VALUATION OF OPTION WITH STREAM OF PAYOFF

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Abstract. The present paper is concerned with the valuation of a derivative which has a stream of payoff contingent on an underlying asset. A general approach to value such a derivative is developed and several examples with economic meanings are provided.

1. Introduction. An important problem in economics is that how should an individual allocate his wealth among consumption and securities optimally when he is expected to receive a stream of uncertain labor income in future working period? To solve such a problem, a key step, as shown by Koo [5], Richard [7], and Svensson and Werner [8], is to value the individual's human capital, his income stream in future. Based on the fact that the average labor income in the real economy tends to co-move with security prices in business-cycles, Bodie, Merton, and W. Samuelson [2] and some other studies consider the cases where an individual's labor income process depends on the price process of a stock, which can be explained either as the index of security markets, or as the stock issued by the firm where the individual works. In such cases, the valuation problem of human capital is in fact an asset pricing problem of a contingent claim with a stream of payoff.

Based on the researches on option pricing, which is originated by Black and Scholes [1] and is surveyed by, for example, Musiela and Rutkowski [6] and Karatzas and Shreve [4], the present paper develops a general approach to value contingent claim with a stream of payoff. Several examples with economic meanings are provided.

2. The Economy. Consider the following economy. In the spot market, a bond and a stock are continuously traded during a period $[0, T]$. The bond price $B(t)$ at time t follows a process $dB(t) = rB(t)dt$ with a constant r and $B(0) = 1$. The stock price $S(t)$ follows a geometric Brownian motion $dS(t) = \mu S(t)dt + \sigma S(t)dZ(t)$, where μ and σ are constants and $Z = \{Z(t); 0 \leq t \leq T\}$ denotes a standard Brownian motion on a filtered probability space $\{\Omega, \mathcal{F}, P\}$. Here, the underlying filtration $\mathcal{F} \equiv \{\mathcal{F}_t; 0 \leq t \leq T\}$ is assumed to be generated by the Brownian motion Z which represents all the uncertainty in this economy. In the derivative market, there is an option which, at each time $t \in [0, T]$, pays its holder an amount of $y(t)$. It is assumed that $y(t)$ purely depends on $\{S(s); 0 \leq s \leq t\}$ and satisfies $\int_0^T \mid y(t) \mid dt < \infty$. For the economic background mentioned in Section 1, this option is called a "human capital option".

The purpose of this paper is to derive the price of the human capital option such that there is no arbitrage opportunity between the two markets. However, for the fact that a European or an American call or put option has payoff only at the exercising date, while the human capital option continuously generates payoff throughout the period $[0, T]$, pricing methods of common options can not be applied directly to the present case. Some extensions to the notions of self-financing strategy, replicating strategy, and so on are needed, which are done in the following sections.

3. Trading and Consumption Strategy. A trading and consumption strategy in the spot market is denoted by a triple $\{m_1, m_2, C\}$ of measurable processes, where $m_1(t)$ is the

shares of the stock that are held by the individual at time t, $m_2(t)$ is that of the bond, and $C(t)$ with $C(0) = 0$ is the cumulative amount of funds that are withdrawn and consumed by the individual during the period $[0, t]$. Accordingly, the definition of self-financing strategy is given as follows, where $V(t) \equiv m_1(t)S(t) + m_2(t)B(t)$ is called the wealth of the strategy at time t.

Definition 3.1. *A trading and consumption strategy $\{m_1, m_2, C\}$ is said to be self-financing if it satisfies*

$$V(t) + C(t) = V(0) + \int_0^t m_1 dS + \int_0^t m_2 dB \tag{3.1}$$

for every $t \in [0, T]$.

Let $G(t) \equiv V(t) + B(t) \int_0^t \frac{dC(s)}{B(s)} = V(t) + C(t) + \int_0^t re^{r(t-s)}C(s)ds$ denote the cumulative gain of strategy $\{m_1, m_2, C\}$, where $\int_0^t re^{r(t-s)}C(s)ds$ can be explained as the opportunity cost of consumption in period $[0, t]$. It should be noted that $G(0) = V(0)$; moreover, if the strategy $\{m_1, m_2, C\}$ is a self-financing strategy, then $G(t) = G(0) + \int_0^t m_1 dS + \int m_2 dB + \int_0^t re^{r(t-s)}C(s)ds$ holds for every $t \in [0, T]$. With the notion of the gain process of a strategy, an arbitrage strategy is defined as follows:

Definition 3.2. *A self-financing strategy $\{m_1, m_2, C\}$ is called an arbitrage strategy if $G(0) < 0$ and $G(T) \geq 0$, or $G(0) = 0$ and $G(T) \geq 0$ with $Pr\{G(T) > 0\} > 0$.*

For convenience of analysis, a martingale measure is introduced into the model as follows:

Definition 3.3. *In the case where the discounted stock process $S^*(t) \equiv S(t)/B(t)$ follows a local martingale under Q, which is a probability measure on $\{\Omega, \mathcal{F}\}$ and is equivalent to P, the measure Q is called to be a martingale measure for the spot market.*

Lemma 3.1. *The probability measure Q defined by the Radon-Nikodym derivativ*

$$\frac{dQ}{dP} = exp(\frac{r-\mu}{\sigma}Z(T) - \frac{1}{2}\frac{(r-\mu)^2}{\sigma^2}T) \tag{3.2}$$

is a martingale measure for the spot market, and the discounted stock price $S^*(t)$ satisfies

$$dS^*(t) = \sigma S^*(t) dZ^*(t) \tag{3.3}$$

where

$$Z^*(t) \equiv Z(t) - \frac{r-\mu}{\sigma}t \tag{3.4}$$

follows a standard Brownian motion on { Ω, $\mathcal{F}$, Q } (from the Girsanov theorem, Section 3.5 in Karatzas and Shreve [3]).

Now, consider a self-financing strategy's gain process under martingale measure Q. Denote by $V^*(t) \equiv \frac{G(t)}{B(t)}$ a strategy's discounted value and $G^*(t) \equiv \frac{G(t)}{B(t)} = V^*(t) + \int_0^t \frac{dC}{B}$ its discounted gain at date t. It follows from Definition 3.1 and Lemma 3.1 that the following result holds under Q.

Lemma 3.2. *A strategy $\{m_1, m_2, C\}$ is self-financing if and only if it satisfies*

$$G^*(t) = G(0) + \int_0^t m_1(s)dS^*(s) \tag{3.5}$$

for every $t \in [0, T]$.

Proof. Sufficiency. (3.5) means that

$$\frac{V(t)}{B(t)} + \int_0^t \frac{dC(s)}{B(s)} = V(0) + \int_0^t m_1(s)d(\frac{S(s)}{B(s)}), \tag{3.6}$$

which implies that

$$d(\frac{V(t)}{B(t)}) + \frac{dC(t)}{B(t)} = m_1(t)d(\frac{S(t)}{B(t)}), \tag{3.7}$$

or, equivalently,

$$dV(t) + dC(t) = V(t)rdt + m_1(t)dS(t) - m_1(t)S(t)rdt. \tag{3.8}$$

Substituting the definition $V(t) \equiv m_1(t)S(t) + m_2(t)B(t)$ into the right side of (3.8), it is obtained that

$$\begin{aligned} dV(t) + dC(t) &= (m_1(t)S(t) + m_2(t)B(t))rdt + m_1(t)dS(t) - m_1(t)S(t)rdt \\ &= m_1(t)dS(t) + m_2(t)B(t)rdt \\ &= m_1(t)dS(t) + m_2(t)dB(t). \end{aligned} \tag{3.9}$$

(3.9) is equivalent to (3.1), the definition of a self-financing strategy. This completes the proof of sufficiency.

Necessity. A proof of necessity can be provided by reversing each step mentioned above.

Lemma 3.2 implies that the discounted gain process G^* of a self-financing strategy follows a local-martingale under Q measure. It is well-known that to restrict investors' strategies to the set of self-financing strategies does not exclude the existence of arbitrage strategies. To exclude arbitrage strategies from the spot market, it is needed to assume that each individual holds admissible strategies, which are defined as follows:

Definition 3.4. *A self-financing strategy $\{m_1, m_2, C\}$ is called admissible if its discounted gain $G^*(t)$ follows a martingale under Q.*

Lemma 3.3. *An admissible self-financing strategy is not an arbitrage strategy.*

Proof. Let $\{m_1, m_2, C\}$ be an arbitrage strategy. By the definition of arbitrage strategy, the gain process G^* of this strategy satisfies $E_0^Q[G^*(T)] > V(0) = G(0)$, which implies that G^* is not a martingale under Q. This suggests that the strategy $\{m_1, m_2, C\}$ is not admissible, and hence completes the proof.

4. Valuation of Human Capital Option. Consider the human capital option with payoff process $y = \{y(t); 0 \le t \le T\}$.

Definition 4.1. *The human capital option is said to be attainable if there is an admissible self-financing strategy* $\{m_1, m_2, C\}$ *such that*

$$C(t) = \int_0^t y(s)ds \qquad \text{for every } t \in [0,T] \tag{4.1}$$

and

$$V(T) = 0. \tag{4.2}$$

Moreover, this admissible self-financing strategy is called a replicating strategy of the human capital option.

Here, (4.1) means that the replicating strategy generates the same payoff process as the human capital option, and (4.2) must hold because the value of the option is zero at terminal date T. It is easy to see that, for any $t \in [0,T]$, an individual who buys the replicating strategy at a cost of $V(t)$ will have a consumption process $\{C(s) = \int_t^s y(u)du \ ; \ s \in [t,T]\}$ which is just the cumulative payoff process on the option. Hence, it is natural to conjecture that, if there is no arbitrage opportunity between the spot market and the contingent claim market, the price of the option at time t should equal to the value of the replicating strategy. In this sense, the no-arbitrage price of the option can be defined as $\pi(t) \equiv V(t)$. This intuition is true, only, for $\pi(t)$ to be well-defined, it is needed to show that $V(t)$ is unique.

Lemma 4.1. *If the human capital option is attainable, then, all its replicating strategies have an unique wealth process.*

Proof. Let $\{m_1, m_2, C\}$ be any replicating strategy of the human capital option. It is easy to see that this strategy's discounted gain at time t satisfies

$$\begin{aligned} G^*(t) &= V^*(t) + \int_0^t \frac{dC(s)}{B(s)} \\ &= V^*(t) + \int_0^t \frac{y(s)}{B(s)} ds, \end{aligned} \tag{4.3}$$

where the first line follows the definition of gain process and the second line follows from (4.1). On the other hand, for this strategy is admissible, the discounted gain process $G^*(t)$ follows a martingale under Q, which implies that

$$\begin{aligned} G^*(t) &= E_t^Q[G^*(T)] \\ &= E_t^Q[V^*(T) + \int_0^T \frac{dC(s)}{B(s)}] \\ &= E_t^Q[\int_0^T \frac{y(s)}{B(s)} ds] \\ &= \int_0^t \frac{y(s)}{B(s)} ds + E_t^Q[\int_t^T \frac{y(s)}{B(s)} ds]. \end{aligned} \tag{4.4}$$

Comparing (4.3) and (4.4), we obtain that

$$V^*(t) = E_t^Q[\int_t^T \frac{y(s)}{B(s)} ds]. \tag{4.5}$$

Since the right side of (4.5) does not depend on the strategy $\{m_1, m_2, C\}$, any replicating strategy of the human capital option satisfies (4.5). This completes the proof.

Lemma 4.1 insures that the no-arbitrage price $\pi(t)$ of the option is well-defined; moreover, (4.5) implies that $\pi(t)$ is simply the discounted present value of the option's payoff in future period. The following theorem states these results formally, and gives a sufficient condition for a human capital option to be attainable.

Theorem 4.1. *If $E^Q[(\int_0^T y^* dt)^2] < \infty$, where $y^*(t) \equiv y(t)/B(t)$, then, the human capital option is attainable and its no-arbitrage price at any time $t \in [0, T]$ is given by*

$$\pi(t) = B(t)E_t^Q[\int_t^T y^*(s)ds]. \tag{4.6}$$

Proof. If $E^Q[(\int_0^T y^*(t)dt)^2] < \infty$, then, by the martingale representation theorem, there is a measurable process $\eta = \{\eta(t); 0 \le t \le T\}$ such that

$$E^Q[\int_0^T \eta^2(t)dt] < \infty, \tag{4.7}$$

$$\int_0^T y^*(t)dt = E_0^Q[\int_0^T y^*(t)dt] + \int_0^T \eta(t)dS^*(t). \tag{4.8}$$

Consider the strategy $\{m_1, m_2, C\}$ where

$$m_1(t) = \eta(t), \tag{4.9}$$

$$m_2(t) = E_t^Q[\int_t^T y^*(s)ds] - m_1(t)S^*(t), \tag{4.10}$$

$$C(t) = \int_0^t y(s)ds. \tag{4.11}$$

Obviously, this strategy satisfies (4.1)-(4.2). Hence, in order to show that the strategy is a replicating strategy of the human capital option, it only needs to show that this strategy is both self-financing and admissible. Recall that by (4.8)-(4.11), the discounted value of the strategy at time t satisfies

$$\begin{aligned}
V^*(t) &= E_t^Q[\int_t^T y^*(s)ds] \\
&= E_t^Q[\int_0^T y^*(s)ds - \int_0^t y^*(s)ds] \\
&= E_t^Q\{E_0^Q[\int_0^T y^*(s)ds] + \int_0^T \eta(s)dS^*(s)\} - \int_0^t y^*(s)ds \\
&= E_0^Q[\int_0^T y^*(s)ds] + \int_0^t \eta(s)dS^*(s) \ - \int_0^t y^*(s)ds \\
&= G(0) + \int_0^t m_1(s)dS^*(s) - \int_0^t \frac{dC(s)}{B(s)},
\end{aligned}$$

which implies

$$G^*(t) = G(0) + \int_0^t +m_1(s)dS^*(s). \tag{4.12}$$

Hence, by Lemma 3.2, the strategy defined by (4.9)-(4.11) is self-financing. Moreover, (4.7) insures that $G^*(t)$ is a martingale under Q, which implies that the strategy is also admissible. Thus, the strategy defined by (4.9)-(4.11) is a replicating strategy of the human capital option, and $V^*(t) = E_t^Q[\int_t^T y^*(s)ds]$, the discounted wealth of the replicating strategy, is the no-arbitrage price of the option at time t. This completes the proof.

5. Examples and Economic Explanations. In this section we present some examples.

Example 5.1. Consider a human capital option with payoff process

$$y(t) = \begin{cases} a & \text{if } S(t) > b, \\ 0 & \text{otherwise,} \end{cases} \tag{5.1}$$

where a and b are positive constants. It can be shown that this option is attainable and the no-arbitrage price at time t is given by

$$\pi(t) = a \int_0^{T-t} e^{-rs}\Phi(\frac{\log S - \log b + s(r - \sigma^2/2)}{\sigma\sqrt{s}}ds \tag{5.2}$$

where $\Phi(\ )$ denotes the cumulative distribution function of standard normal distribution.

Remark 5.1. Assumption (5.1) can be explained as follows: assume that an individual works in a firm whose stock price is S. As long as the stock price is in a high level, which means the firm is in good states, the individual can work and receive a fixed wage a; however, if the firm's state becomes worse and its stock price falls to a certain level, say $S \leq b$, then, the individual loses his job and receives no labor income. That is, the lower the stock price of the firm, the higher the employee's unemployment risk.

Example 5.2. Consider a human capital option with payoff process

$$y(t) = \begin{cases} a & \text{if } \min_{s\in[0,t]} S(s) > b, \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

In this case, the no-arbitrage price is

$$\pi(t) = \begin{cases} a\int_0^{T-t} \bar{F}(s;S)e^{-rs}ds & \text{if } \min_{s\in[0,t]} S(s) > b, \\ 0 & \text{otherwise,} \end{cases} \tag{5.4}$$

where $\bar{F}(s;S) = \int_s^{+\infty} f(l;S)dl$ and $f(s;S) = \frac{\log S - \log b}{\sigma\sqrt{2\pi s^3}}\exp[-\frac{[\log S - \log b + s(r-\sigma^2/2)]^2}{2\sigma^2 s}]$.

Remark 5.2. In Example 5.1, an individual, after being fired, can go back to work when the stock price rises again; however, in the present case, an individual can never work again once losing his job.

Example 5.3. Consider a human capital option with payoff process

$$y(t) = \begin{cases} aS(t) & \text{if } aS(t) > b, \\ b & \text{otherwise.} \end{cases} \tag{5.5}$$

This option is also attainable and its no-arbitrage price is

$$\pi(t) = aS \int_0^{T-t} \Phi(d_1)dt + b \int_0^{T-t} e^{-rs}(1 - \Phi(d_2))dt, \tag{5.6}$$

where $d_{1,2} = \frac{\log S - \log b/a + (r \pm \sigma^2/2)s}{\sigma\sqrt{s}}$.

Remark 5.3. In this case, $aS(t)$ represents the wage rate at time t and b represents the unemployment allowance which an individual can receive from the government when he has no job. An individual determines his labor supply in the following way: if $aS(t) > b$, which means the wage rate is higher than the unemployment allowance, the individual works and receives the wage $aS(t)$; otherwise, the individual prefers not to work, and receives the unemployment allowance b. It is noted that (5.5) can be rewritten as

$$y(t) = a(S(t) - b/a, 0)^+ + b, \tag{5.7}$$

where the first term in the right side is the payoff on a shares of a European call option with expiration date t and exercising price b/a.

6. Concluding Remarks. The present paper develops a general approach to value option with a stream of payoff contingent on an underlying asset. This approach is expected to be useful in solving many economic problems. Several examples of human capital valuation are provided in this paper; further discussions on economic implications, however, are left for future studies.

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