A NUMERICAL CALCULATION METHOD FOR OPTIMAL CONTROL PROBLEMS HAVING CONTROL RESTRICTIONS AND ADJUSTABLE PARAMETERS WITH CONSTRAINT

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ABSTRACT. This paper describes a new numerical calculation method for optimal control problems having control restrictions and adjustable parameters with constraint. The algorithm proposed in this paper is developed by applying the viewpoint of the grandient method. To verify the performance of the algorithm, three cases of computer simulation are provided.

1. INTRODUCTION. Adjustable parameters, for example, time constants in electrical circuits, can be adjusted when the system is being designed, but they are constant when the system is in operation and the optimal solution for them is unknown. As required in any control engineering problem, these parameters will have appropriate control restrictions.

The method of extension of maximum principle[1], the method of variation[2], etc. were proposed as theoretical approaches to optimal control problems including such parameters.

The methods of these works look upon parameters as artificial state variables and succeed in finding the conditions which the optimal solution must satisfy in state spaces with increased dimensions.

But, these methods are used only for problems that are analytically solvable. It is nearly impossible to solve them analytically when the system is nonlinear and the parameters have constraint. In cases like these, a numerical calculation method is effective.

Dyer et al.[3] introduced the function $W(\tau)$ given by the following expression (2.10) in order to assure the terminal state restraint. But, their method was not applicable to problems with control restrictions. (See Dyer et al. [3], for some concrete numerical calculation examples.)

Therefore, in this paper, the authors propose a new algorithm that can be applied to optimal control problems with parameters that have restrictions of upper and lower bounds with control restrictions. This algorithm is made by improving the Dyers' calculation method in continuous system, and has a characteristic of dealing with parameters with control restrictions in particular.

For the control restrictions, a new function η given by the following expression (3.34) is introduced. Inequality restrictions of parameters are converted into equality restrictions by sine functions. Variables of these functions are regarded as new parameters in the calculation of the control.

Moreover, problems of free terminal time are treated as follows. A real time variable is converted into a product of a time coefficient and a pseudo time variable. This coefficient is then adjusted by the gradient method.

Finally, to verify the performance of the algorithm, three cases of computer simulation are provided.

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2. ESTABLISHING THE PROBLEM. We consider the system equations

$$\dot{x} = f(x(t), u(t), w, t) \quad ; \quad x(0) = x_0. \tag{2.1}$$

Here, x is an n-dimensional vector, u is an m-dimensional ($m \leq n$) control vector, t is a time variable, w is an s-dimensional parameter. The parameter w is adjustable before the system starts running, but, once the system is running, this w is constant and does not change.

In this system, we consider the problem of finding the optimal control value and the optimal parameter that satisfy the control restrictions

$$|u_i(t)| \leq 1 \quad , \quad (1 \leq i \leq m) \tag{2.2}$$

and the terminal state restriction

$$\Psi(x(t_f), t_f) = 0, \tag{2.3}$$

and minimize the cost function

$$J = \int_0^{t_f} l(x(t), u(t), w, t)dt + \phi(x(t_f), w, t_f). \tag{2.4}$$

Here, Ψ is an r-dimensional ($1 \leq r \leq n$) function vector. The terminal time t_f is unknown. The functions $f(\cdot), l(\cdot), \phi(\cdot)$ and $\Psi(\cdot)$ are continuously differentiable.

The parameter w has a restriction of $w_{\min} \leq w \leq w_{\max}$. Now, by using a new s-dimensional parameter v, the parameter w is converted as follows.

$$w = \frac{1}{2}(w_{\max} - w_{\min}) \sin v + \frac{1}{2}(w_{\max} + w_{\min}) \tag{2.5}$$

By defining this tranformation, the optimal control problem of the parameter w with restriction

$$\forall v \quad , \quad w_{\min} \leq w \leq w_{\max}$$

is transformed into one of the parameter v without restrictions.

Because terminal time t_f is unknown, a pseudo time variable τ is introduced into this problem. The real time variable t and pseudo time variable τ have the following relation.

$$t = p\tau \quad (0 \leq \tau \leq 1) \tag{2.6}$$

In the case of such a conversion, the control section becomes $0 \leq \tau \leq 1$ for the pseudo time variable. Then, in appearance we can consider the problem of fixed terminal time. Since the time coefficient p also represents terminal time t_f, the optimal terminal time can be obtained by adding the correction to this p, and the problem of free terminal time can be solved.

By using expressions (2.6) and (2.5), the system of differential equations (2.1), the terminal state restraint (2.3) and the cost function (2.4) are rewritten as follows.

$$x'(\tau) = f(x(\tau), u(\tau), v, p, \tau)p \ ; \ x(0) = x_0 \tag{2.7}$$

$$\Psi(x(1), p) = 0 \tag{2.8}$$

$$J = \int_0^1 l(x(\tau), u(\tau), v, p, \tau)pd\tau + \phi(x(1), v, p). \tag{2.9}$$

Here, the symbol " $'$ " means differentiation with respect to the pseudo time variable τ.

Next, a new variable vector $W(\tau)$ that is defined by

$$W(\tau) = \Psi(x(\tau), p, \tau) \tag{2.10}$$

is introduced into the interval $[0,1]$. Because state variable $x(\tau)$ is dependent on the initial values of state variable $x(0)$, control variable $u(\tau)$, parameter v, time coefficient p and pseudo time τ, the new variable $W(\tau)$ is considered to be an implicit function of $x(\tau)$, $u(\tau)$, v and p.

The time coefficient p in expression (2.6) and the parameter v are constant in the interval $[0,1]$, and

$$p' = 0\ , \qquad v' = 0 \tag{2.11}$$

hold.

By including expressions (2.7) and (2.8) in (2.9), the synthetic cost function is defined as

$$\begin{aligned}
\hat{J} &= \int_0^1 \{l(x(\tau),u(\tau),v,p,\tau)p + \mathcal{A}(\tau)[f(x(\tau),u(\tau),v,p,\tau)p - x'(\tau)] \\
&\quad -\mathcal{B}(\tau)p' - \mathcal{C}(\tau)v'\}d\tau + \nu\Psi(1) + \phi(x(1),v,p) \\
&= \int_0^1 \{l(x(\tau),u(\tau),v,p,\tau)p + \mathcal{A}(\tau)[f(x(\tau),u(\tau),v,p,\tau)p - x'(\tau)] \\
&\quad -\mathcal{B}(\tau)p' - \mathcal{C}(\tau)v' + \nu W'(\tau) - \nu W'(\tau)\}d\tau + \nu\Psi(1) + \phi(x(1),v,p) \\
&= \int_0^1 \{H(x(\tau),u(\tau),v,p,\tau) - \mathcal{A}(\tau)x' - \mathcal{B}(\tau)p' \\
&\quad -\mathcal{C}(\tau)v' + \nu W'(\tau)\}d\tau - \nu[W(\tau)]_0^1 + \nu\Psi(1) + \phi(x(1),v,p) \\
&= \int_0^1 \{H(x(\tau),u(\tau),v,p,\tau) - \mathcal{A}(\tau)x' - \mathcal{B}(\tau)p' \\
&\quad -\mathcal{C}(\tau)v' + \nu W'(\tau)\}d\tau + \nu[\Psi(1) - W(1)] + \nu W(0) + \phi(x(1),v,p).
\end{aligned} \tag{2.12}$$

Here, the Hamiltonian $H(x(\tau),u(\tau),v,p,\tau)$ is defined as

$$H(x(\tau),u(\tau),v,p,\tau) = l(x(\tau),u(\tau),p,v,\tau)p + \mathcal{A}(\tau)f(x(\tau),u(\tau),p,v,\tau)p.$$

$\mathcal{A}(\tau)$, $\mathcal{B}(\tau)$, $\mathcal{C}(\tau)$ and ν are Lagrange's multipliers. $\mathcal{A}(\tau)$ is an n-dimensional vector, $\mathcal{B}(\tau)$ is a scalar, $\mathcal{C}(\tau)$ is an s-dimensional vector and ν is an r-dimensional vector.

A variation of expression (2.12) is

$$\begin{aligned}
\delta\hat{J} &= \int_0^1 \{H_x\delta x + H_p\delta p + H_v\delta v + H_u\delta u - \mathcal{A}\delta x' - \mathcal{B}\delta p' - \mathcal{C}\delta v' \\
&\quad +\nu[(W')_x\delta x + (W')_u\delta u + (W')_p\delta p + (W')_v\delta v]\}d\tau \\
&\quad +\nu[\Psi_x(1) - W_x(1)]\delta x(1) + \nu[\Psi_p(1) - W_p(1)]\delta p(1) - \nu W_v(1)\delta v(1) - \nu W_u(1)\delta u(1) \\
&\quad +\nu W_x(0)\delta x(0) + \nu W_u(0)\delta u(0) + \nu W_p(0)\delta p(0) + \nu W_v(0)\delta v(0) \\
&\quad +\phi_x(1)\delta x(1) + \phi_p(1)\delta p(1) + \phi_v(1)\delta v(1).
\end{aligned} \tag{2.13}$$

Here,

$$\begin{aligned}
\int_0^1 \mathcal{A}\delta x' d\tau &= [\mathcal{A}\delta x]_0^1 - \int_0^1 \mathcal{A}'\delta x d\tau \\
&= \mathcal{A}(1)\delta x(1) - \mathcal{A}(0)\delta x(0) - \int_0^1 \mathcal{A}'\delta x d\tau \\
\int_0^1 \mathcal{B}\delta p' d\tau &= [\mathcal{B}\delta p]_0^1 - \int_0^1 \mathcal{B}'\delta p d\tau \\
&= \mathcal{B}(1)\delta p(1) - \mathcal{B}(0)\delta p(0) - \int_0^1 \mathcal{B}'\delta p d\tau \\
\int_0^1 \mathcal{C}\delta v' d\tau &= [\mathcal{C}\delta v]_0^1 - \int_0^1 \mathcal{C}'\delta v d\tau \\
&= \mathcal{C}(1)\delta v(1) - \mathcal{C}(0)\delta v(0) - \int_0^1 \mathcal{C}'\delta v d\tau
\end{aligned}$$

The relations of $(W')_x$, $(W')_u$, $(W')_p$ and $(W')_v$ (see Appendix B)

$$\begin{aligned}
(W')_x &= W'_x + W_x f_x p \\
(W')_u &= W'_u + W_x f_u p \\
(W')_p &= W'_p + W_x(f_p p + f) \\
(W')_v &= W'_v + W_x f_v p
\end{aligned}$$

are substituted into expression (2.13). Then,

$$\begin{aligned}
\delta\hat{J} = \int_0^1 &\{[H_x(\tau) + \mathcal{A}'(\tau)]\delta x(\tau) + [H_u(\tau) + \nu W_x(\tau) f_u(\tau) p]\delta u(\tau) \\
&+[H_p(\tau) + \mathcal{B}'(\tau)]\delta p(\tau) + [H_v(\tau) + \mathcal{C}'(\tau)]\delta v(\tau) \\
&+\nu[W_x(\tau) f_x(\tau) p + W'_x(\tau)]\delta x(\tau) + \nu W'_u(\tau)\delta u(\tau) \\
&+\nu[W_x(\tau)(f_p(\tau)p + f(\tau)) + W'_p(\tau)]\delta p + \nu[W_x(\tau) f_v(\tau) p + W'_v(\tau)]\delta v\} d\tau \\
&+[\phi_x(1) - \mathcal{A}(1)]\delta x(1) + [\mathcal{A}(0) + \nu W_x(0)]\delta x(0) + \nu W_u(0)\delta u(0) \\
&+[\phi_p(1) - \mathcal{B}(1)]\delta p(1) + [\mathcal{B}(0) + \nu W_p(0)]\delta p(0) + [\phi_v(1) - \mathcal{C}(1)]\delta v(1) \\
&+[\mathcal{C}(0) + \nu W_v(0)]\delta v(0) + \nu[\Psi_x(1) - W_x(1)]\delta x(1) - \nu W_u(1)\delta u(1) \\
&+\nu[\Psi_p(1) - W_p(1)]\delta p(1) - \nu W_v(1)\delta v(1)
\end{aligned} \tag{2.14}$$

is obtained.

In expression (2.14), we assume that the following 7 differential equations and terminal conditions hold.

$$\mathcal{A}'(\tau) = -H_x(\tau),\ \mathcal{A}(1) = \phi_x(1) \tag{2.15}$$

$$W'_x(\tau) = -W_x(\tau) f_x p,\ W_x(1) = \Psi_x(1) \tag{2.16}$$

$$\mathcal{B}'(\tau) = -H_p(\tau),\ \mathcal{B}(1) = \phi_p(1) \tag{2.17}$$

$$W'_p(\tau) = -W_x(\tau)\left[f_p(\tau)p + f(\tau)\right],\ W_p(1) = \Psi_p(1) \tag{2.18}$$

$$\mathcal{C}'(\tau) = -H_v(\tau),\ \mathcal{C}(1) = \phi_v(1) \tag{2.19}$$

$$W'_v(\tau) = -W_x(\tau) f_v(\tau) p,\ W_v(1) = 0 \tag{2.20}$$

$$W'_u(\tau) = 0,\ W_u(1) = 0 \tag{2.21}$$

From the differential equation and terminal condition (2.21), $W_u(\tau) \equiv 0$. Under these assumptions, (2.14) can be denoted as

$$\delta\hat{J} = \int_0^1 g_u(\tau)\delta u(\tau)d\tau + g_p(0)\delta p(0) + g_v(0)\delta v(0) + g_x(0)\delta x(0). \tag{2.22}$$

Here,

$$G(\tau) = W_x(\tau)f_u(\tau)p \tag{2.23}$$

$$g_u(\tau) = H_u(\tau) + \nu G(\tau) \tag{2.24}$$

$$g_p(0) = \mathcal{B}(0) + \nu W_p(0) \tag{2.25}$$

$$g_v(0) = \mathcal{C}(0) + \nu W_v(0) \tag{2.26}$$

$$g_x(0) = \mathcal{A}(0) + \nu W_x(0). \tag{2.27}$$

Furthermore, we find that

$$H_u(\tau) = l_u p + \mathcal{A} f_u p. \tag{2.28}$$

$g_u(\tau)$ is called a gradient function. $g_p(0)$, $g_v(0)$ and $g_x(0)$ are called gradient coefficients. In this problem, we let $\delta x(0) = 0$ because the starting point of a state variable is fixed.

In this paper, when revising the laws of control variable $u(\tau)$, time variable p and parameter v are chosen as follows as a gradient method

$$\delta u(\tau) = \alpha g_u^T(\tau) \quad , \quad \alpha < 0 \tag{2.29}$$

$$\delta p = \beta g_p(0) \quad , \quad \beta < 0 \tag{2.30}$$

$$\delta v = \gamma g_v^T(0) \quad , \quad \gamma < 0 \tag{2.31}$$

The variation of the synthetic cost function is

$$\delta\hat{J} = \int_0^1 \alpha g_u(\tau)g_u^T(\tau)d\tau + \beta g_p^2(0) + \gamma g_v(0)g_v^T(0) \leq 0.$$

It is guaranteed that the value of J will decrease.

3. ALGORITHM. To construct the calculation algorithm of the gradient method, the control variable $u(\tau)$, time coefficient p, parameter v and Lagrange's multiplier ν must be calculated.

Assume that $\delta u(\tau)$ in expression (2.29) is the difference between the $i+1$-th correct value $u^{(i+1)}(\tau)$ and the i-th correct value $u^{(i)}(\tau)$. Its size is decided in proportion to the size of the gradient function under the condition of expression (2.2).

$$u^{(i+1)}(\tau) = \underset{|u|\leq 1}{\mathbf{sat}}\{u^{(i)}(\tau) + \alpha g_u^T(\tau)\} \tag{3.32}$$

$$\delta u(\tau) = u^{(i+1)}(\tau) - u^{(i)}(\tau) \tag{3.33}$$

(3.32) means saturating the value of $u^{(i)}(\tau) + \alpha g_u^T(\tau)$ for the sake of keeping $|u| \leq 1$. Here, the authors introduce an m-dimensional diagonal matrix $\eta(\tau)$ in order to satisfy the terminal condition (2.3).

$$\eta(\tau) = Diag\{\eta_1(\tau), \cdots, \eta_m(\tau)\} \tag{3.34}$$

Each element of this matrix is calculated by

$$\eta_k(\tau) = \mathcal{F}(u_k^{(i+1)}(\tau) - u_k^{(i)}(\tau)). \tag{3.35}$$

The scalar function $\mathcal{F}(\cdot)$ is defined by

$$\mathcal{F}(y) = \begin{cases} 0 \ ; & y = 0 \\ 1 \ ; & y \neq 0 \end{cases} . \tag{3.36}$$

If $u_k(\tau)$ is on the boundary values, the function $\eta_k(\tau)$ invalidates the gradient function $g_u(\tau)$, and if $u_k(\tau)$ exists inside of the boundary values, it validates it.

In order to revise the time coefficient p and parameter v, the following expressions are used.

$$\delta p = p^{(i+1)} - p^{(i)} \tag{3.37}$$

$$p^{(i+1)} = p^{(i)} + \beta g_p(0) \tag{3.38}$$

$$\delta v = v^{(i+1)} - v^{(i)} \tag{3.39}$$

$$v^{(i+1)} = v^{(i)} + \gamma g_v^T(0) \tag{3.40}$$

Next, for calculating $g_u(\tau)$, $g_p(0)$ and $g_v(0)$, the Lagrange's multiplier ν must be decided. This calculation is conducted as follows. $\Delta\Psi$ is defined by

$$\Delta\Psi = \Psi(x^{(i+1)}(1), p^{(i+1)}) - \Psi(x^{(i)}, p^{(i)}). \tag{3.41}$$

Because the terminal restraint condition $\Psi(x(t_f), p, t_f)$ is a continuous differentiable function vector, the existence of a differentiation of function vector Ψ depending on the pseudo time variable τ is guaranteed. Therefore, from the defining equation of $W(\tau)$,

$$\Psi(\tau) = \int_0^\tau W'(s)ds + W(0)$$

is obtained. When $\tau = 1$, a variation of $\Psi(\tau)$ becomes

$$\begin{aligned} \Delta\Psi(1) &= \int_0^1 \delta W'(s)ds + \delta W(0) \\ &= \int_0^1 [(W')_u(s)\delta u(s) + (W')_x(s)\delta x(s) + (W')_p(s)\delta p + (W')_v(s)\delta v]ds \\ &\quad + W_u(0)\delta u(0) + W_x(0)\delta x(0) + W_p(0)\delta p + W_v(0)\delta v. \end{aligned}$$

From differential equations (2.16), (2.18) and (2.20),

$$(W')_x = 0 \ , \quad (W')_p = 0 \ , \quad (W')_v = 0$$

are obtained. Therefore,

$$\Psi(x^{(i+1)}(1), p^{(i+1)}) - \Psi(x^{(i)}, p^{(i)}) = \int_0^1 G\delta u d\tau + W_p(0)\delta p + W_v(0)\delta v \tag{3.42}$$

is obtained.

From expression (2.29)

$$\delta u(\tau) = \alpha\eta(\tau)(H_u(\tau) + \nu G(\tau))^T. \tag{3.43}$$

From expression (2.30)

$$\delta p = \beta(\mathcal{B}(0) + \nu W_p(0)). \tag{3.44}$$

From expression (2.31)

$$\delta w = \gamma(\mathcal{C}(0) + \nu W_w(0))^T. \tag{3.45}$$

Expressions (3.43) $\sim$ (3.45) are substituted into expression (3.42) and we put the terminal state restriction as $\Psi(x^{(i+1)}(1), p^{(i+1)}, 1) = 0$. Then,

$$\begin{aligned}
-\Psi(x^{(i)}(1), p^{(i)}) &= \alpha \int_0^1 G(\tau)\eta(\tau)(H_u(\tau) + \nu G(\tau))^T d\tau + W_p(0)\beta(\mathcal{B}(0) + \nu W_p(0)) \\
&\quad + W_v(0)\gamma(\mathcal{C}(0) + \nu W_v(\tau))^T \\
&= \alpha \int_0^1 G(\tau)\eta(\tau)H_u^T(\tau)d\tau + \beta W_p(0)\mathcal{B}(0) + \gamma W_v(0)\mathcal{C}(0) \\
&\quad + \alpha \int_0^1 G(\tau)\eta(\tau)G^T(\tau)\nu^T d\tau + \beta W_p(0)W_p^T(0)\nu^T + \gamma W_v(0)W_v^T(0)\nu^T
\end{aligned}$$

This equation is solved for ν.

$$\begin{aligned}
\nu &= -\{\Psi^T(x^{(i)}(1), p^{(i)}) + \alpha \int_0^1 H_u(\tau)\eta(\tau)G^T(\tau)d\tau \\
&\quad + \beta\mathcal{B}(0)W_p^T(0) + \gamma\mathcal{C}(0)W_v^T(0)\} \cdot M^{-1} \quad ,
\end{aligned} \tag{3.46}$$

$$M = \alpha \int_0^1 G(\tau)\eta(\tau)G^T(\tau)d\tau + \beta W_p(0)W_p^T(0) + \gamma W_v(0)W_v^T(0) \quad . \tag{3.47}$$

Considering the circumstances mentioned above, the algorithm for optimal control with parameter v is as follows.

Algorithm

1) Expected values of the control variable $u^{(0)}(\tau)$, time coefficient $p^{(0)}$ and parameter $v^{(0)}$ are given. Negative values are given to coefficients α, β, γ . The initial value 1 is given to $\eta(\tau)$ $(0 \le \tau \le 1)$. For the cost function and terminal error, the convergence criterion values, δ and ε, of each are established.

2) The state equation of the system (2.7) is integrated in the forward time direction and $x^{(i)}(\tau)$ $(0 \le \tau \le 1)$ is stored. At the same time, the cost function

$$\begin{aligned}
J^{(i)} &= \int_0^1 l(x^{(i)}, u^{(i)}, p^{(i)}, v^{(i)}, \tau)p^{(i)}d\tau \\
&\quad + \phi(x^{(i)}(1), v^{(i)}, p^{(i)})
\end{aligned}$$

and terminal error $\Psi(x^{(i)}(1), p^{(i)}, 1)$ are calculated and stored.

3) Differential equations (2.15) $\sim$ (2.20) are integrated in the reverse time direction. At the same time, $H_u(\tau)$ in expression (2.28) and $G(\tau)$ in expression (2.23) are calculated. By using expressions (3.46) and (3.47), the Lagrange's multiplier ν is calculated and $\mathcal{B}(0)$, $W_p(0)$, $\mathcal{C}(0)$ and $W_v(0)$ are stored.

4) $u(\tau)$ is corrected by (3.32) and p and v are corrected by (3.38) and (3.40). Then, $u^{(i+1)}(\tau)$ $(0 \le \tau \le 1)$ is stored. At the same time, by using expressions (2.7) and (2.9), $x^{(i+1)}(\tau)$, $J^{(i+1)}$ and $\Psi^{(i+1)}(1)$ are calculated and stored.

5) When the convergence conditions

$$\begin{aligned}
&|J^{(i+1)} - J^{(i)}| \le \delta \\
&|\Psi^{(i+1)}(1) - \Psi^{(i)}(1)| \le \varepsilon
\end{aligned}$$

are simultaneously satisfied, the iterative calculation is completed. When the conditions are not met, $u^{(i)}$, $x^{(i)}(\tau)$, $J^{(i)}$ and $\Psi^{(i)}(1)$ are renewed to $u^{(i+1)}$, $x^{(i+1)}(\tau)$, $J^{(i+1)}$ and $\Psi^{(i+1)}(1)$, respectively. Return to 3).

In this calculation algorithm, δ and ε are some infinitesimal constants and are used for judging the convergence condition of the iterative calculation.

4. SIMULATION. To verify the performance of the algorithm, three computer simulations are provided as follows. For the system,

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + w \ , \quad & x_1(0) &= 0 \\ \dot{x}_2(t) &= u(t) \ , & x_2(0) &= 4 \end{aligned}$$

the terminal restriction condition

$$\Psi(t_f) = \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix} = 0$$

and the control restriction

$$|u(t)| \leq 1$$

are satisfied and parameter w is restricted as

$$w_{\min} \leq w \leq w_{\max}.$$

We consider the problem of minimizing the cost function

$$J = t_f = \int_0^{t_f} 1 \cdot dt.$$

Also, when this problem is solved analytically, the parameter w has no restrictions and the optimal solutions u^{opt}, w^{opt} and p^{opt} are

$$\begin{cases} u^{opt} = -1.0 \\ w^{opt} = -2.0 \\ p^{opt}(= t_f = J) = 4.0 \end{cases} .$$

For this problem, by using the proposed algorithm, calculations of three cases are provided.

Case 1) The parameter w has the following restriction

$$-5.0 \leq w \leq 2.0 \ .$$

Let $u^{(0)} = 0.0$, $w^{(0)} = -4.0$ and $p^{(0)} = 1.0$. The results are shown in Fig. 1 $\sim$ Fig. 4.

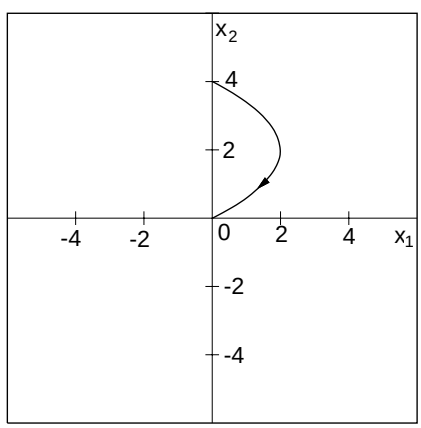

FIGURE 1. **Case** 1) Trajectory of states.

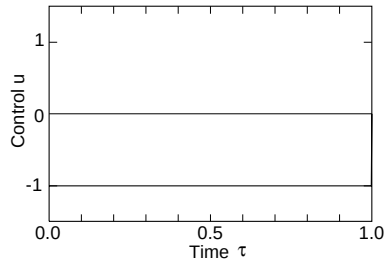

FIGURE 2. **Case** 1) Optimal control output.

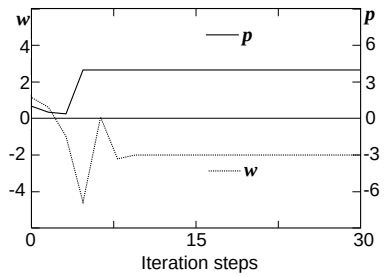

FIGURE 3. **Case** 1) The convergence of time coefficient p and parameter w.

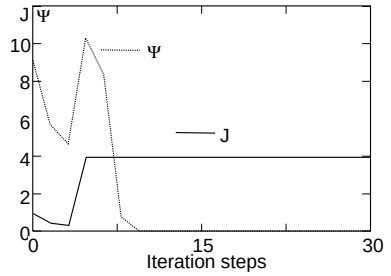

FIGURE 4. **Case** 1) The convergence of cost function and error of constraint.

In this case, the optimal value is included in the restriction of parameter w. Therefore, this case can be solved analytically, and the parameter w converges to the analytical value.

Case 2) The parameter w has the following restriction

$$-8.0 \leq w \leq -3.0.$$

Let $u^{(0)} = 0.0$, $w^{(0)} = -6.0$, $p^{(0)} = 1.0$. The results are shown in Fig. 5 $\sim$ Fig. 8.

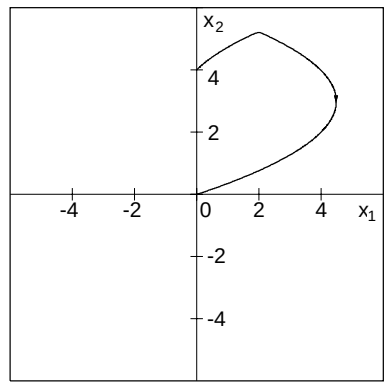

FIGURE 5. **Case** 2) Trajectory of states.

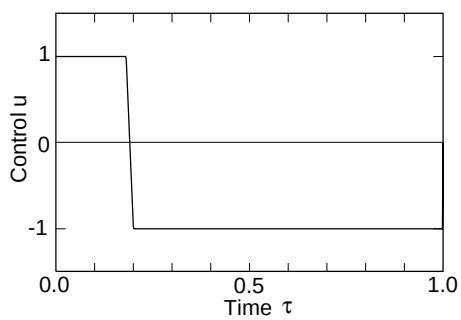

FIGURE 6. **Case** 2) Optimal control output.

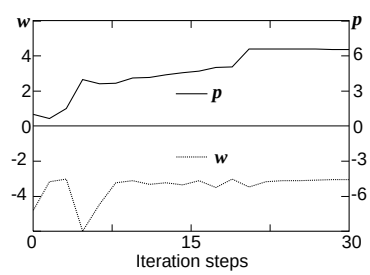

FIGURE 7. **Case** 2) The convergence of time coefficient p and parameter w.

In this case, because of the restriction of parameter w, it converges to the nearest value to the analytical value within the restriction. In our results, $w^{opt} = -3.0$, $p^{opt} = 6.4763$, cost function $J^{opt} = 6.4727$ and terminal error $\Psi = 0.000094$.

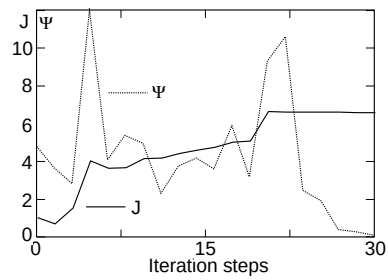

FIGURE 8. **Case** 2) The convergence of cost function and error of constraint.

Case 3) The parameter w has the following restriction

$$-1.0 \leq w \leq 5.0$$

Let $u^{(0)} = 0.0$, $w^{(0)} = 2.0$, $p^{(0)} = 1.0$. The results are shown in Fig. 9 $\sim$ Fig. 12.

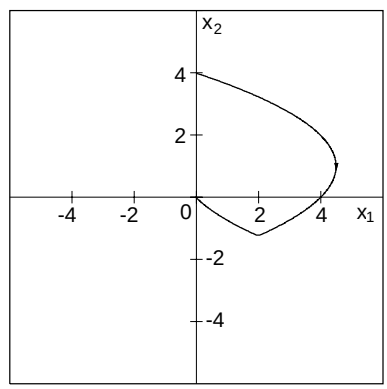

FIGURE 9. **Case** 3) Trajectory of states.

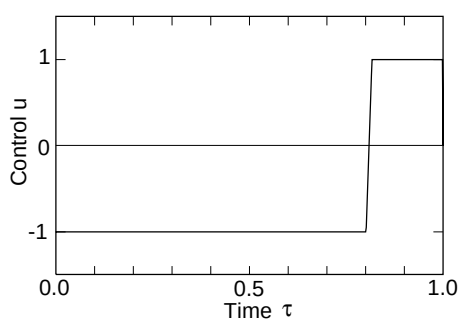

FIGURE 10. **Case** 3) Optimal control output.

In this case, the results are the same as **Case** 2:. $w^{opt} = -0.999988$, $p^{opt} = 6.4729$, cost function $J^{opt} = 6.4725$ and terminal error $\Psi = 0.000032$.

In this simulation, the gradient coefficients are $\alpha = -0.3$, $\beta = -0.03$ and $\gamma = -0.03$.

Fig. 1, Fig. 5 and Fig. 9 show each track of the state variable after 30 iterative calculations.

Fig. 2, Fig. 6 and Fig. 10 show each optimal control after 30 iterative calculations.

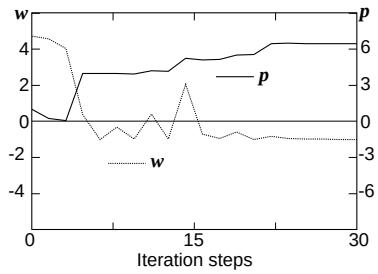

FIGURE 11. **Case** 3) The convergence of time coefficient p and parameter w.

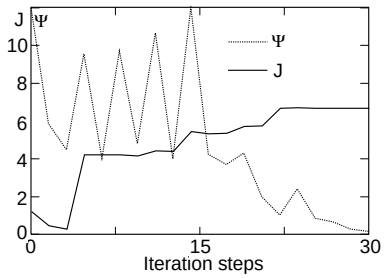

FIGURE 12. **Case** 3) The convergence of cost function and error of constraint.

Fig. 3, Fig. 7 and Fig. 11 track the progress of converging time coefficient p and optimized parameter w.

Fig. 4, Fig. 8 and Fig. 12 show the convergences of the cost function J and terminal error Ψ.

5. CONCLUSION. In this paper, the authors proposed a new numerical calculation method for optimal control problems having adjustable parameters with constraint and unknown terminal time with terminal restriction and control restrictions. The main strengths of this method are that:

1) it is able to deal with problems that have adjustable parameters, restrictions of control input, unknown terminal times, etc. at the same time by introducing a pseudo time variable τ and a new function vector W, and

2) we are able to obtain a stable convergence and sufficiently small terminal error.

However, it is not guaranteed that the solution obtained by this algorithm is the global solution. Therefore, in the case of some local solutions existing, the trial must be repeated.

From the results of our simulation works using the proposed method, the effectiveness of this numerical calculation algorithm is confirmed.

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Appendix A. SYMBOLS.

1) Scalar function g is a partial differentiation with respect to n-dimensional vector function x shown by $g_x = \begin{bmatrix} g_{x_1} & \cdots & g_{x_n} \end{bmatrix}$.
2) m-dimensional function f is a partial differentiation with respect to n-dimensional vector function x shown by

$$f_x = \begin{bmatrix} f_{1x_1} & \cdots & f_{1x_n} \\ \vdots & \ddots & \vdots \\ f_{mx_1} & \cdots & f_{mx_n} \end{bmatrix}.$$

Appendix B. DIFFERENTIAL FUNCTION OF W.

Because W is an r-dimensional vector, the i-th element of W' is

$$\frac{dW_i}{d\tau} = \frac{\partial W_i}{\partial \tau} + \sum_{k=1}^{n} \frac{\partial W_i}{\partial x_k}\frac{\partial x_k}{\partial \tau} \tag{B.48}$$

$(W')_x$ is an $r \times n$-dimensional matrix. The element at the i-th row and j-th rank of $(W')_x$ is

$$\frac{\partial}{\partial x_j}\Big(\frac{dW_i}{d\tau}\Big) = \frac{\partial^2 W_i}{\partial x_j \partial \tau} + \sum_{k=1}^{n} \frac{\partial^2 W_i}{\partial x_j \partial x_k}\frac{\partial x_k}{\partial \tau} + \sum_{k=1}^{n} \frac{\partial W_i}{\partial x_k}\frac{\partial}{\partial x_j}\Big(\frac{\partial x_k}{\partial \tau}\Big) \tag{B.49}$$

W'_x is an $r \times n$-dimensional matrix. The element at the i-th row and j-th rank of W'_x is

$$\frac{d}{d\tau}\Big(\frac{\partial W_i}{\partial x_j}\Big) = \frac{\partial^2 W_i}{\partial x_j \partial \tau} + \sum_{k=1}^{n} \frac{\partial^2 W_i}{\partial x_j \partial x_k}\frac{\partial x_k}{\partial \tau}. \tag{B.50}$$

From expressions $(B.49)$ and $(B.50)$,

$$\frac{\partial}{\partial x_j}\Big(\frac{dW_i}{d\tau}\Big) = \frac{d}{d\tau}\Big(\frac{\partial W_i}{\partial x_j}\Big) + \sum_{k=1}^{n} \frac{\partial W_i}{\partial x_k}\frac{\partial}{\partial x_j}\Big(\frac{\partial x_k}{\partial \tau}\Big) \tag{B.51}$$

is obtained. This is rewritten as a matrix, and the relation

$$\begin{aligned} (W')_x &= W'_x + W_x x'_x \\ &= W'_x + W_x (fp)_x \\ &= W'_x + W_x f_x p \end{aligned}$$

is obtained.

Similarly,

$$(W')_u = W'_u + W_x f_u p$$

$$\begin{aligned} (W')_p &= W'_p + W_x x'_p \\ &= W'_p + W_x (f_p p + f) \end{aligned}$$

$$\begin{aligned} (W')_v &= W'_v + W_x x'_v \\ &= W'_v + W_x f_w p \end{aligned}$$

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