

REPLACEMENT POLICIES FOR CUMULATIVE DAMAGE MODEL WITH MAINTENANCE COST

CUN-HUA QIAN, SYOUJI NAKAMURA AND TOSHIO NAKAGAWA

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ABSTRACT. This paper considers replacement policies for an extended cumulative damage model with minimal maintenance at each shock and minimal repair at failure: A system is replaced at time T or at shock number N , and undergoes maintenance or repair between replacements. The expected cost rate is obtained and each optimal T^* and N^* to minimize the cost rate is discussed. It is shown that this model would be applied to the backup of secondary storage files in a database system as an example.

1. Introduction.

We consider the stochastic model where shocks occur at random times and each shock causes the damage to a system. These damages accumulate additively. A system fails when the total amount of damage exceeds a failure level K . This stochastic model generates a cumulative process [1]. Some aspects of such cumulative damage models from reliability viewpoints were discussed by Esary, Marshall and Proschan [2].

It is of great interest to study the problem when to replace a system before failure as preventive maintenance. Optimal maintenance policies where a system is replaced at time T [3], at shock N [4], or at damage Z [5, 6] were studied. Nakagawa and Kijima [7] applied the periodic replacement with minimal repair [8] to a cumulative damage model and obtained optimal values T^* , N^* and Z^* which minimize the expected cost.

In recent years, the database in computer systems has become very important in a highly information-oriented society. In particular, the reliable database is the most indispensable instrument in on-line transaction processing systems such as real-time systems used for account of bank. The data in a computer system are frequently updated by adding or deleting them, and are stored in floppy disks or other secondary media. However, data files in secondary media are sometimes broken by several errors due to noises, human errors and hardware faults. In this case, we have to reconstruct the same files from the beginning.

The most simple and dependable method to ensure the safety of data would be always to make the backup copies of all files in other places as *total backup*, and to take out them if files in the original secondary media are broken. But, this method would take hours and costs, when files become large. To make the backup copies efficiently, we make the backup copies of only updated files which have changed or are new since the last full backup when the total updated files do not exceed a threshold level K . We call it *incremental backup*. This would reduce significantly both duration time and size of backup [9]. Conversely, we

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perform *full backup* at periodic time T or at backup number N , whichever occurs first. It is assumed that the database system returns to an initial state by the full backup.

This paper considers an extended cumulative damage model with minimal maintenance at each shock and minimal repair at each failure. Reliability measures of this model are derived, using the theory of cumulative processes. Further, this is applied to the backup of files in a database system.

2. Problem Formulation. Suppose that shocks occur in accordance with a nonhomogeneous Poisson process having the intensity function $\lambda(t)$ and the mean-value function $R(t)$, *i.e.*, $R(t) \equiv \int_0^t \lambda(u) du$. Then, the probability that the shocks occur exactly j times during $(0, t]$ is

$$(1) \quad H_j(t) \equiv \frac{[R(t)]^j}{j!} e^{-R(t)} \quad (j = 0, 1, 2, \dots).$$

Further, an amount Y_j of damage due to the j -th shock has a probability distribution $G_j(x) \equiv P_r\{Y_j \leq x\}$ ($j = 1, 2, \dots$) with finite mean. Then, the total damage $Z_j \equiv \sum_{i=1}^j Y_i$ to the j -th shock where $Z_0 \equiv 0$ has a distribution

$$(2) \quad G^{(j)}(x) \equiv \Pr\{Z_j \leq x\} = G_1 * G_2 * \dots * G_j(x) \quad (j = 0, 1, 2, \dots),$$

where $G^{(0)}(x) \equiv 1$ for $x \geq 0$, 0 for $x < 0$, and the asterisk mark represents the Stieltjes convolution, *i.e.*, $a * b(t) \equiv \int_0^t b(t-u) da(u)$ for any functions $a(t)$ and $b(t)$. Then, the probability that the total amount of damage exceeds a failure level K at the j -th shock is $G^{(j-1)}(K) - G^{(j)}(K)$. Let $Z(t)$ be the total amount of damage at time t . Then, the distribution function of $Z(t)$ is

$$(3) \quad \Pr\{Z(t) \leq x\} = \sum_{j=0}^{\infty} H_j(t) G^{(j)}(x).$$

Consider the system which should operate for an infinite time span and assume: When the total damage does not exceed a failure level K , the system undergoes minimal maintenance at each shock. The maintenance cost is $c_2 + c_0(x)$ when the total damage is x ($0 \leq x < K$), where $c_0(x)$ is continuous and strictly increasing and $c_0(0) \equiv 0$. When the total damage exceeds a failure level K , the system undergoes minimal repair at each shock, and the repair cost is c_3 , where $c_3 = c_2 + c_0(K)$. The system is replaced at periodic time T or at shock number N , whichever occurs first, and the replacement cost is c_1 , where $c_3 < c_1$. The maintenance time, the repair time and the replacement time are negligible, *i.e.*, the time considered here is measured only by the total operating time of the system.

Lemma 2.1 *Let P_T denote the probability that the system is replaced at time T , and P_N denote the probability that the system is replaced at shock number N . Then,*

$$(4) \quad P_T = \sum_{j=0}^{N-1} H_j(T),$$

$$(5) \quad P_N = \int_0^T H_{N-1}(t) \lambda(t) dt = \sum_{j=N}^{\infty} H_j(T),$$

and $P_T + P_N = 1$.

Lemma 2.2 *Let $N_1(T, N)$ denote the expected number of minimal maintenances until replacement, and $N_2(T, N)$ denote the expected number of minimal repairs until replacement. Then,*

$$(6) \quad N_1(T, N) = \sum_{j=1}^{N-1} G^{(j)}(K) \sum_{i=j}^{\infty} H_i(T),$$

$$(7) \quad N_2(T, N) = \sum_{j=1}^{N-1} [1 - G^{(j)}(K)] \sum_{i=j}^{\infty} H_i(T).$$

Proof: From Lemma 2.1, we have

$$\begin{aligned} P_T &= \sum_{j=0}^{N-1} H_j(T) G^{(j)}(K) + \sum_{j=1}^{N-1} H_j(T) \sum_{i=0}^{j-1} [G^{(i)}(K) - G^{(i+1)}(K)], \\ P_N &= \int_0^T H_{N-1}(t) \lambda(t) dt G^{(N)}(K) + \int_0^T H_{N-1}(t) \lambda(t) dt \sum_{i=0}^{N-1} [G^{(i)}(K) - G^{(i+1)}(K)]. \end{aligned}$$

Thus,

$$\begin{aligned} N_1(T, N) &= \sum_{j=0}^{N-1} j H_j(T) G^{(j)}(K) + \sum_{j=1}^{N-1} H_j(T) \sum_{i=0}^{j-1} i [G^{(i)}(K) - G^{(i+1)}(K)] \\ &\quad + (N-1) \int_0^T H_{N-1}(t) \lambda(t) dt G^{(N)}(K) \\ &\quad + \int_0^T H_{N-1}(t) \lambda(t) dt \sum_{i=0}^{N-1} i [G^{(i)}(K) - G^{(i+1)}(K)] \\ &= \sum_{j=1}^{N-1} H_j(T) \sum_{i=1}^j G^{(i)}(K) + \sum_{j=N}^{\infty} H_j(T) \sum_{i=1}^{N-1} G^{(i)}(K) \\ &= \sum_{j=1}^{N-1} G^{(j)}(K) \sum_{i=j}^{\infty} H_i(T), \end{aligned}$$

and

$$\begin{aligned} N_2(T, N) &= \sum_{j=1}^{N-1} H_j(T) \sum_{i=0}^{j-1} (j-i) [G^{(i)}(K) - G^{(i+1)}(K)] \\ &\quad + \int_0^T H_{N-1}(t) \lambda(t) dt \sum_{i=0}^{N-1} (N-i-1) [G^{(i)}(K) - G^{(i+1)}(K)] \\ &= \sum_{j=1}^{N-1} H_j(T) \sum_{i=1}^j [1 - G^{(i)}(K)] + \sum_{j=N}^{\infty} H_j(T) \sum_{i=1}^{N-1} [1 - G^{(i)}(K)] \\ &= \sum_{j=1}^{N-1} [1 - G^{(j)}(K)] \sum_{i=j}^{\infty} H_i(T). \end{aligned}$$

Lemma 2.3 *Let $E[U]$ denote the mean time to replacement. Then,*

$$(8) \quad E[U] = \sum_{j=0}^{N-1} \int_0^T H_j(t) dt.$$

Proof: From Lemma 2.1, we have

$$E[U] = T \sum_{j=0}^{N-1} H_j(T) + \int_0^T t H_{N-1}(t) \lambda(t) dt.$$

Then, using the relations

$$\int_0^T H_j(t) dt = T H_j(T) - \int_0^T t H_{j-1}(t) \lambda(t) dt + \int_0^T t H_j(t) \lambda(t) dt \quad (j = 1, 2, \dots),$$

and

$$\int_0^T H_0(t) dt = T H_0(T) + \int_0^T t H_0(t) \lambda(t) dt,$$

we have

$$\begin{aligned} E[U] &= T H_0(T) + \int_0^T t H_0(t) \lambda(t) dt + \sum_{j=1}^{N-1} T H_j(T) + \sum_{j=1}^{N-1} \int_0^T t H_j(t) \lambda(t) dt - \sum_{j=0}^{N-2} \int_0^T t H_j(t) \lambda(t) dt \\ &= T H_0(T) + \int_0^T t H_0(t) \lambda(t) dt + \sum_{j=1}^{N-1} [T H_j(T) - \int_0^T t H_{j-1}(t) \lambda(t) dt + \int_0^T t H_j(t) \lambda(t) dt] \\ &= \sum_{j=0}^{N-1} \int_0^T H_j(t) dt. \end{aligned}$$

From Lemma 2.2, we easily have:

Lemma 2.4 *Let $E[C]$ denote the expected cost to replacement. Then,*

$$(9) \quad E[C] = c_1 + \sum_{j=1}^{N-1} \int_0^T H_{j-1}(t) \lambda(t) dt \left\{ \int_0^K [c_2 + c_0(x)] dG^{(j)}(x) + c_3 [1 - G^{(j)}(K)] \right\}.$$

From Lemma 2.3 and Lemma 2.4, by using the theory of renewal reward process [10], we have:

Theorem 2.5 *Let $C(T, N)$ denote the expected cost per unit time in the steady-state. Then,*

$$(10) \quad C(T, N) = \frac{E[C]}{E[U]},$$

where

$$E[C] = c_1 + \sum_{j=1}^{N-1} \int_0^T H_{j-1}(t) \lambda(t) dt \left\{ \int_0^K [c_2 + c_0(x)] dG^{(j)}(x) + c_3 [1 - G^{(j)}(K)] \right\},$$

and

$$E[U] = \sum_{j=0}^{N-1} \int_0^T H_j(t) dt.$$

Suppose that shocks occur according to a Poisson process with rate λ , i.e., $\lambda(t) = \lambda$, $R(t) = \lambda t$ and $H_j(t) = [(\lambda t)^j / j!] e^{-\lambda t}$ ($j = 0, 1, 2, \dots$). Further, assume that the cost of minimal maintenance is proportional to the total damage, i.e., $c_2 + c_0(x) = c_2 + c_0 x$ ($0 \leq x < K$).

Corollary 2.6 *When $\lambda(t) = \lambda$ and $c_0(x) = c_0 x$, the expected cost per unit time is*

$$(11) \quad C(T, N) = \frac{c_1 + \lambda \sum_{j=1}^{N-1} \int_0^T H_{j-1}(t) dt [c_3 - c_0 \int_0^K G^{(j)}(x) dx]}{\sum_{j=0}^{N-1} \int_0^T H_j(t) dt}.$$

3. Optimal Time T^* .

Lemma 3.1 *Let $C(T)$ denote the expected cost per unit time when the system is replaced only at time T . Then,*

$$(12) \quad C(T) \equiv \lim_{N \rightarrow \infty} C(T, N) = c_3 \lambda + \frac{c_1 - c_0 \lambda \sum_{j=1}^{\infty} \int_0^K G^{(j)}(x) dx \int_0^T H_{j-1}(t) dt}{T}.$$

Theorem 3.2 *If $\int_0^K M(x) dx > c_1 / c_0$ then there exists a finite T^* uniquely which minimizes $C(T)$, and it satisfies*

$$(13) \quad \sum_{j=1}^{\infty} H_j(T) \sum_{i=1}^j \int_0^K [G^{(i)}(x) - G^{(j)}(x)] dx = c_1 / c_0.$$

The resulting cost is

$$(14) \quad C(T^*) / \lambda = c_3 - c_0 \sum_{j=0}^{\infty} H_j(T^*) \int_0^K G^{(j+1)}(x) dx,$$

where $M(x) \equiv \sum_{j=1}^{\infty} G^{(j)}(x)$. If $\int_0^K M(x) dx \leq c_1 / c_0$ then $T^ = \infty$ and $C(\infty) = c_3 \lambda$.*

Proof: We easily have that $C(\infty) \equiv \lim_{T \rightarrow \infty} C(T) = c_3 \lambda$. Thus, there exists a positive T^* ($0 < T^* \leq \infty$) which minimizes $C(T)$. A necessary condition that a finite T^* exists is that it satisfies (13) and is given by differentiating $C(T)$ with respect to T and setting it equal

to zero. Let $U(T)$ be the left-hand side of (13), we have

$$\begin{aligned}
U(0) &\equiv \lim_{T \rightarrow 0} U(T) = 0, \\
U(\infty) &\equiv \lim_{T \rightarrow \infty} U(T) = \lim_{T \rightarrow \infty} \sum_{j=1}^{\infty} \int_0^K G^{(j)}(x) dx \left[\sum_{i=j}^{\infty} H_i(T) - j H_j(T) \right] \\
&= \int_0^K \sum_{j=1}^{\infty} G^{(j)}(x) dx = \int_0^K M(x) dx, \\
U'(T) &= \lambda \sum_{j=1}^{\infty} [H_{j-1}(T) - H_j(T)] \sum_{i=1}^j \int_0^K [G^{(i)}(x) - G^{(j)}(x)] dx \\
&= \lambda \sum_{j=1}^{\infty} H_j(T) \int_0^K \left[\sum_{i=1}^{j+1} G^{(i)}(x) - (j+1) G^{(j+1)}(x) - \sum_{i=1}^j G^{(i)}(x) + j G^{(j)}(x) \right] dx \\
&= \lambda \sum_{j=1}^{\infty} H_j(T) j \int_0^K [G^{(j)}(x) - G^{(j+1)}(x)] dx > 0.
\end{aligned}$$

Thus, $U(T)$ is a strictly increasing function from 0 to $\int_0^K M(x) dx$. If $\int_0^K M(x) dx > c_1/c_0$ then there exists a finite T^* uniquely which satisfies (13), and the resulting cost is given in (14).

Example 3.3 Suppose that a database is updated according to a Poisson process with rate λ . Further, an amount of only files, which have changed or are new at the j -th update since the last full backup, is Y_j . It is assumed that each Y_j has an identical probability distribution function $G_j(x) = 1 - e^{-\mu x}$, i.e., $G^{(j)}(x) = 1 - \sum_{i=0}^{j-1} [(\mu x)^i / i!] e^{-\mu x}$ ($j = 1, 2, \dots$) and $M(K) = \mu K$. We replace *shock* by *update*, *damage* by *updated files*, *minimal maintenance* by *incremental backup*, *minimal repair* by *total backup*, and *replacement* by *full backup*.

In this case, to compute (13), let $g_j(s)$ denote the Laplace-Stieltjes (*LS*) transform of Cdf $G_j(x)$, i.e.,

$$g_j(s) \equiv \int_0^{\infty} e^{-sx} dG_j(x),$$

for $s > 0$, and $g^{(j)}(s)$ denote the LS transform of Cdf $G^{(j)}(x)$. When $G_j(x) = 1 - e^{-\mu x}$ ($j = 1, 2, \dots$), we easily have

$$g^{(j)}(s) = g^{(j-1)}(s) g_j(s),$$

and

$$g_j(s) = \frac{\mu}{s + \mu}.$$

Thus,

$$(15) \quad \frac{1}{s} [g^{(j-1)}(s) - g^{(j)}(s)] = \frac{1}{\mu} g^{(j)}(s).$$

Inverting the *LS* transforms of (15),

$$(16) \quad \int_0^K [G^{(j-1)}(x) - G^{(j)}(x)] dx = \frac{1}{\mu} G^{(j)}(K).$$

From (16), we have

$$\frac{1}{\mu} \sum_{i=j+1}^{\infty} G^{(i)}(K) = \sum_{i=j+1}^{\infty} \int_0^K [G^{(i-1)}(x) - G^{(i)}(x)] dx = \int_0^K G^{(j)}(x) dx,$$

and hence,

$$\sum_{i=1}^j \int_0^K [G^{(i)}(x) - G^{(j)}(x)] dx = \frac{1}{\mu} \sum_{i=1}^{j-1} i G^{(i+1)}(K).$$

Thus, the equation (13) is simplified as

$$(17) \quad \sum_{j=1}^{\infty} H_{j+1}(T) \sum_{i=1}^j i G^{(i+1)}(K) = \frac{\mu c_1}{c_0}.$$

Let $Q(T)$ be the left-hand side of (17),

$$Q(0) \equiv \lim_{T \rightarrow 0} Q(T) = 0,$$

$$Q(\infty) \equiv \lim_{T \rightarrow \infty} Q(T) = \mu U(\infty) = \mu \int_0^K M(x) dx = \frac{(\mu K)^2}{2}.$$

Thus, it is easily proved that $Q(T)$ is a strictly increasing function of T from 0 to $(\mu K)^2/2$. If $(\mu K)^2/2 > \mu c_1/c_0$ then there exists a finite T^* uniquely which satisfies (17), and from (14) and (16), the resulting cost is rewritten as

$$\begin{aligned} C(T^*)/\lambda &= c_2 + c_0 \sum_{j=0}^{\infty} H_j(T^*) \int_0^K [1 - G^{(j+1)}(x)] dx \\ &= c_2 + c_0 \sum_{j=0}^{\infty} H_j(T^*) \sum_{i=0}^j \int_0^K [G^{(i)}(x) - G^{(i+1)}(x)] dx \\ &= c_2 + \frac{c_0}{\mu} \sum_{j=0}^{\infty} H_j(T^*) \sum_{i=1}^{j+1} G^{(i)}(K). \end{aligned}$$

Corollary 3.4 When $G_j(x) = 1 - e^{-\mu x}$ ($j = 1, 2, \dots$), we have the following optimal policy: If $\mu K^2/2 > c_1/c_0$ then there exists a finite T^* ($0 < T^* < \infty$) uniquely which satisfies (17). The resulting cost is

$$(18) \quad C(T^*)/\lambda = c_2 + \frac{c_0}{\mu} \sum_{j=0}^{\infty} H_j(T^*) \sum_{i=1}^{j+1} G^{(i)}(K).$$

If $\mu K^2/2 \leq c_1/c_0$ then $T^* = \infty$, and the resulting cost is $c_3\lambda$.

Example 3.5 It is supposed in example 3.3 that the total volume of files is 5×10^5 trucks and a threshold level K is 3×10^5 trucks which correspond to 60% of the total volume. Table 1 gives the optimal full backup times λT^* , the resulting costs $C(T^*)/\lambda$ for $c_1 = 40, 50, 75, 100, 150, 200$, and $\mu K = 8, 12$ when $c_2 = 10$ and $c_0 = 10^{-4}$. It is found from the optimal policy that if $15\mu K > c_1$ then $T^* < \infty$, and conversely, if $15\mu K \leq c_1$ then $T^* = \infty$ and $C(\infty)/\lambda = 40$. This shows that both optimal T^* and costs $C(T^*)$ are increasing with c_1 , and $C(T^*)$ are decreasing with μK . However, T^* are smaller for small c_1 , and conversely, are greater for large c_1 , when μK is smaller. This reason would be explained that if the

cost c_1 is small then it is better to perform the full backup early, but if c_1 is large then it is better to do it lately, especially when its mean updated file is large.

Table 1. Optimal full backup times λT^* and resulting costs $C(T^*)/\lambda$

c_1		40	50	75	100	150	200
$\mu K = 8$	λT^*	5.144	5.946	8.098	11.149	∞	∞
	$C(T^*)/\lambda$	30.467	32.274	35.878	38.537	40.000	40.000
$\mu K = 12$	λT^*	5.823	6.587	8.371	10.163	14.895	∞
	$C(T^*)/\lambda$	26.537	28.149	31.505	34.213	38.328	40.000

For example, when the mean time of update is $1/\lambda = 1$ day, $c_1 = 50$ and $\mu K = 12$, the optimal full backup time T^* is about 7 days. In this case, $K/(\lambda/\mu) = 12$ days, and note that it represents the mean time until the total updated files exceed a threshold level K .

4. Optimal Shock Number N^* .

Lemma 4.1 *Let $C(N)$ denote the expected cost per unit time when the system is replaced only at shock number N . Then,*

$$(19) \quad \frac{C(N)}{\lambda} \equiv \lim_{T \rightarrow \infty} \frac{C(T, N)}{\lambda} = c_3 + \frac{c_1 - c_2 - c_0 \sum_{j=0}^{N-1} \int_0^K G^{(j)}(x) dx}{N}.$$

Proof: From Corollary 2.6, using the relation

$$\lim_{T \rightarrow \infty} \int_0^T H_j(t) \lambda dt = \lim_{T \rightarrow \infty} \sum_{i=j+1}^{\infty} H_i(T) = 1,$$

we have

$$\begin{aligned} \frac{C(N)}{\lambda} &= \frac{c_1 + c_3(N-1) - c_0 \sum_{j=1}^{N-1} \int_0^K G^{(j)}(x) dx}{N} \\ &= c_3 + \frac{c_1 - c_2 - c_0 \sum_{j=0}^{N-1} \int_0^K G^{(j)}(x) dx}{N}. \end{aligned}$$

Theorem 4.2 *If $\int_0^K M(x) dx > (c_1 - c_3)/c_0$ then there exists a finite N^* uniquely which minimizes $C(N)$ and it satisfies*

$$(20) \quad L(N) \geq (c_1 - c_2)/c_0 \text{ and } L(N-1) < (c_1 - c_2)/c_0,$$

where

$$(21) \quad L(N) \equiv \sum_{j=0}^{N-1} \int_0^K [G^{(j)}(x) - G^{(N)}(x)] dx.$$

If $\int_0^K M(x) dx \leq (c_1 - c_3)/c_0$ then $N^ = \infty$.*

Proof: A necessary condition that a finite N^* minimizes $C(N)$ is given by forming the inequalities $C(N+1) \geq C(N)$ and $C(N) < C(N-1)$. From these inequalities, we have (20). Further,

$$\begin{aligned} L(1) &= \int_0^K [1 - G^{(1)}(x)]dx, \\ L(\infty) &\equiv \lim_{N \rightarrow \infty} L(N) = \int_0^K [1 + M(x)]dx, \\ L(N+1) - L(N) &= (N+1) \int_0^K [G^{(N)}(x) - G^{(N+1)}(x)]dx > 0. \end{aligned}$$

Thus, $L(N)$ is a strictly increasing function from 0 to $\int_0^K [1 + M(x)]dx$. If $\int_0^K [1 + M(x)]dx > (c_1 - c_2)/c_0$ then there exists a finite N^* uniquely which satisfies (20), and if $\int_0^K [1 + M(x)]dx \leq (c_1 - c_2)/c_0$ then $N^* = \infty$.

Example 4.3 In example 3.3, we perform a full backup at backup number N when $G_j(x) = 1 - e^{\mu x}$ and $M(x) = \mu x$. In this case, from (16),

$$\sum_{j=0}^{N-1} \int_0^K [G^{(j)}(x) - G^{(N)}(x)]dx = \frac{1}{\mu} \sum_{j=0}^{N-1} \sum_{i=j+1}^N G^{(i)}(K) = \frac{1}{\mu} \sum_{j=1}^N jG^{(j)}(K).$$

Thus, the equation (20) is simplified as

$$(22) \quad \sum_{j=1}^N jG^{(j)}(K) \geq \frac{\mu(c_1 - c_2)}{c_0} \text{ and } \sum_{j=1}^{N-1} jG^{(j)}(K) < \frac{\mu(c_1 - c_2)}{c_0}.$$

Let $L_1(N) \equiv \sum_{j=1}^N jG^{(j)}(K)$,

$$\begin{aligned} L_1(1) &= 1 - e^{-\mu K}, \\ L_1(\infty) &\equiv \lim_{N \rightarrow \infty} L_1(N) = \mu L(\infty) = \mu K + \frac{(\mu K)^2}{2}. \end{aligned}$$

Thus, $L_1(N)$ is a strictly increasing function of N from $1 - e^{-\mu K}$ to $\mu K + (\mu K)^2/2$. If $\mu K + (\mu K)^2/2 > \mu(c_1 - c_2)/c_0$ then there exists a finite N^* uniquely which satisfies (22), and from (19), the resulting cost is

$$\begin{aligned} \frac{C(N^*)}{\lambda} &= c_2 + \frac{c_1 - c_2 + c_0 \sum_{j=0}^{N^*-1} \int_0^K [1 - G^{(j)}(x)]dx}{N^*} \\ &= c_2 + \frac{c_1 - c_2 + \frac{c_0}{\mu} \sum_{j=1}^{N^*-1} \sum_{i=1}^j G^{(i)}(K)}{N^*} \\ &= c_2 + \frac{c_1 - c_2 + \frac{c_0}{\mu} [N^* \sum_{j=1}^{N^*} G^{(j)}(K) - \sum_{j=1}^{N^*} jG^{(j)}(K)]}{N^*}. \end{aligned}$$

Corollary 4.4 When $G_j(x) = 1 - e^{-\mu x}$ ($j = 1, 2, \dots$), we have the following optimal policy: If $\mu K^2/2 > (c_1 - c_3)/c_0$ then there exists a finite N^* uniquely which minimizes $C(N)$, and it satisfies (22). The resulting cost is

$$(23) \quad \frac{C(N^*)}{\lambda} = c_2 + \frac{c_0}{\mu} \sum_{j=1}^{N^*} G^{(j)}(K) + \frac{c_1 - c_2 - \frac{c_0}{\mu} \sum_{j=1}^{N^*} jG^{(j)}(K)}{N^*}.$$

If $\mu K^2/2 \leq (c_1 - c_3)/c_0$ then $N^* = \infty$.

Example 4.5 It is supposed in example 4.3 that the total volume of files is 5×10^5 trucks and a threshold level K is 3×10^5 trucks which correspond to 60% of the total volume. Table 2 gives the optimal full backup number N^* and the resulting costs $C(N^*)/\lambda$ for $c_1 = 40, 50, 75, 100, 150, 200$ and $\mu K = 8, 12$ when $c_2 = 10$ and $c_0 = 10^{-4}$. It is found from the optimal policy that if $c_1 < 210$ then $N^* < \infty$, and conversely, if $c_1 \geq 210$ then $N^* = \infty$ and $C(\infty)/\lambda = 40$, when $\mu K = 12$. For example, when $c_1 = 50$, the optimal number is $N^* = 6$.

Table 2. Optimal full backup number N^* and resulting costs $C(N^*)/\lambda$

c_1		40	50	75	100	150	200
$\mu K = 8$	N^*	4	5	6	8	12	∞
	$C(N^*)/\lambda$	23.106	25.440	30.058	33.788	38.913	40.000
$\mu K = 12$	N^*	5	6	7	9	11	15
	$C(N^*)/\lambda$	20.998	22.911	26.770	29.931	34.972	38.839

The change of optimal number N^* is similar to the optimal times λT^* . It is noted that λT^* represents the expected number of updates during $(0, T^*]$, and $N^* < \lambda T^*$, $C(N^*)/\lambda < C(T^*)/\lambda$ under the same conditions. In general, a replacement policy for backup number N would be more economical than that for time T under the same conditions.

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DEPARTMENT OF INDUSTRIAL ENGINEERING, AICHI INSTITUTE OF TECHNOLOGY, TOYOTA 470-0392 JAPAN

SYSTEMS DIVISION, THE BANK OF NAGOYA, LTD., NAGOYA 468-0003 JAPAN

DEPARTMENT OF INDUSTRIAL ENGINEERING, AICHI INSTITUTE OF TECHNOLOGY, TOYOTA 470-0392 JAPAN