, using the group isomorphism $K_0(\iota_1)K_0(A_1) \cong K_0(A_2)$ with (3.1) and (3.4), we have

$$\exists \varphi_1 : K_0(A_1) \cong \mathbb{Z}/(n^2 - n)\mathbb{Z} \ \text{ such that } \varphi_1([1]) = [1]. \tag{3.5}$$

Then, by (3.4) and (3.5), we can conclude $A_1 \not\cong A_2$. □

Remark 3.3. The C^*-algebras A_1, A_2 in Proposition 3.1 are non-nuclear, (See Proposition 2.1.2.) and their K_0-group is $\mathbb{Z}/(n^2 - n)\mathbb{Z}$, their K_1-group is trivial. We remark that there is a pair of nuclear C^*-algebras B_1, B_2 such that $B_1 \not\cong B_2$, $B_1 \otimes M_n \cong B_2 \otimes M_n$ and their K_0-group, K_1-group are the same as those of A_1, respectively. In fact, if we put $B_1 = \mathcal{O}_{n^2-n+1}, B_2 = \mathcal{O}_{n^2-n+1} \otimes M_n$, then B_1, B_2 have all the desired properties. (See [12].)

Proposition 3.4. *Let* $(A, E) = (M_{q_1+q_2}, E_1) \underset{\mathbb{C}^2}{*} (M_2, E_2)$ *be as in Definition 1.1. Put* $\{e_{ij}\}_{1\le i,j\le q_1+q_2}$, $\{f_{ij}\}_{1\le i,j\le 2}$ *be systems of matrix units in* $M_{q_1+q_2}$, M_2, *respectively. If* $q_1 < q_2$, *then the* C^**-algebra* $e_{11}Ae_{11}$ *is infinite.*

In particular, the C^**-algebras* A_1, A_2 *which appear in Proposition 3.1 are infinite.*

To prove Proposition 3.4, we need some lemmas.

Lemma 3.5 ([1]). *Let* A *be a* C^**-algebra and let* p, q *be projections with* $\|p - pqp\| < 1$. *Then* p *is equivalent to a subprojection* q' *of* q, *which is given by*

$$q' = q(p + \sum_{n=1}^{\infty}(p - pqp)^n)q.$$

Lemma 3.6 ([1]). *Suppose* A *is a simple, unital* C^**-algebra containing nontrivial projections* p *and* q. *If* A *is generated by* p, q *and some other positive elements each of which is orthogonal to either* p *or* q, *then* $\|q(1-p)q - q\| < 1$.

Lemma 3.7 ([8]). *Consider the reduced free product*

$$(A, \varphi) = (\mathbb{C}^n, \tau_{\alpha_1,\dots,\alpha_n}) \underset{\mathbb{C}}{*} (\mathbb{C}^m, \tau_{\beta_1,\dots,\beta_n}),$$

where $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n > 0$, $\sum_{i=1}^n \alpha_i = \sum_{i=1}^m \beta_i = 1$, *and* $\tau_{\alpha_1,\dots,\alpha_n}$ *and* $\tau_{\beta_1,\dots,\beta_n}$ *are defined by*

$$\tau_{\alpha_1,\dots,\alpha_n}((\lambda_i)_{i=1}^n) = \sum_{i=1}^n \alpha_i \lambda_i,$$
$$\tau_{\beta_1,\dots,\beta_n}((\lambda_i)_{i=1}^n) = \sum_{i=1}^n \beta_i \lambda_i.$$

If $\alpha_i + \beta_j < 1$ *for all* $(i, j) \in \{1, \dots, n\} \times \{1, \dots, m\}$, *then* A *is simple.*

Proof of Proposition 3.4. We shall verify the infiniteness of $e_{11}Ae_{11}$ by constructing a proper subprojection of e_{11} which is equivalent to e_{11}. Consider the C^*-subalgebra B of A which is generated by mutually orthogonal projections $f_{21}e_{11}f_{12}$, $f_{21}(e_{22} + \cdots + e_{q_1q_1})f_{12}$, $e_{q_1+1,q_1+1}, \dots, e_{q_1+q_2,q_1+q_2}$. Moreover, define a state φ on B by $\varphi = \psi \circ E$, where $\psi : \mathbb{C}^2 \ni (\lambda_1, \lambda_2) \mapsto \lambda_2 \in \mathbb{C}$. We remark that, by Lemma 2.1.1, we have that φ is faithful on B.

It is easy to show that

$$\begin{aligned}
(3.6) \qquad & (B, \varphi) \\
& \cong (C^*(f_{21}e_{11}f_{12}, f_{21}(\sum_{j=2}^{q_1} e_{jj})f_{12}), \varphi) \underset{\mathbb{C}}{*} (C^*(e_{q_1+1,q_1+1}, \dots, e_{q_1+q_2,q_1+q_2}), \varphi) \\
& \cong (\mathbb{C}^2, \tau_{\frac{1}{q_1}, 1-\frac{1}{q_1}}) \underset{\mathbb{C}}{*} (\mathbb{C}^{q_2}, \tau_{\frac{1}{q_2},\dots,\frac{1}{q_2}}).
\end{aligned}$$

Furthermore, by Lemma 3.7, the right side hand of (3.6) is a simple C^*-algebra. Therefore, B is simple.

Then applying Lemma 3.6 on B, we get the estimation

$$\|e_{q_1+1,q_1+1} \cdot f_{21}e_{11}f_{12} \cdot e_{q_1+1,q_1+1}\| < 1.$$

Therefore, using Lemma 3.5, we can construct a subprojection q of $f_{21}e_{11}f_{12}$ which is equivalent to e_{q_1+1,q_1+1}.

Moreover, the projection q is not equal to e_{11}. In fact, since

$$\varphi(q) = \varphi(e_{q_1+1,q_1+1}) = \frac{1}{q_2} \text{ and } \varphi(f_{21}e_{11}f_{12}) = \frac{1}{q_1},$$

we have $q \neq e_{11}$ from the assumption $q_1 \neq q_2$.

Finally, define $p = e_{11}f_{12}qf_{21}e_{11}$. From the above arguments about the projection q, we can easily show that p is a proper subprojection of e_{11} which is equivalent to e_{11}. Therefore, $e_{11}Ae_{11}$ is infinite. $\square$

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