A CHARACTERIZATION OF LOG-HYPONORMAL OPERATORS VIA p-PARANORMALITY

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Received October 18, 1999

Abstract. An operator T is said to be log-hyponormal if T is invertible and $\log T^*T \geq \log TT^*$, and p-paranormal for $p > 0$ if $\||T|^pU|T|^px\| \geq \||T|^px\|^2$ for every unit vector x, where the polar decomposition of T is $T = U|T|$. We show that T is log-hyponormal if and only if T is invertible and p-paranormal for all $p > 0$.

1. Introduction

In this paper, an operator means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible.

An operator T is said to be log-hyponormal if T is invertible and

$$\log T^*T \geq \log TT^*,$$

and T is said to be paranormal if

$$\|T^2x\| \geq \|Tx\|^2 \quad \text{for every unit vector } x.$$

It was shown in [1] that every log-hyponormal operator is paranormal. Afterward, another simplified proof of this result was given in [4] by introducing a new class of operators called class A. In fact, T belongs to class A if

$$|T^2| \geq |T|^2,$$

where $|T| = (T^*T)^{\frac{1}{2}}$, and it was shown that every log-hyponormal operator belongs to class A and every class A operator is paranormal.

As generalizations of class A and paranormal operators, class $A(p,p)$ and p-paranormal operators were introduced in [3] and [2], respectively. T belongs to class $A(p,p)$ for $p > 0$ if

$$(|T^*|^p|T|^{2p}|T^*|^p)^{\frac{1}{2}} \geq |T^*|^{2p},$$

and for each $p > 0$, class $AI(p,p)$ is the class of invertible operators which belong to class $A(p,p)$. T is said to be p-paranormal for $p > 0$ if

$$\||T|^pU|T|^px\| \geq \||T|^px\|^2 \quad \text{for every unit vector } x,$$

where the polar decomposition of T is $T = U|T|$. We remark that class $A(1,1)$ equals class A and 1-paranormality equals paranormality. It was shown in [3] that every log-hyponormal operator belongs to class $AI(p,p)$ for all $p > 0$, and every class $A(p,p)$ operator is p-paranormal for each $p > 0$. And it was also shown in [3] that every class $AI(p,p)$ operator belongs to class $AI(p',p')$ and every p-paranormal operator is p'-paranormal for any $p' \geq p > 0$, that is, both class $AI(p,p)$ and the class of p-paranormal operators are monotone increasing for $p > 0$.

1991 *Mathematics Subject Classification.* Primary 47B20, 47A30, 47A63.

Key words and phrases. Log-hyponormal operator, class $A(p,p)$ operator, p-paranormal operator.

Furthermore, in [3], the following result was shown which is a characterization of log-hyponormal operators in terms of class $A(p,p)$.

Theorem A ([3])**.** *T is log-hyponormal if and only if T belongs to class $AI(p,p)$ for all $p>0$.*

Theorem A states that the class of log-hyponormal operators can be considered as the limit of class $AI(p,p)$ as $p \to +0$ since class $AI(p,p)$ is monotone increasing for $p>0$ as mentioned above.

In this paper, we shall give a new characterization of log-hyponormal operators in terms of p-paranormality.

2. Result

Theorem 1. *T is log-hyponormal if and only if T is invertible and p-paranormal for all $p>0$.*

Theorem 1 states that the class of log-hyponormal operators can be considered as the limit of the class of invertible and p-paranormal operators as $p \to +0$ since the class of p-paranormal operators is monotone increasing for $p>0$ as mentioned in the previous section. It is interesting to remark that class $AI(p,p)$ and the class of invertible and p-paranormal operators can be considered to be parallel to each other, but their limits as $p \to +0$ coincide. In fact, Theorem 1 gives a more precise sufficient condition for that an operator T is log-hyponormal than Theorem A since every class $A(p,p)$ operator is p-paranormal for each $p>0$ [3].

Proof. The "only if" part is shown in [3], so that we have only to prove the "if" part. Assume that T is invertible and p-paranormal for all $p>0$. Then for any $x \in H$,

$$\begin{aligned} \||T|^pU|T|^px\|^2\|x\|^2+\|x\|^4 &\geq \||T|^px\|^4+\|x\|^4 \qquad \text{since } T \text{ is } p\text{-paranormal} \\ &\geq 2\||T|^px\|^2\|x\|^2, \end{aligned}$$

so that we have

$$\||T|^pU|T|^px\|^2+\|x\|^2 \geq 2\||T|^px\|^2 \quad \text{for all } x\in H \text{ and } p>0. \tag{2.1}$$

Put $y=U|T|^px$, then $\|x\|=\||T|^{-p}U^*y\|=\||T^*|^{-p}y\|$ since U is unitary and $U|T|^{-2p}U^*=|T^*|^{-2p}$, and $\||T|^px\|=\|y\|$, so that (2.1) is equivalent to the following (2.2):

$$\||T|^py\|^2+\||T^*|^{-p}y\|^2 \geq 2\|y\|^2 \quad \text{for all } y\in H \text{ and } p>0. \tag{2.2}$$

(2.2) is equivalent to the following (2.3):

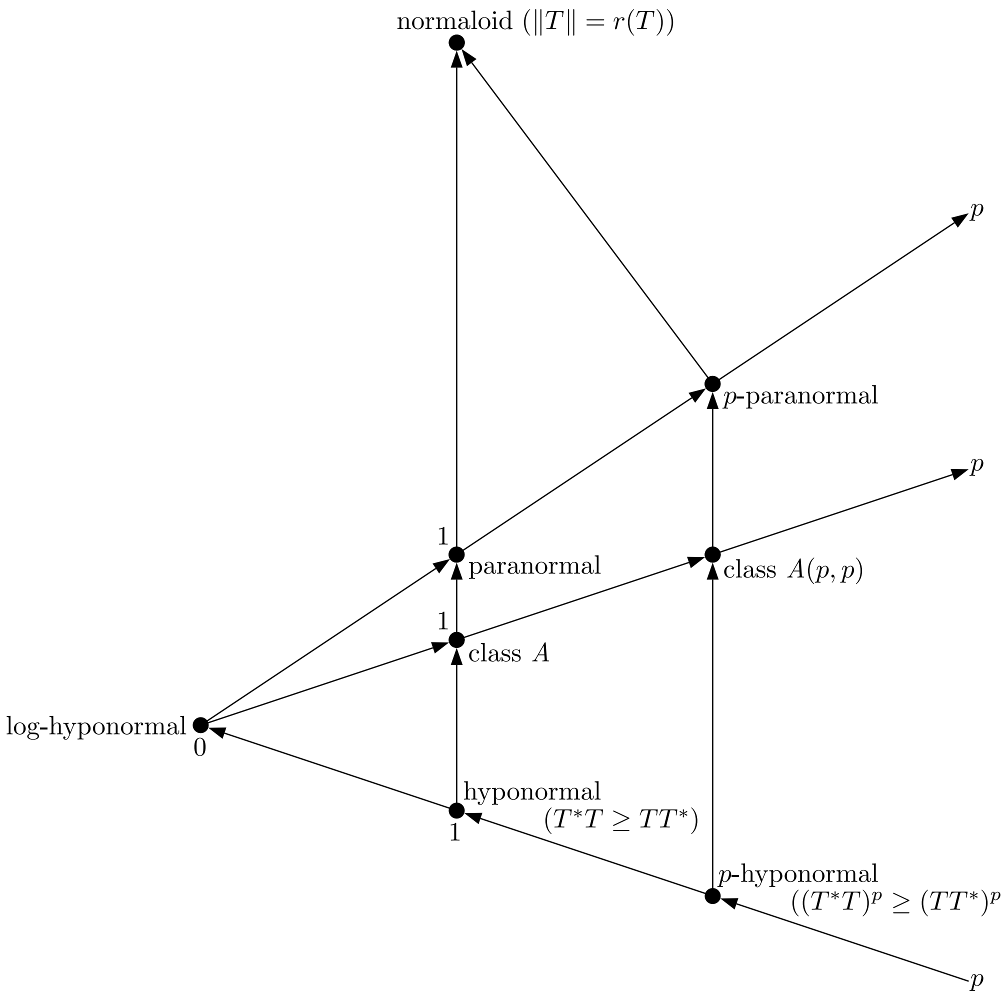
$$|T|^{2p}+|T^*|^{-2p}-2I \geq 0 \quad \text{for all } p>0. \tag{2.3}$$

Noting that $\lim\limits_{p\to+0}\frac{X^p-I}{p}=\log X$ for $X>0$, (2.3) implies the following:

$$\log|T|^2 \geq \log|T^*|^2,$$

that is, T is log-hyponormal. □

We show the following diagram which represents the inclusion relations among the classes discussed in this paper and some related classes.



Acknowledgement.

We would like to express our cordial gratitude to Professor Takayuki Furuta for his valuable advice and suggestions. We would also like to express our cordial thanks to Professor Masatoshi Fujii and the referee for their useful comments.

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