SPECIAL IMPLICATIVE FILTERS IN IMPLICATIVE ALGEBRAS

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Abstract. In this paper we show that in an implicative algebra $\mathcal{A} = (A; V, \Rightarrow)$ inherited from the poset $(A, \leq)$ with the greatest element V, every implicative filter in $\mathcal{A}$ is special. Moreover, we show that the homomorphic inverse image of an (special, resp.) implicative filter in implicative algebras is also an (special, resp.) implicative filter.

1. Introduction

Implicative algebras are closely related to ordered sets with a greast element. In [4], it was concerned with elementary properties of implicative algebras and implicative filters. The notion of fuzzy sets was formulated by Zadeh [6] and since then fuzzy sets have been applied to various branches of mathematics and computer science. Rosenfeld [5] inspired the development of fuzzy algebraic structures. In [3], the fuzzification of an implicative filter in an implicative algebra was discussed. Recently Font [2] showed that Rasiowa's claim is not true, and solved several questions concerning special implicative filters, taking the theory of algebraizable logics of Blok and Pigozzi as a framework to approach the question in a systematic way. In this paper we show that in an implicative algebra $\mathcal{A} = (A; V, \Rightarrow)$ inherited from the poset $(A, \leq)$ with the greatest element V, every implicative filter in $\mathcal{A}$ is special. Moreover, we show that the homomorphic inverse image of an (special, resp.) implicative filter in implicative algebras is also an (special, resp.) implicative filter.

2. Preliminaries

An abstract algebra $\mathcal{A} = (A, V, \Rightarrow)$, where V is a 0-argument operation and $\Rightarrow$ is a two-argument operation, is said to be an *implicative algebra*, provided the following conditions are satisfied for all $a, b, c \in A$:

(i_1) $a \Rightarrow a = V$,
(i_2) if $a \Rightarrow b = V$ and $b \Rightarrow c = V$, then $a \Rightarrow c = V$,
(i_3) if $a \Rightarrow b = V$ and $b \Rightarrow a = V$, then $a = b$,
(i_4) $a \Rightarrow V = V$.

Let $\mathcal{A} = (A, V, \Rightarrow)$ be an implicative algebra. Then the equivalence

$$a \leq b \text{ if and only if } a \Rightarrow b = V$$

defines an ordering on A. The element V is the greatest element in the ordered set $(A, \leq)$.

3. Spcecial implicative filters in implictive algebras

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Definition 3.1 ([4]). A subset $\bigtriangledown$ of the set A of all elements of an implicative algebra $\mathcal{A} = (A, V, \Rightarrow)$ is said to be an *implicative filter* provided the following conditions are satisfied:

(f_1) $V \in \bigtriangledown$,

(f_2) if $a \in \bigtriangledown$ and $a \Rightarrow b \in \bigtriangledown$, then $b \in \bigtriangledown$.

Example 3.2. Let $A := \{V, a, b, c, d\}$ be a set with the following tables:

$\Rightarrow_1$	V	a	b	c	d
V	V	a	b	c	d
a	V	V	b	c	d
b	V	a	V	c	d
c	V	V	V	V	d
d	V	V	V	V	V

Table 1

$\Rightarrow_2$	V	a	b	c	d
V	V	a	b	c	d
a	V	V	b	c	d
b	V	V	V	c	d
c	V	V	V	V	d
d	V	V	V	V	V

Table 2

Then $\mathcal{A} = (A, V, \Rightarrow_i)$ $(i = 1, 2)$ are implicative algebras, and the subsets $\bigtriangledown_1 = \{V, a\}$, $\bigtriangledown_2 = \{V, b\}$ and $\bigtriangledown_3 = \{V, a, b\}$ are implicative filters in $\mathcal{A} = (A, V, \Rightarrow_1)$, while the subset $\bigtriangledown_4 = \{V, a, d\}$ is not an implicative filter in $\mathcal{A} = (A, V, \Rightarrow_1)$, since $d \in \bigtriangledown_4, d \Rightarrow_1 c = V \in \bigtriangledown_4$, but $c \notin \bigtriangledown_4$.

Definition 3.3 ([4]). An implicative filter $\bigtriangledown$ in an implicative algebra $\mathcal{A} = (A, V, \Rightarrow)$ is said to be *special* provided the following conditions are satisfied: for any a, b, c, d in A

(f_3) if $a \in \bigtriangledown$, then $b \Rightarrow a \in \bigtriangledown$,

(f_4) if $a \Rightarrow b$, $b \Rightarrow c \in \bigtriangledown$, then $a \Rightarrow c \in \bigtriangledown$,

(f_5) if $b \Rightarrow a$, $c \Rightarrow d \in \bigtriangledown$, then $(a \Rightarrow c) \Rightarrow (b \Rightarrow d) \in \bigtriangledown$.

Example 3,4. In Example 3.2 we can easily see that $\bigtriangledown_1$ is a special implicative filter in $\mathcal{A} = (A, V, \Rightarrow_1)$.

Example 3.5. Consider the following implicative algebra $\mathcal{A} = (A, V, \Rightarrow)$:

$\Rightarrow$	V	a	b	c	d
V	V	a	b	c	d
a	V	V	b	c	d
b	V	V	V	c	d
c	V	V	V	V	d
d	V	V	c	c	V

Then the implicative filter $\bigtriangledown = \{V, a, b\}$ is not special, since $b \in \bigtriangledown$, but $d \Rightarrow b = c \notin \bigtriangledown$.

Proposition 3.6. *Given a poset $(A, \leq)$ with the greatest element V, there exists at least one implicative algebra $\mathcal{A} = (A; V, \Rightarrow)$.*

Proof. If we define a binary operation $\Rightarrow$ on A by

$$a \Rightarrow b := \begin{cases} V & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

Then $\mathcal{A} = (A; V, \Rightarrow)$ is an implicative algebra, called the *implicative algebra inherited from the poset* $A(\leq)$. $\square$

Note that any positive implicative algebra is an implicative algebra, but the converse need not be true. In [4, pp. 36] each implicative filter in every positive implicative is special. We consider this fact in an implicative algebra inherited from the poset $(A, \leq)$ with the greatest element V as follows:

Theorem 3.7. *In an implicative algebra $\mathcal{A} = (A; V, \Rightarrow)$ inherited from the poset $(A, \leq)$ with the greatest element V, every implicative filter in $\mathcal{A}$ is special.*

Proof. Assume that there is $a \in \bigtriangledown$ such that $b \Rightarrow a \notin \bigtriangledown$ for some $b \in \bigtriangledown$. Since $\mathcal{A}$ is an implicative algebra inherited from the poset $(A, \leq)$, either $b \Rightarrow a = V \in \bigtriangledown$ or $b \Rightarrow a = a \in \bigtriangledown$, which are contradictions to $b \Rightarrow a \notin \bigtriangledown$. This proves (f_3).

Assume that $a \Rightarrow b, b \Rightarrow c \in \bigtriangledown$, but $a \Rightarrow c \notin \bigtriangledown$ for some $a, b, c \in A$. Since $\mathcal{A}$ is an implicative algebra and $V \in \bigtriangledown$, $a \Rightarrow c = c \notin \bigtriangledown$. Since $b \Rightarrow c \in \bigtriangledown$, it follows from (f_2) that $b \notin \bigtriangledown$. Similarly, since $a \Rightarrow b \in \bigtriangledown$, we obtain $a \notin \bigtriangledown$. This means that $a \Rightarrow b = V$ and $b \Rightarrow c = V$, and hence $a \Rightarrow c = V \in \bigtriangledown$, a contradiction. This proves (f_4).

Assume that $b \Rightarrow a, c \Rightarrow d \in \bigtriangledown$, but $(a \Rightarrow c) \Rightarrow (b \Rightarrow d) \notin \bigtriangledown$ for some $a, b, c, d \in A$. Since $V \in \bigtriangledown$, $(a \Rightarrow c) \Rightarrow (b \Rightarrow d) = b \Rightarrow d$. If $b \Rightarrow d = V$, then $(a \Rightarrow c) \Rightarrow (b \Rightarrow d) = (a \Rightarrow c) \Rightarrow V = V \in \bigtriangledown$, a contradiction. Hence $b \Rightarrow d = d$ and so $(a \Rightarrow c) \Rightarrow (b \Rightarrow d) = d \notin \bigtriangledown$. We claim that $c \notin \bigtriangledown$. Since $c \Rightarrow d \in \bigtriangledown$, if $c \in \bigtriangledown$ then by (f_2) $d \in \bigtriangledown$, a contradiction. This means that $c \Rightarrow d = V$. Consider $a \Rightarrow c$. Assume $a \Rightarrow c = V$. Then $a \notin \bigtriangledown$. Indeed, if $a \in \bigtriangledown$, then by (f_2) $c \in \bigtriangledown$, a contradiction. Since $b \Rightarrow a \in \bigtriangledown$, this leads to $b \notin \bigtriangledown$. ¿From the facts that $a, b \notin \bigtriangledown$ we have $b \Rightarrow a = V$. Therefore $b \Rightarrow d = V$, contradiction. Assume $a \Rightarrow c = c$. Then $c \Rightarrow d = (a \Rightarrow c) \Rightarrow (b \Rightarrow d) \notin \bigtriangledown$, a contradiction. This proves (f_5), ending the proof. $\square$

4. Homomorphisms

Let $\mathcal{A} = (A; V, \Rightarrow)$ and $\mathcal{B} = (B; V', \Rightarrow)$ be any implicative algebras. A mapping $f : \mathcal{A} \to \mathcal{B}$ is said to be a *homomorphism* if $f(a \Rightarrow b) = f(a) \Rightarrow f(b)$ for any $a, b \in A$. Note that $f(V) = V'$, since $f(V) = f(a \Rightarrow a) = f(a) \Rightarrow f(a) = V'$.

Example 4.1. Let $(A; *, \Rightarrow_1)$ be an implicative algebra descrived in Example 3.2. Let $(B; *, \Rightarrow)$ be an implicative algebra with the following table:

$\Rightarrow$	V	a	b	c	d
V	V	a	b	c	d
a	V	V	V	V	d
b	V	d	V	c	d
c	V	a	V	V	d
d	V	V	c	c	V

Define a mapping f from $\mathcal{A}$ to $\mathcal{B}$ by $f(V) = f(a) = f(b) = V'$, $f(c) = c$ and $f(d) = a$, and $g(V) = g(a) = g(b) = V'$, $f(c) = b$ and $f(d) = c$. Then the mappings f and g are homomorphisms.

Theorem 4.2. *The homomorphic inverse image of an (special, resp.) implicative filter in implicative algebras is also an (special, resp.) implicative filter.*

Proof. Let $f : \mathcal{A} \to \mathcal{B}$ be a homomorphism of implicative algebras. Assume $\bigtriangledown$ is an implicative filter of $\mathcal{B}$. If $a \in f^{-1}(\bigtriangledown)$ and $a \Rightarrow b \in f^{-1}(\bigtriangledown)$, then $f(a) \in \bigtriangledown$ and $f(a) \Rightarrow f(b) = f(a \Rightarrow b) \in \bigtriangledown$. Since $\bigtriangledown$ is an implicative filter of $\mathcal{B}$, we obtain $b \in f^{-1}(\bigtriangledown)$. This proves (f_2).

Assume that $\bigtriangledown$ is a special implicative filter of $\mathcal{B}$. Let $a \in f^{-1}(\bigtriangledown)$ and let $b \in \mathcal{A}$. Then $f(a) \in \bigtriangledown$ and $f(b \Rightarrow a) = f(b) \Rightarrow f(a) \in \bigtriangledown$, since $\bigtriangledown$ is special. This means that $b \Rightarrow a \in f^{-1}(\bigtriangledown)$. This proves (f_3).

Let $a \Rightarrow b, b \Rightarrow c \in f^{-1}(\bigtriangledown)$. Then $f(a) \Rightarrow f(b), f(b) \Rightarrow f(c) \in \bigtriangledown$. Since $\bigtriangledown$ is special, $f(a) \Rightarrow f(c) \in \bigtriangledown$, i.e., $a \Rightarrow c \in f^{-1}(\bigtriangledown)$. This proves (f_4).

Finally, if $b \Rightarrow a, c \Rightarrow d \in f^{-1}(\bigtriangledown)$, then $f((a \Rightarrow c) \Rightarrow (b \Rightarrow d)) = (f(a) \Rightarrow f(c)) \Rightarrow (f(b) \Rightarrow f(d)) \in \bigtriangledown$, since $\bigtriangledown$ is special. This means that $(a \Rightarrow c) \Rightarrow (b \Rightarrow d) \in f^{-1}(\bigtriangledown)$. This proves (f_5). Hence $f^{-1}(\bigtriangledown)$ is also an (special) filter of $\mathcal{A}$. $\square$

Notice that *the homomorphic image of an implicative filter in implicative algebras need not be an implicative filter.* For example, if we take a mapping f in Example 4.1, and if $\bigtriangledown := \{V, a, b, c\}$, then $\bigtriangledown$ is an implicative filter of $\mathcal{B}$, but its homomorphic image $f(\bigtriangledown) = \{V, c\}$ is not an implicative filter of $\mathcal{A}$, since $c \in f(\bigtriangledown), c \Rightarrow b = V \in f(\bigtriangledown)$, but $b \notin f(\bigtriangledown)$.

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