SOME CLASSES OF OPERATORS DERIVED FROM FURUTA INEQUALITY

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Abstract. We introduce a new family of classes of operators which is derived from the Furuta inequality. It is closely related to log-hyponormal operators defined by Tanahashi, the classes $A(p)$ and absolute p-paranormal operators defined by Furuta-Ito-Yamazaki. We discuss some properties of operators in such classes and relations among them and extend results due to Furuta-Ito-Yamazaki.

1. Introduction. Throughout this note, an operator means a bounded linear operator acting on a Hilbert space. An operator A on H is positive, $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in H$. In particular, we denote $A > 0$ if $A \geq 0$ is invertible.

A real valued continuous function f on $[0, \infty)$ is called operator monotone if f is order-preserving, i.e., $f(A) \geq f(B)$ for $A \geq B \geq 0$. A typical example is the α-power function $x \to x^\alpha$ for $\alpha \in [0, 1]$, which is the famous Löwner-Heinz inequality;

$$A \geq B \geq 0 \quad \text{implies} \quad A^\alpha \geq B^\alpha \quad \text{for} \quad \alpha \in [0, 1]. \tag{1.1}$$

Another example is the logarithmic function $\log x$, which induces a weaker order than the usual order $\geq$. In [12], the chaotic order $A \gg B$ for $A,\ B > 0$ is defined by $\log A \geq \log B$.

Recall that an operator T is hyponormal if $T^*T - TT^* \geq 0$. Based on the Löwner-Heinz inequality, Aluthge [1] introduced the p-hyponormal operators for $p \in (0, 1]$ by

$$(T^*T)^p \geq (TT^*)^p, \tag{1.2}$$

cf. [28] and [13]. Recently Tanahashi [26] introduced the log-hyponormality for invertible operators by $T^*T \gg TT^*$, i.e., $\log T^*T \geq \log TT^*$, see [27]. Note that log-hyponormality is regarded as 0-hyponormality sometimes. As a matter of fact, it is essential in the Putnam inequality [27] and [6], cf. [4]:

If T is a log-hyponormal operator, then

$$\| \log T^*T - \log TT^* \| \leq \frac{1}{\pi} \iint_{\sigma(T)} r^{-1}\ dr d\theta,$$

where $\sigma(T)$ is the spectrum of T.

Now we have to state the celebrated order-preserving operator inequality, that is, the Furuta inequality [15] and [16] for a one-page proof, see also [5] and [22]:

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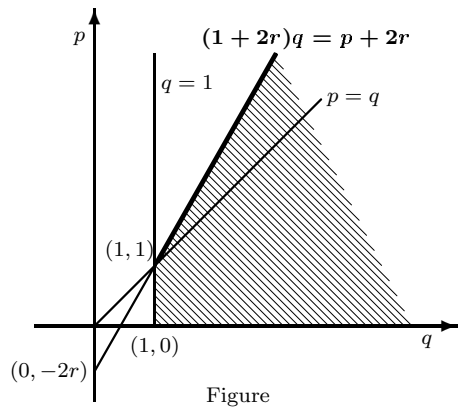
The Furuta inequality. *If $A \geq B \geq 0$, then for each $r \geq 0$,*

$$(B^r A^p B^r)^{1/q} \geq (B^r B^p B^r)^{1/q} \tag{1.3}$$

holds for $p \geq 0$ and $q \geq 1$ with

$$(1+2r)q \geq p + 2r. \tag{*}$$

The domain representing (*) is drawn in the right and it is shown in [25] that this domain is the best possible one for the Furuta inequality.



Figure

It is a historical extension of the Löwner-Heinz inequality and gives us the following characterizations of the chaotic order:

Theorem A. *The following statements are mutually equivalent for $A, B > 0$:*

(i) $A \gg B$, *i.e.*, $\log A \geq \log B$.

(ii) $(B^p A^{2p} B^p)^{\frac{1}{2}} \geq B^{2p}$ *for all* $p > 0$.

(iii) $(B^r A^{2p} B^r)^{\frac{r}{p+r}} \geq B^{2r}$ *for all* $p, r > 0$.

We remark that (ii) is due to Ando [3] and (iii) in [7] and [17], and that (iii) is regarded as "the Furuta inequality for chaotic order". We also refer [8].

Based on such a recent development of operator inequalities, Furuta, Ito and Yamazaki [19] introduced new families of classes of operators; they are defined by operator inequalities and norm inequalities, and named class $A(k)$ and absolute k-paranormal operators respectively.

In our preceding note [11], we continued their discussion. For this, we introduced a new class $A(p,p)$ of operators in order to make clear interrelation among such classes of operators mentioned above: For $p, r > 0$, an operator T belongs to the class $A(p,r)$ if it satisfies an operator inequality

$$(|T^*|^r |T|^{2p} |T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r}. \tag{1.4}$$

We here note that the definition of $A(p,r)$ is based on Theorem A and that $A(k,1)$ is nothing but $A(k)$ due to Furuta-Ito-Yamazaki.

The purpose of this note is to develop such discussion; we introduce a new family of classes of operators derived from the Furuta inequality. For $p > 0$, $r \geq 0$ and $q \geq 1$, an operator T belongs to $F(p,r,q)$ if it satisfies an operator inequality

$$(|T^*|^r |T|^{2p} |T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}}. \tag{1.5}$$

Thus we consider some properties of operators belonging to $F(p,r,q)$ and relations among these classes. Clearly our new family includes the class of p-hyponormal operators and $A(p,r)$ in (1.4). Precisely, $A(p,r) = F(p,r,\frac{p+r}{r})$ for $p, r > 0$.

2. Preliminaries. An operator T on H is paranormal if it satisfies a norm inequality

$$\|T^2 x\| \|x\| \geq \|Tx\|^2 \quad \text{for all } x \in H, \tag{2.1}$$

see [14],[18] and [21]. Ando [2] showed that every log-hyponormal is paranormal. To explain it, Furuta-Ito-Yamazaki [19] introduced new families of classes of operators as follows:

Definition B. *Let $k > 0$. (i) An operator T belongs to the class $A(k)$ if it satisfies*

$$(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2. \tag{2.2}$$

(ii) An operator T on H is absolute k-paranormal if it satisfies

$$\||T|^kTx\|\|x\|^k \geq \|Tx\|^{k+1} \tag{2.3}$$

for all $x \in H$.

It is clear that the absolute 1-paranormality is nothing but the paranormality. It is proved in [19] that these two families have monotonicity on k, e.g., $A(k) \subseteq A(l)$ if $k < l$, and that every operator in $A(k)$ is absolute k-paranormal. Namely one is determined by operator inequalities and the other norm inequalities; they constitute clearly parallel and increasing lines.

On the other hand, we introduced p-paranormal operators for $p > 0$ by a norm inequality

$$\||T|^pU|T|^px\|\|x\| \geq \||T|^px\|^2 \tag{1.6}$$

for all $x \in H$, where U is the partial isometry appeared in the polar decomposition $T = U|T|$ of T. We proved that every p-paranormal operator is paranormal for $0 < p < 1$, [10; Theorem 4]. The background of p-paranormal operators is the following Hölder-McCarthy inequality [23], see [11]. It will be used in the below.

Hölder-McCarthy inequality. *For $A \geq 0$ on H, the following inequalities hold for all $x \in H$;*

$$(Ax, x)^r \geq \|x\|^{2(r-1)}(A^rx, x) \quad \text{if } 0 \leq r \leq 1 \tag{2.4}$$

and

$$(Ax, x)^r \leq \|x\|^{2(r-1)}(A^rx, x) \quad \text{if } r \geq 1. \tag{2.5}$$

Consequently, if $0 < t \leq s$ and $\|x\| = 1$, then

$$\|A^tx\|^s \leq \|A^sx\|^t. \tag{2.6}$$

In addition, the p-paranormality is based on the fact that $T = U|T|$ is p-hyponormal if and only if $S = U|T|^p$ is hyponormal, [9; Lemma 1]. Actually, $T = U|T|$ is p-paranormal if and only if $S = U|T|^p$ is paranormal.

In our preceding note [11], we discussed some relations among $A(k)$, $A(p, p)$, absolute k-paranormal and p-paranormal operators. We showed another parallelism between $A(p, p)$ and p-paranormal operators which is similar to parallelism between $A(k)$ and absolute k-paranormal operators obtained by Furuta-Ito-Yamazaki [19]. Among others, we gave an approach to log-hyponormal operators from $A(p, p)$ as $p \to 0$, and proved that every absolute k-paranormal operator is k-paranormal for $k > 1$, and that every k-paranormal operator is normaloid.

3. Operators in $F(p, r, q)$. In [11; Theorem 3.1], we considered the monotonicity of $A(p, r)$. So we first discuss that of $F(p, r, q)$.

Theorem 3.1. *If $0 < r < r'$ and $1 \le q < q'$, then $F(p,r,q) \subseteq F(p,r',q)$ and $F(p,r,q) \subseteq F(p,r,q')$ for all $p > 0$.*

Proof. Suppose that $T \in F(p,r,q)$ and $\epsilon > 0$ is given. Putting $A = |T|$ and $B = |T^*|$, we have

$$A_1 = (B^r A^{2p} B^r)^{\frac{1}{q}} \ge B^{\frac{2(p+r)}{q}} = B_1$$

by the assumption, so that the Furuta inequality ensures that

$$(B_1^{r_1} A_1^{2p_1} B_1^{r_1})^{\frac{1}{q}} \ge B_1^{\frac{2(p_1+r_1)}{q}} \tag{3.1}$$

for all $p_1, r_1 \ge 0$ with $(1+2r_1)q \ge 2(p_1+r_1)$. Take $p_1 = \frac{q}{2}$ and $r_1 = \frac{q\epsilon}{2(p+r)}$ in (3.1). Since $(1+2r_1)q \ge 2(p_1+r_1)$ clearly, (3.1) is arranged as

$$(B^{r+\epsilon} A^{2p} B^{r+\epsilon})^{\frac{1}{q}} \ge B^{\frac{2(p+r+\epsilon)}{q}},$$

that is, $T \in F(p, r+\epsilon, q)$.

In addition, the latter follows from the Löwner-Heinz inequality.

The following characterization of k-hyponormal operators is a simple application of the Furuta inequality, cf. [11; Theorem 3.2]:

Theorem 3.2. *For a fixed $k > 0$, an operator T is k-hyponormal if and only if $T \in F(2kp, 2kr, q)$ for all $p, r \ge 0$ and $q \ge 1$ with $(1+2r)q \ge 2(p+r)$.*

We now define a new family of classes of operators corresponding to the family of $F(p,r,q)$ which is determined by norm inequalities: For $p, r, q \ge 0$, an operator T on H is (p,r,q)-paranormal if

$$\||T|^{\frac{p+r}{q}} x\| \le \||T|^p U |T|^r x\|^{\frac{1}{q}} \tag{3.2}$$

for all unit vectors $x \in H$, where $T = U|T|$ is the polar decomposition of T. It is easily seen that the p-paranormality is the $(p,p,2)$-paranaormality. That is, it is a generalization of the p-paranormality. Thus we have the following extension of [11; Theorem 3.4].

Theorem 3.3. *If $T \in F(p,r,q)$ for $p, r > 0$ and $q \ge 1$, then T is (p,r,q)-paranormal. In particular, if $T \in F(p,p,2)$, then T is p-paranormal.*

Proof. For a given unit vector $x \in H$ and $T = U|T|$, we have

$$\begin{aligned}
|T|^{\frac{2(p+r)}{q}} &= U^*|T^*|^{\frac{2(p+r)}{q}} U \\
&\le U^*(|T^*|^r |T|^{2p} |T^*|^r)^{\frac{1}{q}} U \quad \text{by } T \in F(p,r,q) \\
&= U^*(U|T|^r U^* |T|^{2p} U |T|^r U^*)^{\frac{1}{q}} U \\
&= (|T|^r U^* |T|^{2p} U |T|^r)^{\frac{1}{q}}.
\end{aligned}$$

Hence it follows from the Hölder-McCarthy inequality (2.4) that

$$\begin{aligned}
\||T|^{\frac{p+r}{q}} x\|^2 &\le ((|T|^r U^* |T|^{2p} U |T|^r)^{\frac{1}{q}} x, x) \\
&\le (|T|^r U^* |T|^{2p} U |T|^r x.x)^{\frac{1}{q}} \\
&= \||T|^p U |T|^r x\|^{\frac{2}{q}},
\end{aligned}$$

so that T is (p,r,q)-paranormal.

4. (p,r,q)-paranormal operators. In this section, we first discuss the monotonicity of the (p,r,q)-paranormality.

Theorem 4.1. *The (p,r,q)-paranormality is monotone increasing on $q>0$. In addition, the $(p,r,1)$-paranormality is monotone increasing on r.*

Proof. Suppose that T is (p,r,q)-paranormal and $q' \geq q > 0$. For a given unit vector $x \in H$, it follows from (2.4) that

$$\||T|^{\frac{p+r}{q'}}x\| = \||T|^{\frac{p+r}{q}\cdot\frac{q}{q'}}x\| \leq \||T|^{\frac{p+r}{q}}x\|^{\frac{q}{q'}} \leq \||T|^pU|T|^rx\|^{\frac{1}{q}\cdot\frac{q}{q'}} = \||T|^pU|T|^rx\|^{\frac{1}{q'}},$$

so that T is (p,r,q')-paranormal.

Next suppose that T is $(p,r,1)$-paranormal and $\epsilon > 0$. Then we have

$$\||T|^pU|T|^{r+\epsilon}x\| = \||T|^pU|T|^r|T|^\epsilon x\| \geq \||T|^{p+r}|T|^\epsilon x\| = \||T|^{p+r+\epsilon}x\|,$$

so that T is $(p,r+\epsilon,1)$-paranormal.

Next we consider relations between $(p,r,1)$-paranormality and p-paranormality.

Theorem 4.2. *If T is $(p,r,1)$-paranormal, then T is $\max\{p,r\}$-paranormal.*

Proof. Let x be a given unit vector. If $p \geq r$, then

$$\||T|^pU|T|^px\| = \||T|^pU|T|^r|T|^{p-r}x\| \geq \||T|^{p+r}|T|^{p-r}x\| = \||T|^{2p}x\| \geq \||T|^px\|^2,$$

so that T is p-paranormal.

On the other hand, if $p < r$, then we note that

$$\||T|^rx\|^{1+\frac{r}{p}} \leq \||T|^{p+r}x\|^{\frac{r}{p}}$$

because $\||T|^rx\|^{p+r} \leq \||T|^{p+r}x\|^r$ by (2.6). Hence it follows that

$$\begin{aligned}
\||T|^rU|T|^rx\| &= \|(|T|^p)^{\frac{r}{p}}U|T|^rx\| \\
&\geq \||T|^pU|T|^rx\|^{\frac{r}{p}}\||T|^rx\|^{1-\frac{r}{p}} \text{ by (2.4)} \\
&\geq \||T|^{p+r}x\|^{\frac{r}{p}}\||T|^rx\|^{1-\frac{r}{p}} \text{ by the assumption} \\
&\geq \||T|^rx\|^{(1+\frac{r}{p})+(1-\frac{r}{p})} \text{ by the above remark} \\
&= \||T|^rx\|^2,
\end{aligned}$$

so that T is r-paranormal.

From the viewpoint of the (p,r,q)-paranormality, we see the absolute p-paranormality in [19]. Namely we generalize it as follows: An operator T on H is absolute (p,r)-paranormal if it is $(p,r,p+r)$-paranormal, i.e., it satisfies

$$\||T|^pU|T|^rx\| \geq \||T|x\|^{p+r} \tag{4.1}$$

for all unit vectors $x \in H$. Clearly the absolute $(p,1)$-paranormality is the absolute p-paranormality.

In [19], it is shown that every $T \in A(p)$ is absolute p-paranormal. We now have the following variant:

Theorem 4.3. *If $T \in F(p,r,1)$ and $p+r \geq 1$, then T is absolute (p,r)-paranormal.*

Proof. Since $|T^*|^r = U|T|^r U^*$, $T \in F(p,r,1)$ if and only if

$$|T|^r U^* |T|^{2p} U |T|^r \geq |T|^{2(p+r)}.$$

Therefore it follows from (2.6) that for each unit vector $x \in H$

$$\||T|^p U |T|^r x\| \geq \||T|^{p+r} x\| \geq \||T|x\|^{p+r}.$$

Next we show that the absolute (p,r)-paranormality has the monotone property as well as the absolute p-paranormality.

Theorem 4.4. *Suppose that T is absolute (p,r)-paranormal. If $0 < r \leq 1$, then T is absolute (p',r)-paranormal for $p' > p$. If $p+r \geq 1$, then T is absolute (p,r')-paranormal for $r' > r$.*

Proof. Let x be a given unit vector. If $\frac{p'}{p} > 1$, then (2.5) implies that

$$\begin{aligned}\||T|^{p'} U |T|^r x\| &\geq \||T|^p U |T|^r x\|^{\frac{p'}{p}} \|U|T|^r x\|^{1-\frac{p'}{p}} \\ &\geq \||T|x\|^{(p+r)\cdot\frac{p'}{p}} \||T|x\|^{r(1-\frac{p'}{p})} \\ &= \||T|x\|^{(p'+r)},\end{aligned}$$

that is, T is absolute (p',r)-paranormal.

To prove the latter, we may assume that $0 < \epsilon = r' - r \leq 1$. Then it follows from (2.6) that

$$\begin{aligned}\||T|^p U |T|^{r'} x\| &= \||T|^p U |T|^r |T|^\epsilon x\| \geq \||T||T|^\epsilon x\|^{p+r} \||T|^\epsilon x\|^{1-(p+r)} \\ &\geq \||T|x\|^{(1+\epsilon)(p+r)} \||T|x\|^{\epsilon(1-(p+r))} = \||T|x\|^{p+r+\epsilon}\end{aligned}$$

because $1-(p+r) \leq 0$. That is, T is absolute (p,r')-paranormal.

Following our preceding note [11], we investigated the relation between p-paranormality and absolute p-paranormality; we showed that every p-paranormal operator is absolute p-paranormal for $0 < p < 1$ and every absolute p-paranormal operator is p-paranormal for $p > 1$. So we discuss relations between the (p,r)-paranormality and the absolute (p,r)-paranormality.

Theorem 4.5. (1) *If T is (p,r,q)-paranormal, then T is absolute (p,s)-paranormal for $s \geq 0$ with $r \leq s \leq 1+r$ and $p+r \geq q(1-s+r)$.*

(2) *If T is absolute (p,s)-paranormal and $p+s \geq 1$, then T is (p,r,q)-paranormal for $r \geq s$ and $q \geq 1$ with $q(1+r-s) \geq p+r \geq q(r-s)$.*

Corollary 4.6. (1) *If T is $(p,r,2)$-paranormal, then T is absolute (p,s)-paranormal for $s \geq r$ with $s-r \leq 1$ and $p+2s-r \geq 2$.*

(2) *If T is absolute (p,s)-paranormal and $p+s \geq 1$, then T is $(p,r,2)$-paranormal for $r \geq s$ with $0 \leq p+2s-r \leq 2$.*

Remark. Corollary 4.6 is a direct generalization of our preceding result [11 ; Theorem 4.2]. As a matter of fact, we obtain it by taking $r = p$ and $s = 1$ in above.

Proof of Theorem 4.5. (1) Since T is (p,r,q)-paranormal and $1-q \le 0$, it follows from the Hölder-McCarthy inequality that for a given unit vector x

$$\begin{aligned}\||T|^pU|T|^sx\| &= \||T|^pU|T|^r|T|^{s-r}x\| \\ &\ge \||T|^{\frac{p+r}{q}}|T|^{s-r}x\|^q\||T|^{s-r}x\|^{1-q} \\ &\ge \||T|x\|^{(\frac{p+r}{q}+s-r)q}\||T|x\|^{(s-r)(1-q)} \\ &= \||T|x\|^{p+s},\end{aligned}$$

so that T is absolute (p,s)-paranormal.

(2) Since T is absolute (p,s)-paranormal and $p+s \ge 1$, it follows that for a given unit vector x

$$\begin{aligned}\||T|^pU|T|^rx\| &= \||T|^pU|T|^s|T|^{r-s}x\| \\ &\ge \||T||T|^{r-s}x\|^{p+s}\||T|^{r-s}x\|^{1-(p+s)} \\ &= \|(|T|^{\frac{p+r}{q}})^{\frac{q(1+r-s)}{p+r}}x\|^{p+s}\|(|T|^{\frac{p+r}{q}})^{\frac{q(r-s)}{p+r}}x\|^{1-(p+s)} \\ &\ge \||T|^{\frac{p+r}{q}}x\|^{\frac{q(1+r-s)(p+s)}{p+r}}\||T|^{\frac{p+r}{q}}x\|^{\frac{q(r-s)(p+s)}{p+r}} \\ &= \||T|^{\frac{p+r}{q}}x\|^q,\end{aligned}$$

so that T is (p,r,q)-paranormal.

Remark. As in the proof of (2) in Theorem 4.5, we have a complementary result to it: If T is absolute (p,s)-paranormal and $p+s \le 1$, then T is (p,r,q)-paranormal for $r \ge s$ and $q \ge 1$ with $q(r-s) \ge p+r$.

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