A FURTHER GENERALIZATION OF PARANORMAL OPERATORS

TAKEAKI YAMAZAKI AND MASAHIRO YANAGIDA

ABSTRACT. As a further generalization of paranormal operators, we shall introduce a new class "absolute-(p, r)-paranormal" operators for $p > 0$ and $r > 0$ such that $\||T|^p|T^*|^r x\|^r \geq \||T^*|^r x\|^{p+r}$ for every unit vector x. And we shall show several properties on absolute-(p, r)-paranormal operators as generalizations of the results on absolute-k-paranormal and p-paranormal operators introduced in [10] and [6], respectively.

1. INTRODUCTION

In this paper, an operator means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and also T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible.

An operator T is said to be hyponormal if $T^*T \geq TT^*$. As extensions of it, p-hyponormal and log-hyponormal operators are defined. An operator T is said to be p-hyponormal for $p > 0$ if $(T^*T)^p \geq (TT^*)^p$, and T is said to be log-hyponormal if T is invertible and $\log T^*T \geq \log TT^*$. It is easily seen that every p-hyponormal operator is q-hyponormal for $p \geq q > 0$ by the celebrated Löwner-Heinz theorem "$A \geq B \geq 0$ *ensures* $A^\alpha \geq B^\alpha$ *for any* $\alpha \in [0, 1]$," and every invertible p-hyponormal operator for $p > 0$ is log-hyponormal since $\log t$ is an operator monotone function.

On the other hand, T is said to be paranormal if

$$(1.1) \qquad \|T^2 x\| \geq \|Tx\|^2 \quad \text{for every unit vector } x.$$

Paranormal operators have been studied by many researchers, for example, [4], [8] and [11]. Particularly, Ando [4] showed that every log-hyponormal operator is paranormal. Afterward, in [10], we gave another simplified proof of this result by introducing class A as a new class of operators given by an operator inequality. In fact, T belongs to class A if

$$|T^2| \geq |T|^2,$$

where $|T| = (T^*T)^{\frac{1}{2}}$, and we showed that every log-hyponormal operator belongs to class A and every class A operator is paranormal.

We introduced class $A(k)$ and absolute-k-paranormal operators for $k > 0$ in [10] as generalizations of class A and paranormal operators, respectively. T belongs to class $A(k)$ if

$$(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2,$$

and T is said to be absolute-k-paranormal if

$$(1.2) \qquad \||T|^k Tx\| \geq \|Tx\|^{k+1} \quad \text{for every unit vector } x.$$

1991 *Mathematics Subject Classification.* Primary 47B20, 47A30, 47A63.

Key words and phrases. Paranormal operator, absolute-k-paranormal operator, p-paranormal operator, class A operator, class $A(k)$ operator, class $A(p, r)$ operator, Hölder-McCarthy inequality.

It is clear that class $A(1)$ equals class A and absolute-1-paranormality equals paranormality since $\||S|y\| = \|Sy\|$ for any $S \in B(H)$ and $y \in H$. T is said to be normaloid if $\|T\| = r(T)$. We showed inclusion relations among these classes in [10]. Class A and class $A(k)$ operators have been studied in [12], [13] and [15].

On the other hand, Fujii, Izumino and Nakamoto [6] introduced p-paranormal operators for $p > 0$ as another generalization of paranormal operators. T is said to be p-paranormal if

$$(1.3) \qquad \||T|^p U|T|^p x\| \geq \||T|^p x\|^2 \quad \text{for every unit vector } x,$$

where the polar decomposition of T is $T = U|T|$. It is clear that 1-paranormality equals paranormality. p-Paranormality is based on the following fact [5]: $T = U|T|$ is p-hyponormal if and only if $S = U|T|^p$ is hyponormal for $p > 0$. Actually, it was shown in [6] that $T = U|T|$ is p-paranormal if and only if $S = U|T|^p$ is paranormal for $p > 0$.

Fujii, Jung, S.H.Lee, M.Y.Lee and Nakamoto [7] introduced class $A(p,r)$ as a further generalization of class $A(k)$. T belongs to class $A(p,r)$ for $p > 0$ and $r > 0$ if

$$(1.4) \qquad (|T^*|^r |T|^{2p} |T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r},$$

and class $AI(p,r)$ is the class of all invertible operators which belong to class $A(p,r)$. It was pointed out in [15] that class $A(k,1)$ equals class $A(k)$ for each $k > 0$.

In this paper, we shall introduce a further generalization of the classes of both absolute-k-paranormal and p-paranormal operators as a parallel concept to class $A(p,r)$. Then we shall generalize the results on absolute-k-paranormal and p-paranormal operators for this new class.

2. Definition and properties of absolute-(p,r)-paranormal operators

We introduce the following new class of operators.

Definition. For positive real numbers $p > 0$ and $r > 0$, T is absolute-(p,r)-paranormal if

$$(2.1) \qquad \||T|^p |T^*|^r x\|^r \geq \||T^*|^r x\|^{p+r} \quad \text{for every unit vector } x,$$

or equivalently,

$$(2.2) \qquad \||T|^p |T^*|^r x\|^r \|x\|^p \geq \||T^*|^r x\|^{p+r} \quad \text{for all } x \in H.$$

We remark that the definition of absolute-(p,r)-paranormal operators (2.1) and (2.2) are expressed in terms of only T and T^*, without U which appears in the polar decomposition of $T = U|T|$.

To consider the relations to absolute-k-paranormality and p-paranormality, we show another expression of absolute-(p,r)-paranormality as follows.

Proposition 1. *For each $p > 0$ and $r > 0$, T is absolute-(p,r)-paranormal if and only if*

$$(2.3) \qquad \||T|^p U|T|^r x\|^r \geq \||T|^r x\|^{p+r} \quad \text{for every unit vector } x,$$

where the polar decomposition of T is $T = U|T|$.

The following result is easily obtained as a corollary of Proposition 1.

Corollary 2.
(i) *For each $k > 0$, T is absolute-k-paranormal iff T is absolute-$(k,1)$-paranormal.*
(ii) *For each $p > 0$, T is p-paranormal iff T is absolute-(p,p)-paranormal.*
(iii) *T is paranormal iff T is absolute-$(1,1)$-paranormal.*

It turns out by Corollary 2 that absolute-(p,r)-paranormality is a further generalization of paranormality than both absolute-k-paranormality and p-paranormality.

Proof of Proposition 1. It is well known that $|T^*|^r = U|T|^r U^*$ for $r > 0$, so that (2.2) is equivalent to the following (2.4):

$$(2.4) \qquad \||T|^p U|T|^r U^* x\|^r \|x\|^p \geq \|U|T|^r U^* x\|^{p+r} \quad \text{for all } x \in H.$$

It is also well known that $N(S^r) = N(S)$ for any $S \geq 0$ and $r > 0$. By using this fact, we have $R(|T|^r) \subseteq \overline{R(|T|^r)} = N(|T|^r)^\perp = N(|T|)^\perp = N(U)^\perp$, so that $\|U|T|^r U^* x\| = \||T|^r U^* x\|$ for all $x \in H$. Hence (2.4) is equivalent to the following (2.5):

$$(2.5) \qquad \||T|^p U|T|^r U^* x\|^r \|x\|^p \geq \||T|^r U^* x\|^{p+r} \quad \text{for all } x \in H.$$

Put $x = Uy$ in (2.5), then we have the following (2.6) since $|T|^r U^* U = |T|^r$:

$$(2.6) \qquad \||T|^p U|T|^r y\|^r \|Uy\|^p \geq \||T|^r y\|^{p+r} \quad \text{for all } y \in H.$$

(2.6) yields the following (2.7) since $\|y\| \geq \|Uy\|$ for all $y \in H$:

$$(2.7) \qquad \||T|^p U|T|^r y\|^r \|y\|^p \geq \||T|^r y\|^{p+r} \quad \text{for all } y \in H.$$

Hence (2.5) implies (2.7). Here we show that (2.7) implies (2.5) conversely. Put $y = U^* x$ in (2.7), then we have

$$(2.8) \qquad \||T|^p U|T|^r U^* x\|^r \|U^* x\|^p \geq \||T|^r U^* x\|^{p+r} \quad \text{for all } x \in H.$$

(2.8) yields (2.5) since $\|x\| \geq \|U^* x\|$ for all $x \in H$. Hence (2.7) implies (2.5), so that (2.5) is equivalent to (2.7). Consequently, the proof of Proposition 1 is complete since (2.7) is equivalent to (2.3). $\qquad \square$

Proof of Corollary 2. We remark that $\||S|y\| = \|Sy\|$ holds for any $S \in B(H)$ and $y \in H$.
  (i) Put $p = k > 0$ and $r = 1$ in (2.3), then we have (1.2).
  (ii) Put $r = p > 0$ in (2.3), then we have (1.3).
  (iii) Put $r = p = 1$ in (2.3), then we have (1.1). $\qquad \square$

Ando [4] gave a characterization of paranormal operators via an operator inequality as follows: T is paranormal if and only if

$$T^{2^*} T^2 - 2\lambda T^* T + \lambda^2 I \geq 0$$

for all $\lambda > 0$. A generalization of this result for absolute-k-paranormal operators was shown in [10, Theorem 6]. Here we show a further generalization for absolute-(p, r)-paranormal operators as follows.

Proposition 3. *The following assertions hold for each $p > 0$ and $r > 0$:*
  (i) *T is absolute-(p, r)-paranormal if and only if*

$$(2.9) \qquad r|T^*|^r |T|^{2p} |T^*|^r - (p+r)\lambda^p |T^*|^{2r} + p\lambda^{p+r} I \geq 0 \quad \text{for all } \lambda > 0.$$

  (ii) *T is p-paranormal if and only if*

$$(2.10) \qquad |T^*|^p |T|^{2p} |T^*|^p - 2\lambda |T^*|^{2p} + \lambda^2 I \geq 0 \quad \text{for all } \lambda > 0.$$

We use the following well-known fact in the proof of Proposition 3.

Lemma A. *For positive real numbers $a > 0$ and $b > 0$,*

$$\lambda a + \mu b \geq a^\lambda b^\mu$$

holds for $\lambda > 0$ and $\mu > 0$ such that $\lambda + \mu = 1$.

Proof of Proposition 3.
Proof of (i). (2.2) is equivalent to the following (2.11):

(2.11) $(|T^*|^r|T|^{2p}|T^*|^rx,x)^{\frac{r}{p+r}}(x,x)^{\frac{p}{p+r}} \geq (|T^*|^{2r}x,x)$ for all $x \in H$.

By Lemma A,

$$(|T^*|^r|T|^{2p}|T^*|^rx,x)^{\frac{r}{p+r}}(x,x)^{\frac{p}{p+r}} = \left\{\lambda^{-p}(|T^*|^r|T|^{2p}|T^*|^rx,x)\right\}^{\frac{r}{p+r}}\left\{\lambda^r(x,x)\right\}^{\frac{p}{p+r}}$$
$$\leq \frac{r}{p+r}\cdot\lambda^{-p}(|T^*|^r|T|^{2p}|T^*|^rx,x) + \frac{p}{p+r}\cdot\lambda^r(x,x)$$

holds for all $x \in H$ and $\lambda > 0$, so that (2.11) implies the following (2.12):

(2.12) $\dfrac{r}{p+r}\cdot\lambda^{-p}(|T^*|^r|T|^{2p}|T^*|^rx,x) + \dfrac{p}{p+r}\cdot\lambda^r(x,x) \geq (|T^*|^{2r}x,x)$

for all $x \in H$ and $\lambda > 0$.

Conversely, (2.11) follows from (2.12) by putting $\lambda = \left\{\frac{(|T^*|^r|T|^{2p}|T^*|^rx,x)}{(x,x)}\right\}^{\frac{1}{p+r}} > 0$ in case $(|T^*|^r|T|^{2p}|T^*|^rx,x) \neq 0$, and letting $\lambda \to +0$ in case $(|T^*|^r|T|^{2p}|T^*|^rx,x) = 0$. Hence (2.11) is equivalent to (2.12). Consequently, the proof of Proposition 3 is complete since (2.12) is equivalent to (2.9).
Proof of (ii). Put $r = p > 0$ and replace λ^p with λ in (i), then we have (ii) by (ii) of Corollary 2.   ∎

It was shown in [8] and [11] that if T is invertible and paranormal, then T^{-1} is also paranormal. Here we show the following generalization of this well-known result.

Proposition 4. *The following assertions hold for each $p > 0$ and $r > 0$:*
 (i) *If T is invertible and absolute-(p,r)-paranormal, then T^{-1} is absolute-(r,p)-paranormal.*
 (ii) *If T is invertible and p-paranormal, then T^{-1} is also p-paranormal.*

We prepare the following lemma to give a proof of Proposition 4.

Lemma 5. *Let T be an invertible operator. For each $p > 0$ and $r > 0$, T is absolute-(p,r)-paranormal if and only if*

(2.13) $\||T|^px\|^r\||T^{-1}|^rx\|^p \geq 1$ *for every unit vector x.*

Proof. (2.2) is equivalent to the following (2.14) by putting $y = |T^*|^rx$ since $R(|T^*|^r) = H$:

(2.14) $\||T|^py\|^r\||T^*|^{-r}y\|^p \geq \|y\|^{p+r}$ for all $y \in H$.

(2.14) is equivalent to the following (2.15):

(2.15) $\||T|^py\|^r\||T^*|^{-r}y\|^p \geq 1$ for every unit vector y.

(2.15) is equivalent to (2.13) since $|T^*|^{-1} = |T^{-1}|$, so that the proof is complete.   ∎

Proof of Proposition 4.
 (i) Obvious by Lemma 5.
 (ii) Put $r = p > 0$ in (i), then we have (ii) by (ii) of Corollary 2.   ∎

At the end of this section, we show the following parallel result to Proposition 4 for class $AI(p,r)$ operators.

Proposition 6. *The following assertions hold for each $p > 0$ and $r > 0$:*
 (i) *If T belongs to class $AI(p,r)$, then T^{-1} belongs to class $AI(r,p)$.*
 (ii) *If T belongs to class $AI(p,p)$, then T^{-1} also belongs to class $AI(p,p)$.*
 (iii) *If T is invertible and belongs to class A, then T^{-1} also belongs to class A.*

We use the following lemma in the proof of Proposition 6.

Lemma F ([9]). *Let $A > 0$ and B be an invertible operator. Then*

$$(BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

holds for any real number λ.

Proof of Proposition 6.
Proof of (i). Assume that T belongs to class $AI(p,r)$ for $p > 0$ and $r > 0$, i.e.,

(1.4) $$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r}.$$

By Lemma F, we have

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} = |T^*|^r|T|^p(|T|^p|T^*|^{2r}|T|^p)^{\frac{-p}{p+r}}|T|^p|T^*|^r,$$

so that (1.4) implies the following (2.16):

(2.16) $$|T|^{2p} \geq (|T|^p|T^*|^{2r}|T|^p)^{\frac{p}{p+r}}.$$

We remark that $|T| = |T^{-1*}|^{-1}$ and $|T^*| = |T^{-1}|^{-1}$. Applying these facts to (2.16), we have

$$|T^{-1*}|^{-2p} \geq (|T^{-1*}|^{-p}|T^{-1}|^{-2r}|T^{-1*}|^{-p})^{\frac{p}{p+r}}$$

$$= (|T^{-1*}|^p|T^{-1}|^{2r}|T^{-1*}|^p)^{\frac{-p}{p+r}},$$

so that

$$(|T^{-1*}|^p|T^{-1}|^{2r}|T^{-1*}|^p)^{\frac{p}{p+r}} \geq |T^{-1*}|^{2p}.$$

Hence T^{-1} belongs to class $AI(r,p)$.
Proof of (ii). Put $r = p > 0$ in (i), then we have (ii).
Proof of (iii). Put $p = 1$ in (ii), then we have (iii) since class $A(1,1)$ equals class A. □

Remark. Aluthge and Wang [2] introduced w-hyponormal operators such that

$$|\tilde{T}| \geq |T| \geq |(\tilde{T})^*|,$$

where the polar decomposition of T is $T = U|T|$ and $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. w-Hyponormal operators are studied in [1], [3] and [12]. It was shown in [12] [15] that the class of invertible w-hyponormal operators equals class $AI(\frac{1}{2},\frac{1}{2})$, so that it turns out by Proposition 6 that if T is invertible and w-hyponormal, then T^{-1} is also w-hyponormal.

3. INCLUSION RELATIONS AMONG THE RELATED CLASSES

We cite the following result which plays an important role to give proofs of the results in this section.

Theorem H-M (Hölder-McCarthy inequality [14]). *Let A be a positive operator. Then the following inequalities hold for all $x \in H$:*
  (i) $(A^r x, x) \leq (Ax, x)^r \|x\|^{2(1-r)}$ *for $0 < r \leq 1$.*
  (ii) $(A^r x, x) \geq (Ax, x)^r \|x\|^{2(1-r)}$ *for $r \geq 1$.*

We remark that (i) and (ii) of Theorem H-M can be rewritten as follows:
  (i)' $\|A^r x\| \leq \|Ax\|^r \|x\|^{1-r}$ *for $0 < r \leq 1$.*
  (ii)' $\|A^r x\| \geq \|Ax\|^r \|x\|^{1-r}$ *for $r \geq 1$.*

Firstly, we show the monotonicity of the classes of absolute-(p,r)-paranormal operators for $p > 0$ and $r > 0$ as generalizations of [7, Theorem 4.1] and [10, Theorem 4].

Theorem 7. *Let T be absolute-(p_0, r_0)-paranormal for $p_0 > 0$ and $r_0 > 0$. Then T is absolute-(p, r)-paranormal for any $p \geq p_0$ and $r \geq r_0$. Moreover, for each $r \geq r_0$ and unit vector x,*

$$(3.1) \qquad\qquad f_r(p) = \||T|^p |T^*|^r x\|^{\frac{r}{p+r}}$$

is increasing for $p \geq p_0$.

Theorem 7 can be considered as a parallel result to the following Theorem B which states the monotonicity of class $AI(p, r)$ for $p > 0$ and $r > 0$.

Theorem B ([7])**.** *If T belongs to class $AI(p_0, r_0)$ for $p_0 > 0$ and $r_0 > 0$, then T belongs to class $AI(p, r)$ for any $p \geq p_0$ and $r \geq r_0$.*

Proof of Theorem 7. Assume that T is absolute-(p_0, r_0)-paranormal for $p_0 > 0$ and $r_0 > 0$, i.e.,

$$(3.2) \qquad \||T|^{p_0}|T^*|^{r_0}y\|^{r_0} \geq \||T^*|^{r_0}y\|^{p_0+r_0}\|y\|^{-p_0} \quad \text{for all } y \in H.$$

Then for each $r \geq r_0$ and unit vector x,

$$\||T|^{p_0}|T^*|^r x\|^{r_0}$$
$$=\||T|^{p_0}|T^*|^{r_0}|T^*|^{r-r_0}x\|^{r_0}$$
$$\geq\||T^*|^{r_0}|T^*|^{r-r_0}x\|^{p_0+r_0}\||T^*|^{r-r_0}x\|^{-p_0} \qquad \text{by (3.2)}$$
$$=\||T^*|^r x\|^{p_0+r_0}\||T^*|^{r-r_0}x\|^{-p_0}$$
$$\geq\||T^*|^r x\|^{p_0+r_0}\||T^*|^r x\|^{\frac{r-r_0}{r}\cdot(-p_0)} \qquad \text{by (i)' of Theorem H-M for } \frac{r-r_0}{r} \in [0,1)$$
$$=\||T^*|^r x\|^{\frac{(p_0+r)r_0}{r}},$$

so that we have

$$(3.3) \qquad\qquad \||T|^{p_0}|T^*|^r x\|^{\frac{r}{p_0+r}} \geq \||T^*|^r x\|.$$

Hence for each $p \geq p_0$, $r \geq r_0$ and unit vector x,

$$\||T|^p|T^*|^r x\|$$
$$\geq\||T|^{p_0}|T^*|^r x\|^{\frac{p}{p_0}}\||T^*|^r x\|^{1-\frac{p}{p_0}} \qquad\qquad \text{by (ii)' of Theorem H-M for } \frac{p}{p_0} \geq 1$$
$$\geq\||T|^{p_0}|T^*|^r x\|^{\frac{p}{p_0}}\||T|^{p_0}|T^*|^r x\|^{\frac{r}{p_0+r}\cdot\frac{p_0-p}{p_0}} \qquad \text{by (3.3)}$$
$$=\||T|^{p_0}|T^*|^r x\|^{\frac{p+r}{p_0+r}}$$
$$\geq\||T^*|^r x\|^{\frac{p+r}{r}} \qquad\qquad\qquad\qquad \text{by (3.3),}$$

so that we have

$$(3.4) \qquad \||T|^p|T^*|^r x\|^{\frac{r}{p+r}} \geq \||T|^{p_0}|T^*|^r x\|^{\frac{r}{p_0+r}} \geq \||T^*|^r x\|.$$

(3.4) assures that T is absolute-(p, r)-paranormal for any $p \geq p_0$ and $r \geq r_0$, and for each $r \geq r_0$ and unit vector x, $f_r(p) = \||T|^p|T^*|^r x\|^{\frac{r}{p+r}}$ is increasing for $p \geq p_0$. Consequently, the proof of Theorem 7 is complete. $\qquad\square$

Secondly, we show inclusion relations among the class of absolute-(p, r)-paranormal operators and the related classes.

Theorem 8. *The following assertions hold for each $p > 0$ and $r > 0$:*
  (i) *Every class $A(p, r)$ operator is absolute-(p, r)-paranormal.*
  (ii) *Every absolute-(p, r)-paranormal operator is normaloid.*

(i) of Theorem 8 is a generalization of [7, Theorem 3.5] and [10, Theorem 4], and (ii) is a generalization of [10, Theorem 5] and the following result.

Theorem C ([7]). *Every p-paranormal operator is normaloid for $p > 0$.*

Proof of Theorem 7.
Proof of (i). Assume that T belongs to class $A(p, r)$ for $p > 0$ and $r > 0$, i.e.,

$$(1.4) \qquad (|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r}.$$

Then for every unit vector x,

$$
\begin{aligned}
\||T^*|^r x\|^2 &= (|T^*|^{2r}x, x) \\
&\leq ((|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}}x, x) \qquad \text{by (1.4)} \\
&\leq (|T^*|^r|T|^{2p}|T^*|^r x, x)^{\frac{r}{p+r}} \qquad \text{by (i) of Theorem H-M for } \frac{r}{p+r} \in (0,1) \\
&= \||T|^p|T^*|^r x\|^{\frac{2r}{p+r}},
\end{aligned}
$$

so that we have

$$(2.1) \qquad \||T|^p|T^*|^r x\|^r \geq \||T^*|^r x\|^{p+r} \quad \text{for every unit vector } x,$$

i.e., T is absolute-(p, r)-paranormal.
Proof of (ii). Assume that T is absolute-(p, r)-paranormal. Put $q = \max\{p, r\} > 0$, then T is absolute-(q, q)-paranormal by Theorem 7, i.e., T is q-paranormal by (ii) of Corollary 2. Hence T is normaloid by Theorem C. □

Lastly, we introduce a characterization of log-hyponormal operators via absolute-(p, r)-paranormality as an extension of [16, Theorem 1].

Theorem 9. *The following assertions are mutually equivalent:*

  (i) *T is log-hyponormal.*
 (ii) *T is invertible and p-paranormal for all $p > 0$.*
(iii) *T is invertible and absolute-(p, r)-paranormal for all $p > 0$ and $r > 0$.*

In [16], we gave a proof in terms of norm inequalities. Here we give a proof in terms of operator inequalities by using Proposition 3.

Proof of Theorem 9. (i) $\iff$ (ii) is [16, Theorem 1] itself. It is pointed out in [7] that every log-hyponormal operator belongs to class $AI(p, r)$ for all $p > 0$ and $r > 0$, so that (i) $\implies$ (iii) holds by (i) of Theorem 8. Hence we have only to prove (iii) $\implies$ (i).

Assume that T is absolute-(p, r)-paranormal for all $p > 0$ and $r > 0$. By (i) of Proposition 3, (2.9) holds particularly for $\lambda = 1$, that is,

$$(3.5) \qquad r|T^*|^r|T|^{2p}|T^*|^r - (p+r)|T^*|^{2r} + pI \geq 0 \quad \text{for all } p > 0 \text{ and } r > 0.$$

Since T is invertible, (3.5) can be rewritten as the following (3.6):
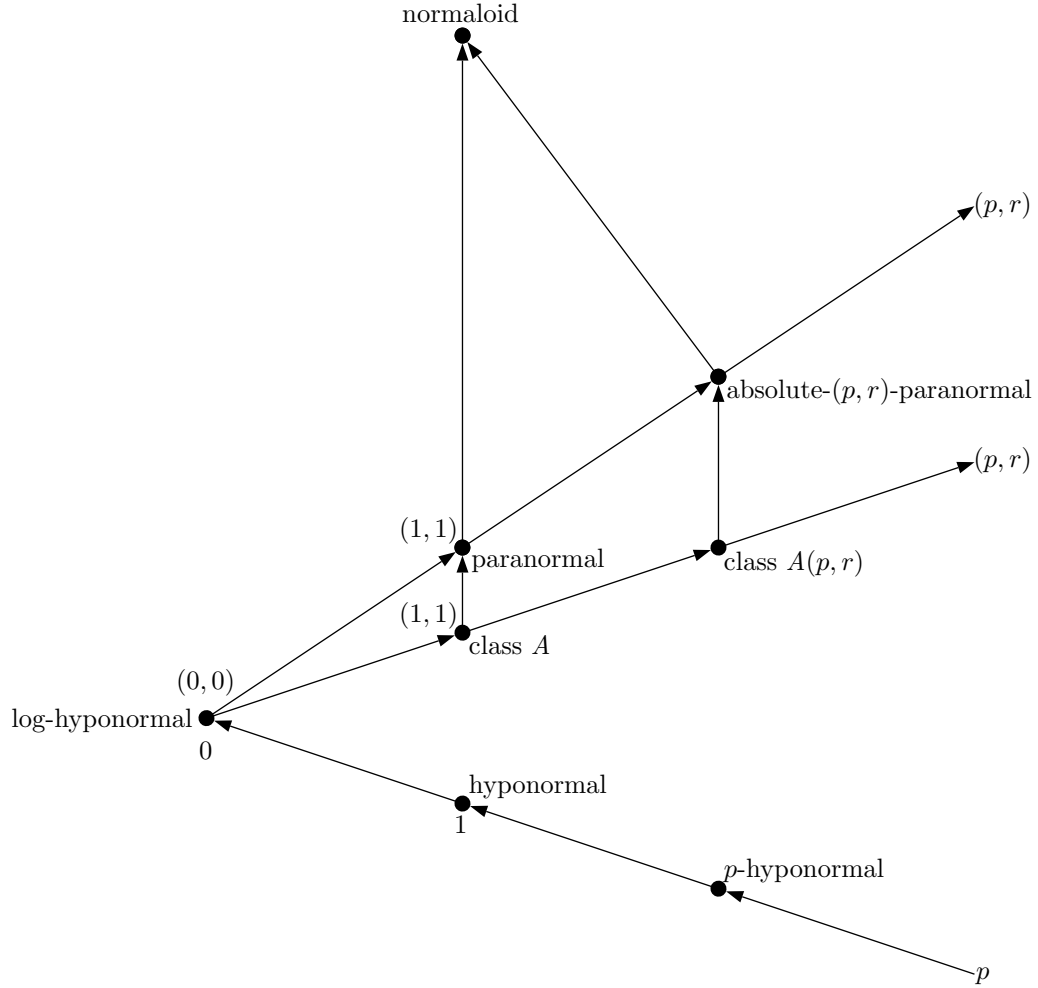
$$(3.6) \qquad \frac{|T|^{2p} - I}{p} \geq \frac{|T^*|^{-2r} - I}{-r} \quad \text{for all } p > 0 \text{ and } r > 0.$$

By letting $p \to +0$ and $r \to +0$ in (3.6), we have

$$\log |T|^2 \geq \log |T^*|^2,$$

i.e., T is log-hyponormal. □

The following diagram represents the inclusion relations among the classes discussed in this section.

References

[1] A.Aluthge and D.Wang, *An operator inequality which implies paranormality*, Math. Inequal. Appl. **2** (1999), 113–119.

[2] A.Aluthge and D.Wang, *w-Hyponormal operators*, to appear in Integral Equations Operator Theory.

[3] A.Aluthge and D.Wang, *w-Hyponormal operators II*, preprint.

[4] T.Ando, *Operators with a norm condition*, Acta Sci. Math. (Szeged) **33** (1972), 169–178.

[5] M.Fujii, C.Himeji and A.Matsumoto, *Theorems of Ando and Saito for p-hyponormal operators*, Math. Japon. **39** (1994), 595–598.

[6] M.Fujii, S.Izumino and R.Nakamoto, *Classes of operators determined by the Heinz-Kato-Furuta inequality and the Hölder-McCarthy inequality*, Nihonkai Math. J. **5** (1994), 61–67.

[7] M.Fujii, D.Jung, S.H.Lee, M.Y.Lee and R.Nakamoto, *Some classes of operators related to paranormal and log-hyponormal operators*, to appear in Math. Japon.

[8] T.Furuta, *On the class of paranormal operators*, Proc. Japan Acad. **43** (1967), 594–598.

[9] T.Furuta, *Extension of the Furuta inequality and Ando-Hiai log-majorization*, Linear Algebra Appl. **219** (1995), 139–155.

[10] T.Furuta, M.Ito and T.Yamazaki, *A subclass of paranormal operators including class of log-hyponormal and several related classes*, Scientiae Mathematicae **1** (1998), 389–403.

[11] V.Istrăţescu, T.Saito and T.Yoshino, *On a class of operators*, Tohoku Math. J. **18** (1966), 410–413.

[12] M.Ito, *Some classes of operators associated with generalized Aluthge transformation*, SUT J. Math. **35** (1999), 149–165.

[13] M.Ito, *Several properties on class A including p-hyponormal and log-hyponormal operators*, to appear in Math. Inequal. Appl.

[14] C.A.McCarthy, c_p, Israel J. Math. **5** (1967), 249–271.

[15] T.Yamazaki, *On powers of class $A(k)$ operators including p-hyponormal and log-hyponormal operators*, to appear in Math. Inequal. Appl.

[16] T.Yamazaki and M.Yanagida, *A characterization of log-hyponormal operators via p-paranormality*, preprint.

Department of Applied Mathematics, Faculty of Science, Science University of Tokyo, 1-3 Kagurazaka, Shinjuku, Tokyo 162-8601, Japan

E-mail address, Takeaki Yamazaki: `tyamaz@am.kagu.sut.ac.jp`

E-mail address, Masahiro Yanagida: `yanagida@am.kagu.sut.ac.jp`