BESSEL TYPE EXTENSION OF THE BERNSTEIN INEQUALITY

Masatoshi Fujii*, Chia-Shiang Lin** and Ritsuo Nakamoto***

Received December 2, 1999

Abstract. In our preceding note, we pointed out that eigenvalues and eigenvectors of adjoint operators are essential in Bernstein's inequality. Based on this idea, we give a Bessel type extension of Bernstein's inequality.

1. Introduction.

Throughout this note, an operator means a bounded linear operator acting on a Hilbert space. In [1], Bernstein proved the following inequality, which is used in testing convergence of eigenvector calculations: If e is a unit vector corresponding to an eigenvalue λ of a selfadjoint operator S on a Hilbert space H, then

$$|(x,e)|^2 \leq \frac{\|x\|^2\|Sx\|^2 - (x,Sx)^2}{\|(S-\lambda)x\|^2} \tag{1}$$

for every $x \in H$ for which $Sx \neq \lambda x$.

The Bernstein inequality (1) is extended to non-normal operators, precisely dominant operators [4] and operators with normal eigenvalues [2]. An operator T is called dominant if for each λ there exists a real number M_λ such that $\|(T-\lambda)^*x\| \leq M_\lambda\|(T-\lambda)x\|$ for all $x \in H$. If a dominant operator T satisfies $(T-\lambda)e = 0$, then we have $(T-\lambda)^*e = 0$. This says that λ is normal eigenvalue, i.e., there exists a nonzero vector x in H such that $Tx = \lambda x$ and $T^*x = \bar{\lambda}x$. Furthermore we pointed out that eigenvalues and its corresponding eigenvectors of adjoint operators are essential [3]. That is,

Theorem A. *If e is a unit eigenvector corresponding to an eigenvalue $\bar{\lambda}$ of S^* on a Hilbert space H, then*

$$|(x,e)|^2 \leq \frac{\|x\|^2\|Sx\|^2 - |(x,Sx)|^2}{\|(S-\lambda)x\|^2} \tag{2}$$

for all x in H for which $Sx \neq \lambda x$.

On the other hand, the Bernstein inequality is expected to have an extension of Bessel type. As such an extension of it, one of the authors recently proposed in [5; Theorem 5(2) and Corollary 5] as follows:

1991 Mathematics Subject Classifications. 47A30 and 47A63.

Key words and phrases. Bernstein inequality, Bessel inequality, eigenvalue.

Theorem B. *Let S be a selfadjoint operator on a Hilbert space H. If $\{e_1, \cdots, e_n\}$ is a set of unit eigenvectors corresponding to a set $\{\lambda_1, \cdots, \lambda_n\}$ of eigenvalues of S, then*

$$\sum_{i=1}^{n} |(u_{i-1}, e_i)|^2 \leq \frac{\|x\|^2\|Sx\|^2 - (x, Sx)^2}{\|(S-\lambda_j)x\|^2} \tag{3}$$

for every $x \in H$ for which $Sx \neq \lambda_j x$, where $u_i = u_{i-1} - (u_{i-1}, e_i)e_i, i = 1, \cdots, n$, with $u_0 = x$ and $j \in \{1, \cdots, n\}$.

In particular, if $\{e_1, \cdots, e_n\}$ is a set of orthonormal vectors, then

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \frac{\|x\|^2\|Sx\|^2 - (x, Sx)^2}{\|(S-\lambda_j)x\|^2}. \tag{4}$$

We have to say that there is some overlook in Theorem B; however we will make it clear and moreover the improved one includes Theorem A. To do this, we review the original proof of Theorem B and consequently recognize that it is essentially true. Furthermore we generalize our result to approximate eigenvalues of operators.

2. Bernstein inequality of Bessel type.

We present the following modified version of Theorems A and B.

Theorem 1. *If $(S-\lambda_i)^* e_i = 0$ and $\|e_i\| = 1$ for $i = 1, \cdots, n$, then*

$$\sum_{i=1}^{n} |(u_{i-1}, e_i)|^2 \leq \frac{\|x\|^2 \|\prod_{i=1}^{n}(S-\lambda_i)x\|^2 - |(x, \prod_{i=1}^{n}(S-\lambda_i)x)|^2}{\|\prod_{i=1}^{n}(S-\lambda_i)x\|^2} \tag{5}$$

for $u_i = u_{i-1} - (u_{i-1}, e_i)e_i$ with $u_0 = x$ and $\prod_{i=1}^{n}(S-\lambda_i)x \neq 0$.

In particular, if $\{e_1, \cdots, e_n\}$ is an orthonormal set, then

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \frac{\|x\|^2 \|\prod_{i=1}^{n}(S-\lambda_i)x\|^2 - |(x, \prod_{i=1}^{n}(S-\lambda_i)x)|^2}{\|\prod_{i=1}^{n}(S-\lambda_i)x\|^2}. \tag{6}$$

Proof. As $u_i = u_{i-1} - (u_{i-1}, e_i)e_i$, we have $\|u_{i-1}\|^2 = \|u_i\|^2 + |(u_{i-1}, e_i)|^2$. Also by assumption, we have $(S-\lambda_i)^* u_i = (S-\lambda_i)^* u_{i-1}$ for $i = 1, \cdots, n$.

Therefore, using these relations and $u_0 = x$, we have

$$\begin{aligned}
&\|x\|^2 \|\prod(S-\lambda_i)x\|^2 - |(x, \prod(S-\lambda_i)x)|^2 \\
&\quad = (\|u_1\|^2 + |(u_0, e_1)|^2)\|\prod(S-\lambda_i)x\|^2 - |(x, \prod(S-\lambda_i)x)|^2 \\
&\quad = |(u_0, e_1)|^2 \|\prod(S-\lambda_i)x\|^2 + \|u_1\|^2 \|\prod(S-\lambda_i)x\|^2 - |(u_0, \prod(S-\lambda_i)x)|^2 \\
&\quad = |(u_0, e_1)|^2 \|\prod(S-\lambda_i)x\|^2 + \|u_1\|^2 \|\prod(S-\lambda_i)x\|^2 - |(u_1, \prod(S-\lambda_i)x)|^2,
\end{aligned}$$

$$\begin{aligned}
&\|u_1\|^2 \|\prod(S-\lambda_i)x\|^2 - |(u_1, \prod(S-\lambda_i)x)|^2 \\
&\quad = (\|u_2\|^2 + |(u_1, e_2)|^2)\|\prod(S-\lambda_i)x\|^2 - |(u_1, \prod(S-\lambda_i)x)|^2 \\
&\quad = |(u_1, e_2)|^2 \|\prod(S-\lambda_i)x\|^2 + \|u_2\|^2 \|\prod(S-\lambda_i)x\|^2 - |(u_2, \prod(S-\lambda_i)x)|^2
\end{aligned}$$

and generally

$$\begin{aligned}
&\|u_{j-1}\|^2\|\prod(S-\lambda_i)x\|^2 - |(u_{j-1},\prod(S-\lambda_i)x)|^2 \\
&\quad = |(u_{j-1},e_j)|^2\|\prod(S-\lambda_i)x\|^2 + \|u_j\|^2\|\prod(S-\lambda_i)x\|^2 - |(u_j,\prod(S-\lambda_i)x)|^2
\end{aligned}$$

for $j = 1, \cdots, n$. It follows that

(7)

$$\begin{aligned}
&\|x\|^2\|\prod(S-\lambda_i)x\|^2 - |(x,\prod(S-\lambda_i)x)|^2 \\
&= \|\prod(S-\lambda_i)x\|^2\sum_{i=1}^{n}|(u_{i-1},e_i)|^2 + \{\|u_n\|^2\|\prod(S-\lambda_i)x\|^2 - |(u_n,\prod(S-\lambda_i)x)|^2\} \\
&\geq \|\prod(S-\lambda_i)x\|^2\sum|(u_{i-1},e_i)|^2,
\end{aligned}$$

so that we obtain (5).

If $\{e_1, \cdots, e_n\}$ is an orthonormal set, then for each $i = 1, \cdots, n$

$$(u_{i-1}, e_i) = (u_{i-2}, e_i) = \cdots = (u_1, e_i) = (u_0, e_i) = (x, e_i).$$

Consequently, the inequality (6) holds.

Remark 1. As a special case $n = 1$ of Theorem 1, Theorem A is obtained by the translation-invariance;

$$\|x\|^2\|Sx\|^2 - |(x,Sx)|^2 = \|x\|^2\|(S-\lambda)x\|^2 - |(x,(S-\lambda)x)|^2$$

for every complex number λ.

Remark 2. Consider a matrix $S = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$. Then the eigenvalues of S are $\lambda_1 = 5$ and $\lambda_2 = 1$, whose unit eigenvectors are $e_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $e_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, respectively. By direct caluculations, we have

$$\begin{aligned}
&|(x,e_1)|^2 + |(x,e_2)|^2 = 5, \\
&\|x\|^2\|Sx\|^2 - (x,Sx)^2 = 5\cdot 113 - 23^2 = 36, \\
&\|(S-\lambda_1)x\|^2 = 8 \\
&\|(S-\lambda_2)x\|^2 = 72
\end{aligned}$$

for $x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. So the theorem B does not hold.

3. A generalization to approximate eigenvalues.

Following our preceding paper [3], we generalize Theorem 1 to approximate eigenvalues, which is also a generalization of [3; Theorem 3].

Theorem 2. *If $\{e_i^{(k)}\}$ be a sequence of unit vectors corresponding to approximate eigenvalue $\bar{\lambda_i}$ $(i = 1, \cdots, n)$ of S^*, then*

$$\sum_{i=1}^{n}\overline{\lim}|(u_{i-1}^{(k)}, e_i^{(k)})|^2 \leq \frac{\|x\|^2\|\prod_{i=1}^{n}(S-\lambda_i)x\|^2 - |(x,\prod_{i=1}^{n}(S-\lambda_i)x)|^2}{\|\prod_{i=1}^{n}(S-\lambda_i)x\|^2}$$

for $u_i^{(k)} = u_{i-1}^{(k)} - (u_{i-1}^{(k)}, e_i^{(k)})e_i^{(k)}$ *with* $u_0^{(k)} = x$ *and* $\prod_{i=1}^n (S-\lambda_i)x \neq 0$.

Proof. First of all, we review the Berberian representation $A \to A^\circ$ briefly, cf. [2,3].

It is induced by a generalized limit Lim as follows: The vector space V of all bounded sequences in H has a semi-inner product $< x^\circ, y^\circ >= \mathrm{Lim}(x_n, y_n)$, so that a Hilbert space H° is given by the completion of V/N, where $N = \{x \in V; < x^\circ, x^\circ >= 0\}$. For an operator A on H, A° is defined by

$$A^\circ(\{x_n\} + N) = \{Ax_n\} + N.$$

Then it is well known that it is an isometric $*$-isomorophism and converts the approximate eigenvalues of A to the eigenvalues of A°.

Using the Berberian representation, we have by Theorem 1 (or (7))

$$\|x\|^2 \| \prod (S-\lambda_i)x\|^2 - |(x, \prod (S-\lambda_i)x)|^2 \geq \| \prod (S-\lambda_i)x\|^2 \sum_{i=1}^n \overline{\lim} |(u_{i-1}^{(k)}, e_i^{(k)})|^2.$$

It implies the desired inequality.

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*Department of Mathematics, Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan

**Department of Mathematics, Bishop University, Lennoxville, P.Q., J1M 1Z7, Canada

***Faculty of Engineering, Ibaraki University, Hitachi, Ibaraki 316-0033, Japan