SOME PROBLEMS AND COUNTER-EXAMPLES ON BCI (BCK)-ALGEBRAS

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Abstract. In this note we first show that the answers of the open problems posed in the paper [3], are positive. Then we give some counter-examples of Theorems 6 of [4], 2.3 of [5] and 1 of [8].

A BCI-algebra is a non-empty set X with a binary operation $*$ and a constant 0 satisfying the axioms:

(1) $\{(x*y)*(x*z)\}*(z*y)=0$,

(2) $\{x*(x*y)\}*y=0$,

(3) $x*x=0$,

(4) $x*y=0$ and $y*x=0$ imply that $x=y$, for all $x,y,z\in X$.

A BCI-algebra X satisfying (5) $0*x=0$ for all $x\in X$, is called a BCK-algebra.

From now on X is a BCI-algebra.

Definition 1. A non-empty subset A of X is called an ideal if

(i) $0\in A$

(ii) $x*y\in A$ and $y\in A$ imply that $x\in A$, for all $x,y\in X$.

Definition 2. An ideal A of X is said to be closed if $0*x\in A$, for all $x\in A$.

Notation. For any elements x,y in X and positive integer n, let us write $x*y^n$ for $(\ldots((x*y)*y)*\ldots)*y$, where y occurs n times.

Definition 3. An element x in X is said to be a nilpotent element if $0*x^n=0$, for some positive integer n.

Definition 4. Let A be any non-empty subset of X. Then for any positive integer k, we define

$$N_k(A)=\{x\in A: 0*x^k=0\},$$

and

$$N(A)=\{x\in A: 0*x^n=0,\ \text{ for some } k\in \mathbf{N}\}.$$

Open problems ([3]). (1) Is there an infinite BCI-algebra X such that $\{0\}\subset N(X)\subset X$?

(2) Are there an infinite BCI-algebra X and an ideal A of X such that $\{0\}\subset N(A)\subset A\subset X$?

Affirmative answers 5. Let $\mathbf{C}^\circ=\mathbf{C}\backslash\{0\}$, where $\mathbf{C}$ is the set of all complex numbers.

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Define $*$ on $\mathbf{C}^\circ$ as division as general. Then as it has mentioned in [2], $(\mathbf{C}^\circ, *, 1)$ is an infinite BCI-algebra. Consider a subset $\mathbf{R}^\circ = \mathbf{R}\backslash\{0\}$ of $\mathbf{C}^\circ$, where $\mathbf{R}$ is the set of all real numbers. Then $(\mathbf{R}^\circ, *, 1)$ is a closed ideal of $\mathbf{C}^\circ$. Now it is easy to check that

$$N(\mathbf{C}^\circ) = \{z \in \mathbf{C}^\circ : z^k = 1, \text{ for some } k \in \mathbf{N}\}.$$

Thus $\{1\} \subset N(\mathbf{C}^\circ) \subset \mathbf{C}^\circ$. Note that here 1 is the zero of the BCI-algebra $(\mathbf{C}^\circ, *, 1)$.

On the other hand we can see that

$$N(\mathbf{R}^\circ) = \{1, -1\}.$$

So

$$\{1\} \subset \{1, -1\} \subset \mathbf{R}^\circ \subset \mathbf{C}^\circ.$$

Hence the answeres are complete.

At present we give some counter-examples which shows that Theorems 6 of [4], 2.3 of [5] and 1 of [8] are not correct in general.

Theorem 6 ([8, Theorem 4]). Any Abelian group X is a p-semisimple BCI-algebra under the operation $-$, that is

$$x * y = x - y, \quad \forall x, y \in X.$$

Counter-example 7. Consider the Abelian group $\mathbf{Z}_2[x]$, of all polynomials with cofficients in $\mathbf{Z}_2$. Now consider the BCI-algebra $(\mathbf{Z}_2[x], *, 0)$, where $f * g$ means $f - g$ in $\mathbf{Z}_2[x]$ (see Theorem 6). Since $0 * f^2 = (0 - f) - f = 0 - 2f = 0$, for all $f \in \mathbf{Z}_2[x]$, then each element of $\mathbf{Z}_2[x]$ is nilpotent, that is

$$N(\mathbf{Z}_2[x]) = \mathbf{Z}_2[x],$$

but $(\mathbf{Z}_2[x], *, 0)$ is not a finite BCI-algebra. Thus Theorem 6 of [4] is not correct.

Open problem 8. If $N(X) = X$, then under what conditions is X finite?

Lemma 9 ([7, Theorem 1]). Let X be a BCI-algebra. Then

$$N_k(X) = \{x \in X | 0 * x^k = 0\}$$

is a subalgebra of X for each $k \in \mathbf{N}$.

Theorem 10. (An answer for the open problem 8) Let X be a BCI-algebra such that $N(X) = X$. If there is a positive integer n such that $|N_k(X)| \leq n$ for all $k \in \mathbf{N}$, then X is finite.

Proof. Assume that $N(X) = X$ and there exists $n \in \mathbf{N}$ such that $|N_k(X)| \leq n$ for all $K \in \mathbf{N}$. Then $X = \bigcup_{k=1}^{\infty} N_k(X)$. Let $x \in N_k(X)$ for $k \in \mathbf{N}$. Then $0 * x, 0 * x^2, \ldots, 0 * x^{n+1} \in N_k(X)$ since $N_k(X)$ is a subalgebra of X (see Lemma 9). It follows from $|N_k(X)| \leq n$ that there exists $s, r \in \mathbf{N}$ such that $0 \leq r \leq s \leq n + 1$ and $0 * x^s = 0 * x^r$. Hence we have

$$\begin{aligned} 0 &= (0 * x^s) * (0 * x^r) \\ &= ((0 * x^{s-1}) * x) * ((0 * x^{r-1}) * x) \\ &\leq (0 * x^{s-1}) * (0 * x^{r-1}), \end{aligned}$$

and so $(0 * x^{s-1}) * (0 * x^{r-1}) = 0$ because $(0 * x^{s-1}) * (0 * x^{r-1})$ is an atom of X. Continuning this process one can shows that $0 * x^{s-r} = 0$, i.e., $x \in N_{s-r}(X)$ where $0 \leq s - r \leq n + 1$.

Therefore $N_k(X) \subseteq N_{s-r}(X) \subseteq \bigcup_{k=1}^{n+1} N_k(X)$ and thus $X = \bigcup_{k=1}^{n+1} N_k(X)$. This means that X is finite.Λ

The following example shows that the condition $t_1 < \mu(x) < t_2$ must replaced by $t_1 \leq \mu(x) < t_2$ in Theorem 2.3 of [5]. For more details see [7].

Counter-example 11. Let X be a BCK-algebra. Define the fuzzy subset μ of X as follows:

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise, } \forall x \in X. \end{cases}$$

Then $\mu_1 = \{0\}$ and $\mu_0 = X$, thus μ is a fuzzy ideal of X and there is not $x \in X$ such that $0 < \mu(x) < 1$.

Finally we give a counter-example of Theorem 1 of [8].

Counter-example 12. Let $X = \{0, a, b, 1\}$. Consider the following table

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
1	1	b	a	0

Then according to [6], $(X, *, 0)$ is a BCK-algebra. Define the fuzzy subset $\mu : X \longrightarrow [0, 1]$ as follows:

$$\mu(x) = \begin{cases} \frac{1}{2} & \text{if } x = 1 \\ 1 & \text{otherwise.} \end{cases}$$

Then for any $t \in [0, 1], t \leq \mu(1)$, we have $\mu_t = X$, and hence μ_t is a dual ideal of X. But μ is not a fuzzy dual ideal of X, because $\mu(1) < \mu(x)$, $\forall x \in X$, $x \neq 1$.

Now we give a correct version of Theorem 1 of [8].

Theorem 13. Let X be a BCK-algebra and μ be a fuzzy subset of X. Then μ is a fuzzy dual ideal of X if and only if μ_t is a dual ideal of X, for all non-empty level subset μ_t of μ.

Proof. The proof is not difficult.

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