FURTHER EXTENSIONS OF CHARACTERIZATIONS OF CHAOTIC ORDER ASSOCIATED WITH KANTOROVICH TYPE INEQUALITIES

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Abstract. We showed characterizations of chaotic order via Kantorovich inequality in our previous paper. Recently as a nice application of generalized Furuta inequality, Furuta and Seo showed an extension of one of our results and a related result on operator equations. In this paper, by using essentially the same idea as theirs, we shall show further extensions of both their results and the following our another previous result which is a characterization of chaotic order via Specht's ratio. "*Let A and B be positive invertible operators satisfying $MI \geq A \geq mI > 0$. Then $\log A \geq \log B$ is equivalent to $M_h(p)A^p \geq B^p$ holds for all $p > 0$, where $h = \frac{M}{m} > 1$ and*

$$M_h(p) = \frac{h^{\frac{p}{h^p-1}}}{e \log h^{\frac{p}{h^p-1}}}."$$

1. Introduction

An operator means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$ and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. The following Löwner-Heinz theorem is well known: $A \geq B \geq 0$ *ensures* $A^\alpha \geq B^\alpha$ *for any* $\alpha \in [0, 1]$. For the sake of convenience on application, the following Theorem F was established.

Theorem F (Furuta inequality [9])**.**

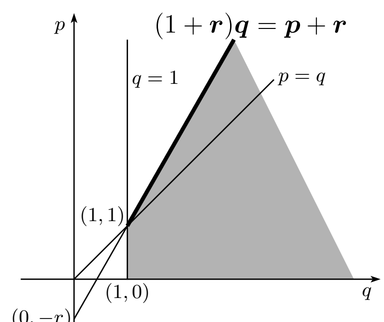
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p + r$.



Figure

We remark that Theorem F yields Löwner-Heinz theorem when we put $r = 0$ in (i) or (ii) stated above. Alternative proofs of Theorem F are given in [6][18] and also an elementary one-page proof in [10]. It is shown in [23] that the domain drawn for p, q and r in the Figure is best possible one for Theorem F.

As an extension of Theorem F, the following Theorem G was obtained in [13].

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Theorem G ([13]). *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,*

$$F_{p,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r \geq t$ and $s \geq 1$, and $F_{p,t}(A,A,r,s) \geq F_{p,t}(A,B,r,s)$, that is, for each $t \in [0,1]$ and $p \geq 1$,

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{1.1}$$

holds for any $s \geq 1$ and $r \geq t$.

Ando-Hiai [2] established excellent log majorization results and proved the following useful inequality equivalent to the main log majorization theorem: *If $A \geq B \geq 0$ with $A > 0$, then*

$$A^r \geq \{A^{\frac{r}{2}}(A^{\frac{-1}{2}}B^pA^{\frac{-1}{2}})^rA^{\frac{r}{2}}\}^{\frac{1}{p}}$$

holds for any $p \geq 1$ and $r \geq 1$. Theorem G interpolates the inequality stated above by Ando-Hiai and Theorem F itself, and also extends results of [7][11] and [12]. A nice mean theoretic proof of Theorem G is shown in [8] and one-page proof of (1.1) is shown in [15]. In [16], we showed equivalence relation among the inequality (1.1), monotonicity of the function $F_{p,t}(A,B,r,s)$ in Theorem G and related results. The best possibility of the outside exponents of both sides in (1.1) is shown in [24] and [26].

On the other hand, related to Löwner-Heinz theorem, the following proposition is also well known: $A \geq B \geq 0$ *does not always assure $A^\alpha \geq B^\alpha$ for any $\alpha > 1$.* As a way to settle this inconvenient, the following result is given in [14].

Theorem A.1 ([14]). *If $A \geq B \geq 0$ and $MI \geq A \geq mI > 0$, then*

$$\left(\frac{M}{m}\right)^{p-1} A^p \geq K_+(m,M,p)A^p \geq B^p \text{ for } p \geq 1,$$

where

$$K_+(m,M,p) = \frac{(p-1)^{p-1}}{p^p}\frac{(M^p-m^p)^p}{(M-m)\,(mM^p-m^pM)^{p-1}}. \tag{1.2}$$

We remark that Theorem A.1 is related to both Hölder-McCarthy inequality [19] and Kantorovich inequality: *If $MI \geq A \geq mI > 0$, then $\left(A^{-1}x,x\right)(Ax,x) \leq \frac{(m+M)^2}{4mM}$ holds for every unit vector x in H.* The number $\frac{(m+M)^2}{4mM}$ is called Kantorovich constant and $K_+(m,M,2) = \frac{(m+M)^2}{4mM}$ where $K_+(m,M,p)$ is stated in (1.2), so that $K_+(m,M,p)$ is a generalization of Kantorovich constant. Many authors investigated a lot of papers on Kantorovich inequality, among others, there is a long research series of Mond-Pečarić, some of them are [20] and [21].

The order between positive invertible operators A and B defined by $\log A \geq \log B$ is said to be chaotic order which is a weaker order than usual order $A \geq B$. As an application of Theorem F, the following characterization of chaotic order is well known.

Theorem A.2 ([7][12]). *Let A and B be positive invertible operators. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $A^p \geq (A^{\frac{p}{2}}B^pA^{\frac{p}{2}})^{\frac{1}{2}}$ *for all $p \geq 0$.*

(iii) $A^u \geq (A^{\frac{u}{2}}B^pA^{\frac{u}{2}})^{\frac{u}{p+u}}$ *for all $p \geq 0$ and $u \geq 0$.*

(i)$\Leftrightarrow$(ii) of Theorem A.2 is shown in [1]. Recently a simple and excellent proof of (i)$\Rightarrow$(iii) is shown in [25] by only applying Theorem F, and a simplified proof of (ii)$\Rightarrow$(i) is shown in [17].

As other characterizations of chaotic order, we prove the following two results as applications of Theorem A.1 and Theorem A.2 in [27].

Theorem B.1 ([27])**.** *Let A and B be positive invertible operators satisfying $MI \geq A \geq mI > 0$. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $\dfrac{(m^p + M^p)^2}{4m^p M^p} A^p \geq B^p$ *for all* $p \geq 0$.

Theorem B.2 ([27])**.** *Let A and B be positive invertible operators satisfying $MI \geq A \geq mI > 0$. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $M_h(p)A^p \geq B^p$ *for all $p \geq 0$, where $h = \frac{M}{m} > 1$ and*

$$M_h(p) = \frac{h^{\frac{p}{h^p - 1}}}{e \log h^{\frac{p}{h^p - 1}}}. \tag{1.3}$$

Theorem B.2 gives a more precise sufficient condition for chaotic order than Theorem B.1 since $\frac{(m^p+M^p)^2}{4m^p M^p} \geq M_h(p)$ holds for all $p \geq 0$ by the following two lemmas.

Lemma B.3 ([27])**.** *Let $K_+(m, M, p)$ be defined in* (1.2)*. Then*

$$F\,(p, r, m, M) = K_+\left(m^r, M^r, \frac{p+r}{r}\right)$$

is an increasing function of p, r and M, and also a decreasing function of m for $p > 0$, $r > 0$ and $M > m > 0$. And the following inequality holds:

$$\left(\frac{M}{m}\right)^p \geq K_+\left(m^r, M^r, \frac{p+r}{r}\right) \geq 1. \tag{1.4}$$

Lemma B.4 ([27])**.** *Let $M > m > 0$, $p > 0$ and $K_+(m, M, p)$ be defined in* (1.2)*. Then*

$$\lim_{r \to +0} K_+\left(m^r, M^r, \frac{p+r}{r}\right) = M_h(p),$$

where $h = \frac{M}{m} > 1$ and $M_h(p)$ be defined in (1.3)*.*

We remark that $M_h(1) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \log h}$ is called Specht's ratio [4][22], which is the best upper bound of the ratio of the arithmetic mean to the geometric mean of numbers x_i satisfying $M \geq x_i \geq m > 0$ ($i = 1, 2, \cdots, n$), that is, the following inequality holds:

$$\frac{(h-1)h^{\frac{1}{h-1}}}{e \log h} \sqrt[n]{x_1 x_2 \cdots x_n} \geq \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

In [3], we showed a simplified proof of Theorem B.2 by using determinant for positive operators defined in [4] and [5]. Moreover we showed the following result which interpolates (i)$\Rightarrow$(ii) of both Theorem B.1 and Theorem B.2 in [27].

Theorem B.5 ([27])**.** *Let A and B be positive invertible operators satisfying $MI \geq A \geq mI > 0$. If $\log A \geq \log B$, then*

$$K_+\left(m^r, M^r, \frac{p+r}{r}\right) A^p \geq B^p \qquad \textit{holds for } p > 0 \textit{ and } r > 0,$$

where $K_+(m, M, p)$ is defined in (1.2).

As a nice application of Theorem G, Furuta and Seo established the following result in [17].

Theorem C.1 ([17]). *Let A and B be positive invertible operators. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each $\alpha \in [0,1]$, $p \geq 0$, $u \geq 0$ and $s \geq 1$ such that $(p+\alpha u)s \geq (1-\alpha)u$, there exists the unique invertible positive contraction T satisfying*
$$TA^{(p+\alpha u)s}T = (A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^s.$$

(iii) *For each $\alpha \in [0,1]$, $p \geq u \geq 0$ and $s \geq 1$, there exists the unique invertible positive contraction T satisfying*
$$TA^{(p+\alpha u)s}T = (A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^s.$$

(iv) *For each $p \geq 0$, there exists the unique invertible positive contraction T satisfying*
$$TA^pT = B^p.$$

Moreover as an extension of Theorem B.1, Furuta and Seo also showed the following result based on Theorem C.1 in [17].

Theorem C.2 ([17]). *Let A and B be positive invertible operators satisfying $MI \geq A \geq mI > 0$. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each $\alpha \in [0,1]$, $p \geq 0$ and $u \geq 0$,*
$$\frac{\left(M^{(p+\alpha u)s} + m^{(p+\alpha u)s}\right)^2}{4M^{(p+\alpha u)s}m^{(p+\alpha u)s}}A^{(p+\alpha u)s} \geq (A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^s$$
holds for $s \geq 1$ such that $(p+\alpha u)s \geq (1-\alpha)u$.

(iii) *For each $\alpha \in [0,1]$ and $p \geq u \geq 0$,*
$$\frac{\left(M^{(p+\alpha u)s} + m^{(p+\alpha u)s}\right)^2}{4M^{(p+\alpha u)s}m^{(p+\alpha u)s}}A^{(p+\alpha u)s} \geq (A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^s$$
holds for $s \geq 1$.

(iv) $\dfrac{(M^p + m^p)^2}{4M^pm^p}A^p \geq B^p$ *holds for $p \geq 0$.*

In this paper, we shall show a further extension of Theorem C.1. And also we shall show a further extension of Theorem C.2 which interpolates both Theorem B.1 and Theorem B.2.

2. Extensions of the results by Furuta and Seo

Firstly, as an extension of Theorem C.1, we have the following characterization of chaotic order via operator equations.

Theorem 1. *Let A and B be positive invertible operators. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each natural number n, $\alpha \in [0,1]$, $p \geq 0$, $u \geq 0$, $s \geq 1$ and $r \geq 1-\alpha$ such that $\{nr+(n+1)\alpha\}u \geq (p+\alpha u)s$, there exists the unique invertible positive contraction $T = T(n,\alpha,p,u,r,s)$ satisfying*

$$T(A^{\frac{(p+\alpha u)s+ru}{n+1}}T)^n = A^{\frac{-(p+\alpha u)s+nru}{2(n+1)}}(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^sA^{\frac{-(p+\alpha u)s+nru}{2(n+1)}}. \tag{2.1}$$

(iii) *For each natural number n, $\alpha \in [0,1]$, $p \geq nu \geq 0$, $s \geq 1$ and real numbers r such that $\{nr+(n+1)\alpha\}u \geq (p+\alpha u)s$, there exists the unique invertible positive contraction $T = T(n,\alpha,p,u,r,s)$ satisfying*

$$T(A^{\frac{(p+\alpha u)s+ru}{n+1}}T)^n = A^{\frac{-(p+\alpha u)s+nru}{2(n+1)}}(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^sA^{\frac{-(p+\alpha u)s+nru}{2(n+1)}}.$$

(iv) *For each natural number n and $p \geq 0$, there exists the unique invertible positive contraction $T = T(n,p)$ satisfying*

$$T(A^{\frac{p}{n}}T)^n = B^p.$$

The following Corollary 2 is easily obtained by Theorem 1.

Corollary 2. *Let A and B be positive invertible operators. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each natural number n, $\alpha \in [0,1]$, $p \geq 0$, $u \geq 0$ and $s \geq 1$ such that $(p+\alpha u)s \geq n(1-\alpha)u$, there exists the unique invertible positive contraction $T = T(n,\alpha,p,u,s)$ satisfying*

$$T(A^{\frac{(p+\alpha u)s}{n}}T)^n = (A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^s. \tag{2.2}$$

(iii) *For each natural number n, $\alpha \in [0,1]$, $p \geq nu \geq 0$ and $s \geq 1$, there exists the unique invertible positive contraction $T = T(n,\alpha,p,u,s)$ satisfying*

$$T(A^{\frac{(p+\alpha u)s}{n}}T)^n = (A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^s. \tag{2.3}$$

(iv) *For each natural number n and $p \geq 0$, there exists the unique invertible positive contraction $T = T(n,p)$ satisfying*

$$T(A^{\frac{p}{n}}T)^n = B^p.$$

Remark 1. Corollary 2 can be proved by Theorem 1. And (ii), (iii) and (iv) of Corollary 2 implies (ii), (iii) and (iv) of Theorem C.1 respectively when we put $n = 1$, that is, Theorem 1 includes Theorem C.1 as a special case.

Secondly, as an extension of Theorem C.2, we have the following Kantorovich type characterization of chaotic order.

Theorem 3. *Let A and B be positive invertible operators satisfying $MI \geq A \geq mI > 0$ and $K_+(m,M,p)$ be defined in (1.2). Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each natural number n, $\alpha \in [0,1]$, $p \geq 0$ and $u \geq 0$,*

$$K_+\left(m^{\frac{(p+\alpha u)s+ru}{n+1}}, M^{\frac{(p+\alpha u)s+ru}{n+1}}, n+1\right)A^{(p+\alpha u)s} \geq (A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^s$$

holds for all $s \geq 1$ and $r \geq 1-\alpha$ such that $\{nr+(n+1)\alpha\}u \geq (p+\alpha u)s$.

(iii) *For each natural number n, $\alpha \in [0,1]$ and $p \geq nu \geq 0$,*

$$K_+\left(m^{\frac{(p+\alpha u)s+ru}{n+1}}, M^{\frac{(p+\alpha u)s+ru}{n+1}}, n+1\right) A^{(p+\alpha u)s} \geq (A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}})^s \tag{2.4}$$

holds for all $s \geq 1$ and real number r such that $\{nr+(n+1)\alpha\}u \geq (p+\alpha u)s$.

(iv) *For each natural number n and $p \geq nu \geq 0$,*

$$K_+\left(m^{\frac{p+ru}{n+1}}, M^{\frac{p+ru}{n+1}}, n+1\right) A^p \geq B^p$$

holds for real number r such that $nru \geq p$.

Remark 2. Theorem 3 implies Theorem C.2 as follows. We have (ii) [resp. (iii)] of Theorem C.2 when we put $n = 1$ and $r = \frac{(p+\alpha u)s}{u}$ in (ii) [resp. (iii)] of Theorem 3. And put $n = 1$ and $r = \frac{p}{u}$ in (iv) of Theorem 3, then we have (iv) of Theorem C.2.

As mentioned above, Theorem 3 yields Theorem C.2 and Theorem C.2 yields Theorem B.1. Moreover Theorem 3 also yields the following Theorem 4 and Theorem 4 yields Theorem B.2, which is a more precise estimation than Theorem B.1.

Theorem 4. *Let A and B be positive invertible operators satisfying $MI \geq A \geq mI > 0$, and $K_+(m,M,p)$ and $M_h(p)$ be defined in (1.2) and (1.3), respectivery. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each natural number n, $\alpha \in [0,1]$, $p \geq 0$ and $u \geq 0$*

$$K_+\left(m^{\frac{(p+\alpha u)s-\alpha u}{n}}, M^{\frac{(p+\alpha u)s-\alpha u}{n}}, n+1\right) A^{(p+\alpha u)s} \geq (A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}})^s$$

holds for all $s \geq 1$ such that $(p+\alpha u)s \geq (n+\alpha)u$.

(iii) *For each natural number n, $\alpha \in [0,1]$ and $p \geq nu \geq 0$,*

$$K_+\left(m^{\frac{(p+\alpha u)s-\alpha u}{n}}, M^{\frac{(p+\alpha u)s-\alpha u}{n}}, n+1\right) A^{(p+\alpha u)s} \geq (A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}})^s$$

holds for all $s \geq 1$.

(iv) *$M_h(p)\, A^p \geq B^p$ holds for $p \geq 0$, where $h = \frac{M}{m} > 1$.*

3. Proofs of results

Proof of Theorem 1. We shall show (i)⇒(ii)⇒(iii)⇒(iv)⇒(i) as follows.

(i)⟹(ii): In case $u = 0$, (ii) holds obviously since the assumption of (ii) ensures $ps = 0$ and (2.1) turns out to be $T^{n+1} = I$, so that we have only to show the case $u > 0$ as follows. By Theorem A.2, (i) ensures the following inequality:

$$A^u \geq (A^{\frac{u}{2}} B^p A^{\frac{u}{2}})^{\frac{u}{p+u}} \text{ for } p \geq 0 \text{ and } u > 0.$$

Put $A_1 = A^u$ and $B_1 = (A^{\frac{u}{2}} B^p A^{\frac{u}{2}})^{\frac{u}{p+u}}$, then A_1 and B_1 satisfy $A_1 \geq B_1 > 0$. By applying Theorem G, we have

$$A_1^{\frac{(p_1-t)s+r}{q}} \geq \{A_1^{\frac{r}{2}} (A_1^{\frac{-t}{2}} B_1^{p_1} A_1^{\frac{-t}{2}})^s A_1^{\frac{r}{2}}\}^{\frac{1}{q}} \tag{3.1}$$

holds for $t \in [0,1]$, $p_1 \geq 1$, $s \geq 1$, $r \geq t$ and $q \geq 1$ such that $(1-t+r)q \geq (p_1 - t)s + r$. We can put $t = 1-\alpha \in [0,1]$, $p_1 = \frac{p+u}{u} \geq 1$ and $q = n+1 \geq 1$ since the hypotheses of (ii) $r \geq 1-\alpha$ and $\{nr + (n+1)\alpha\}u \geq (p+\alpha u)s$ ensure the inequalities

$$r \geq 1 - \alpha = t$$

and

$$(1-t+r)q = (\alpha + r)(n+1) = nr + (n+1)\alpha + r \geq \frac{(p+\alpha u)s}{u} + r = (p_1 - t)s + r$$

respectively. Then (3.1) can be rewritten as follows:

$$A^{\frac{(p+\alpha u)s+ru}{n+1}} \geq \{A^{\frac{ru}{2}}(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^sA^{\frac{ru}{2}}\}^{\frac{1}{n+1}}. \tag{3.2}$$

Let T be defined as follows:

$$T = A^{\frac{-\{(p+\alpha u)s+ru\}}{2(n+1)}}(A^{\frac{ru}{2}}D^sA^{\frac{ru}{2}})^{\frac{1}{n+1}}A^{\frac{-\{(p+\alpha u)s+ru\}}{2(n+1)}}, \tag{3.3}$$

where $D = A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}}$. Then T is an invertible positive contraction and

$$A^{\frac{(p+\alpha u)s+ru}{2(n+1)}}TA^{\frac{(p+\alpha u)s+ru}{2(n+1)}} = (A^{\frac{ru}{2}}D^sA^{\frac{ru}{2}})^{\frac{1}{n+1}} \tag{3.4}$$

holds by (3.3). Therefore we have

$$(A^{\frac{(p+\alpha u)s+ru}{2(n+1)}}TA^{\frac{(p+\alpha u)s+ru}{2(n+1)}})^{n+1} = A^{\frac{ru}{2}}D^sA^{\frac{ru}{2}}. \tag{3.5}$$

It is equivalent to

$$A^{\frac{(p+\alpha u)s+ru}{2(n+1)}}T(A^{\frac{(p+\alpha u)s+ru}{n+1}}T)^nA^{\frac{(p+\alpha u)s+ru}{2(n+1)}} = A^{\frac{ru}{2}}(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^sA^{\frac{ru}{2}},$$

that is, we have (2.1).

Uniqueness of T can be shown as follows: Assume that for each natural number n, $\alpha \in [0,1]$, $p \geq 0$, $u \geq 0$, $s \geq 1$ and $r \geq 1-\alpha$ such that $\{nr + (n+1)\alpha\}u \geq (p+\alpha u)s$, there exists an invertible positive contraction S satisfying

$$S(A^{\frac{(p+\alpha u)s+ru}{n+1}}S)^n = A^{\frac{-(p+\alpha u)s+nru}{2(n+1)}}(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^sA^{\frac{-(p+\alpha u)s+nru}{2(n+1)}}. \tag{3.6}$$

By (2.1) and (3.6), we have

$$S(A^{\frac{(p+\alpha u)s+ru}{n+1}}S)^n = T(A^{\frac{(p+\alpha u)s+ru}{n+1}}T)^n. \tag{3.7}$$

(3.7) is equivalent to

$$(A^{\frac{(p+\alpha u)s+ru}{2(n+1)}}SA^{\frac{(p+\alpha u)s+ru}{2(n+1)}})^{n+1} = (A^{\frac{(p+\alpha u)s+ru}{2(n+1)}}TA^{\frac{(p+\alpha u)s+ru}{2(n+1)}})^{n+1}.$$

Then we have $S = T$.

Hence the proof of (i)$\Longrightarrow$(ii) is complete.

(ii)$\Longrightarrow$(iii): (iii) holds in case $u = 0$ obviously by the same discussion as (ii). Let $p \geq nu > 0$ in (ii), then the condition $r \geq 1-\alpha$ follows from $p \geq nu > 0$ and the other assumption of (ii) since

$$r \geq \frac{(p+\alpha u)s}{nu} - \frac{n+1}{n}\alpha \geq \frac{p+\alpha u}{nu} - \frac{n+1}{n}\alpha = \frac{p}{nu} - \alpha \geq 1 - \alpha,$$

so that we have (iii).

(iii)$\Longrightarrow$(iv): Put $r = \frac{(p+\alpha u)s}{nu}$, $\alpha = 0$ and $s = 1$ in (iii), then we have

$$T(A^{\frac{p}{n}}T)^n = B^p$$

holds for each nutural number n and $p \geq nu > 0$, i.e., $p > 0$. (iv) holds in case $p = 0$ obviously, so that the proof of (iii)$\Longrightarrow$(iv) is complete.

(iv)$\Longrightarrow$(i): Put $n=1$ in (iv), then we have (i) by using (iv)$\Longrightarrow$(i) in Theorem C.1.

Consequently the proof of Theorem 1 is complete.

Proof of Corollary 2. Put $r=\frac{(p+\alpha u)s}{nu}$ in Theorem 1, then the condition $\{nr+(n+1)\alpha\}u \geq (p+\alpha u)s$ in (ii) is satisfied and $r \geq 1-\alpha$ in (ii) can be rewritten as $(p+\alpha u)s \geq n(1-\alpha)u$. Then we have Corollary 2.

In order to prove Theorem 3, we prepare the following lemma.

Lemma 5. *Let A be a positive invertible operator satisfying $MI \geq A \geq mI > 0$ and T be an invertible positive contraction. Then*

$$K_+(m,M,p+1)\,A^p \geq T^{\frac{1}{2}}(T^{\frac{1}{2}}AT^{\frac{1}{2}})^pT^{\frac{1}{2}}$$

holds for $p \geq 0$, where $K_+(m,M,p)$ is defined in (1.2).

We need the following Lemma D.1 to prove Lemma 5.

Lemma D.1 ([13])**.** *Let A be a positive invertible operator and B be an invertible operator. Then*

$$(BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

holds for any real number λ.

Proof of Lemma 5. The condition $I \geq T > 0$ asserts $A \geq A^{\frac{1}{2}}TA^{\frac{1}{2}} > 0$. Put $A_1 = A$ and $B_1 = A^{\frac{1}{2}}TA^{\frac{1}{2}}$, then A_1 and B_1 satisfy $A_1 \geq B_1 > 0$ with $MI \geq A_1 \geq mI > 0$. Applying Theorem A.1,

$$K_+(m,M,p+1){A_1}^{p+1} \geq {B_1}^{p+1} \tag{3.8}$$

holds for $p \geq 0$, where $K_+(m,M,p)$ is defined in (1.2). (3.8) is equivalent to the following by Lemma D.1.

$$\begin{aligned} K_+(m,M,p+1)A^{p+1} &\geq (A^{\frac{1}{2}}TA^{\frac{1}{2}})^{p+1} \\ &= A^{\frac{1}{2}}T^{\frac{1}{2}}(T^{\frac{1}{2}}AT^{\frac{1}{2}})^pT^{\frac{1}{2}}A^{\frac{1}{2}}. \end{aligned} \tag{3.9}$$

Multiplying $A^{\frac{-1}{2}}$ on both sides of (3.9), the proof is complete.

Proof of Theorem 3.

(i)$\Longrightarrow$(ii): Let n be a nutural number, $\alpha \in [0,1]$, $p \geq 0$, $u \geq 0$, $s \geq 1$ and $r \geq 1-\alpha$ such that $\{nr+(n+1)\alpha\}u \geq (p+\alpha u)s$. By (i)$\Rightarrow$(ii) of Theorem 1, there exists the unique invertible positive contraction T satisfying the following (2.1):

$$T(A^{\frac{(p+\alpha u)s+ru}{n+1}}T)^n = A^{\frac{-(p+\alpha u)s+nru}{2(n+1)}}(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^sA^{\frac{-(p+\alpha u)s+nru}{2(n+1)}}. \tag{2.1}$$

By scrutinizing the proof of Theorem 1, (2.1) is equivalent to the following (3.5):

$$(A^{\frac{(p+\alpha u)s+ru}{2(n+1)}}TA^{\frac{(p+\alpha u)s+ru}{2(n+1)}})^{n+1} = A^{\frac{ru}{2}}D^sA^{\frac{ru}{2}}, \tag{3.5}$$

where $D = A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}}$. (3.5) can be rewritten as

$$A^{\frac{(p+\alpha u)s+ru}{2(n+1)}}T^{\frac{1}{2}}(T^{\frac{1}{2}}A^{\frac{(p+\alpha u)s+ru}{n+1}}T^{\frac{1}{2}})^nT^{\frac{1}{2}}A^{\frac{(p+\alpha u)s+ru}{2(n+1)}} = A^{\frac{ru}{2}}D^sA^{\frac{ru}{2}}. \tag{3.10}$$

Let $A_1 = A^{\frac{(p+\alpha u)s+ru}{n+1}}$. Then $MI \geq A \geq mI > 0$ ensures $M^{\frac{(p+\alpha u)s+ru}{n+1}}I \geq A_1 \geq m^{\frac{(p+\alpha u)s+ru}{n+1}}I > 0$ and

$$K_+\left(m^{\frac{(p+\alpha u)s+ru}{n+1}}, M^{\frac{(p+\alpha u)s+ru}{n+1}}, n+1\right){A_1}^n \geq T^{\frac{1}{2}}(T^{\frac{1}{2}}A_1T^{\frac{1}{2}})^nT^{\frac{1}{2}} \tag{3.11}$$

holds for each nutural number n by Lemma 5. (3.11) can be rewritten as

$$K_+\left(m^{\frac{(p+\alpha u)s+ru}{n+1}}, M^{\frac{(p+\alpha u)s+ru}{n+1}}, n+1\right) A^{\frac{\{(p+\alpha u)s+ru\}n}{n+1}} \geq T^{\frac{1}{2}}(T^{\frac{1}{2}}A^{\frac{(p+\alpha u)s+ru}{n+1}}T^{\frac{1}{2}})^n T^{\frac{1}{2}}. \tag{3.12}$$

Multiplying $A^{\frac{(p+\alpha u)s+ru}{2(n+1)}}$ on both sides of (3.12), we have

$$\begin{aligned} &K_+\left(m^{\frac{(p+\alpha u)s+ru}{n+1}}, M^{\frac{(p+\alpha u)s+ru}{n+1}}, n+1\right) A^{(p+\alpha u)s+ru} \\ &\geq A^{\frac{(p+\alpha u)s+ru}{2(n+1)}} T^{\frac{1}{2}}(T^{\frac{1}{2}}A^{\frac{(p+\alpha u)s+ru}{n+1}}T^{\frac{1}{2}})^n T^{\frac{1}{2}} A^{\frac{(p+\alpha u)s+ru}{2(n+1)}} \\ &= A^{\frac{ru}{2}} D^s A^{\frac{ru}{2}}. \end{aligned} \tag{3.13}$$

Hence the proof of (i)$\Longrightarrow$(ii) is complete.

(ii)$\Longrightarrow$(iii): (iii) holds in case $u = 0$ since the assumption of (iii) ensures $ps = 0$ and (2.4) turns out to be $K_+(1, 1, n+1)I = I$ by (1.4) in Lemma B.3. Let $p \geq nu > 0$ in (ii), then the condition $r \geq 1 - \alpha$ follows from $p \geq nu > 0$ and the other assumption of (ii) since

$$r \geq \frac{(p+\alpha u)s}{nu} - \frac{n+1}{n}\alpha \geq \frac{p+\alpha u}{nu} - \frac{n+1}{n}\alpha = \frac{p}{nu} - \alpha \geq 1 - \alpha,$$

so that we have (iii).

(iii)$\Longrightarrow$(iv): Put $\alpha = 0$ and $s = 1$ in (iii).

Proof of (iv)$\Longrightarrow$(i). Put $n = 1$ and $r = \frac{p}{u}$ in (iv). Then $K_+(m^p, M^p, 2) = \dfrac{(M^p + m^p)^2}{4m^pM^p}$ by (1.2), so that

$$\frac{(M^p + m^p)^2}{4m^pM^p} A^p \geq B^p \tag{3.14}$$

holds for all $p \geq nu > 0$, i.e., $p > 0$. By Theorem B.1, (3.14) implies (i).

Whence the proof of Theorem 3 is complete.

Proof of Theorem 4.
In case $u = 0$, (i)$\Rightarrow$(ii)$\Rightarrow$(iii) holds by Theorem B.5, because (ii) and (iii) can be rewritten as follows: For each nutural number n,

$$K_+\left(m^{\frac{ps}{n}}, M^{\frac{ps}{n}}, \frac{ps + \frac{ps}{n}}{\frac{ps}{n}}\right) A^{ps} \geq B^{ps}$$

holds for $ps \geq 0$.

(i)$\Longrightarrow$(ii): In the proof of (i)$\Rightarrow$(ii) of Theorem 3, we can put $r = \frac{(p+\alpha u)s}{nu} - \frac{n+1}{n}\alpha$ since $(p + \alpha u)s \geq (n + \alpha)u$ yields $r = \frac{(p+\alpha u)s}{nu} - \frac{n+1}{n}\alpha \geq 1 - \alpha$. Hence the proof of (i)$\Longrightarrow$(ii) is complete.

(ii)$\Longrightarrow$(iii): Put $p \geq nu \geq 0$, then the required condition $(p + \alpha u)s \geq (n + \alpha)u$ is satisfied.

(iii)$\Longrightarrow$(iv): Put $u = 0$ in (iii), we have, for each nutural number n,

$$K_+\left(m^{\frac{ps}{n}}, M^{\frac{ps}{n}}, n+1\right) A^{ps} \geq B^{ps} \tag{3.15}$$

holds for $ps \geq 0$. (3.15) is equivalent to

$$K_+\left(m^{\frac{ps}{n}}, M^{\frac{ps}{n}}, \frac{ps + \frac{ps}{n}}{\frac{ps}{n}}\right) A^{ps} \geq B^{ps}.$$

Tending $n \to \infty$ (i.e., $\frac{ps}{n} \to 0$), we have (iv) by Lemma B.4.

(iv)⇒(i) is already shown in Theorem B.2.

Hence the proof of Theorem 4 is complete.

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