IDEAL OPERATOR COMPACTIFICATION OF TOPOLOGICAL SPACES

JORGE VIELMA[1] ENNIS ROSAS[2]

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Abstract. We shall introduce the concepts of ideal α-locally compact spaces and ideal α-compactification and prove that every α-T_2 ideal α-locally compact space has an ideal-α-compactification.

In [1] Kasahara introduced the concepts of operator α associated to a topology, in [4] Rosas E.and Vielma Jorge defined α-locally compact spaces and α-T_2 spaces . Rose David A. and Hamlelt T.R in [3] introduced the concepts of ideal-compactification of (X, Γ, I) . Rosas E and Vielma J. in [2] showed that every α-T_2, α-locally compact space has an α-compactification.

In this paper, we try to generalize the results given in [3] and [4] .In order to do this, we need the following definitions.

Definition 1. [1] *Let (X,Γ) be a topological space and α be an operator from $\Gamma \to P(X)$. We say that α is an operator on Γ if $U \subseteq \alpha(U)$ for all $U \in \Gamma$.*

Definition 2. [4]. *Let (X, Γ) be a topological space and α an operator on Γ . The space (X, Γ) is said to be α-locally compact at the point x, if there exists an α-compact subset C of X and an α-open neighborhood U of x, such that $\alpha(U) \subset C$. The space (X, Γ) is said to be α-locally compact if it is α-locally compact at each of its points.*

Definition 3. *Given a nonempty set X, an ideal I is defined to be a nonempty collection of subsets of X such that.*

(1) $B \in I$ and $A \subset B$, then $A \in I$, and
(2) $A \in I$ and $B \in I$, then $A \cup B \in I$.

If, in addition, I satisfies the following condition:
If $\{A_n : n = 1, 2 \ldots\} \subset I$, then $\cup_n A_n \in I$. Then I is said to be a σ-ideal.

Definition 4. [4]. *Let (X, Γ) and (Y, Ψ) be two topological spaces, α and β operators on Γ and Ψ , respectively. We say that (Y, Ψ) is a β-compactification of X if:*

(1) (Y, Ψ) is β-compact

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(2) (X, Γ) is a subspace of (Y, Ψ).
(3) The operator $\beta/_X$ is equal to α.
(4) The Ψ-closure of X is Y.

Definition 5. [3]. *A space (Y, Ψ, J) is said to be a I -compactification of (X, Γ, I) if :*

(1) $X \subset Y$.
(2) $\Gamma = \Psi/_X = \{V \cap X : V \in \Psi\}$
(3) $J/_X = \{J_1 \cap X : J_1 \in J\} = I$, and
(4) (Y, Ψ, J) is J-compact.
(5) $cl_\Psi(X) = Y$,

Furthermore if $Y \setminus X = \{r\}$,then (Y, Ψ, J) is said to be a one- point J-compactification of (X, Γ, I).

We are going to consider a space (X, Γ, I, α) where (X, Γ) is a topological space, I an ideal on X and α an operator associated to Γ.

Definition 6. *Let (X, Γ, I, α) and (Y, Ψ, J, β) be two spaces. We say that (Y, Ψ, J, β) is a J-β-compactification of X if:*

(1) (X, Γ) is a subspace of (Y, Ψ).
(2) (Y, Ψ, J, β) is, J-β-compact
(3) The operator $\beta/_X$ is equal to α.
(4) The Ψ-closure of X is Y.
(5) $J/_X = I$.

Definition 7. *The space (X, Γ, I, α) is said to be I-α-locally compact at the point x if there exists an I-α-compact subset C of X and an α-open neighborhood U of x such that $\alpha(U) \subset C$ .The space (X, Γ, I, α) is said to be I-α-locally compact if it is I-α-locally compact at each of its points.*

Definition 8. [1] . *Let (X, Γ) be a topological space and α be an operator on Γ. We say that (X, Γ) is an α-T_2 space if for every pair x, y of distinct points of X, there exists Γ-open sets U, V such that $x \in U$, $y \in V$ and $\alpha(U) \cap \alpha(V) = \phi$.*

Definition 9. [1]. *Let (X, Γ) be a topological space and α be an operator on Γ. We say that α is a regular operator if for every $x \in X$ and every pair U, V of Γ-open neigborhood of x, there exists a Γ-open neigborhood W of x such that $\alpha(W) \subseteq \alpha(U) \cap \alpha(V)$.*

Theorem 1. *Let (X, Γ) be a topological space, α be a regular operator on Γ, and I an ideal on X . If X is α-T_2 and $K \subset X$ is I-α-compact, then K is α-closed.*

Proof. We need to prove that $X \setminus K$ is α-open. So let $x_0 \in X \setminus K$. For each $y \in K$, there exists Γ-open neighborhoods U_y and V_y such that $x_0 \in U_y$, $y \in V_y$ and $\alpha(\ U_y) \cap \alpha(\ V_y) = \phi$. In this way we construct an open cover $\{V_y : y \in K\}$ of K. Since K is I-α-compact, there exists a finite set $\{y_1, \dots, y_n\}$ of K such that $K \setminus U_{i=1}^{n} \alpha(\ V_{y_i}) \in I$. Since $V_{y_1}, \dots, V_{y_n}$ are Γ-open neighborhood of x_0. $\cap_{J=1}^{n} V_{y_J} \subseteq X \setminus K$ is a Γ-open neighborhood of x_0. Then using the regularity of α, there exists a Γ-open neighborhood W of x_0 such that $W \subset \alpha(W) \subset X \setminus K$. This implies that $X \setminus K$ is α open and hence K is α-closed

Theorem 2. *Let (X, Γ, I, α) be a space, where α is a regular, subadditive, stable operator with respect to all α-closed subset of (X, Γ) and $\alpha(\emptyset) = \emptyset$. If (X, Γ, I, α) is I-α-locally compact not I-α-compact and α-T_2, then there exists a space (Y, Ψ) and an operator β on Ψ such that (Y, Ψ, I, β) is an I-β-compactification of (X, Γ, I, α), $|Y \setminus X| = 1$ and (Y, Ψ) is β-T_2.*

Proof. Define $Y = X \cup \{\infty\}$, where ∞ is an object not in X. On Y , we define a topology as follows:

(1) If $U \in \Gamma$, then $U \in \Psi$
(2) If C is an I-α-compact subset of X, then $X \setminus C \cup \{\infty\} \in \Psi$.

To prove that Ψ is a topology on Y . look at the proof of the theorem 4 in [4] and the fact that finite union and arbitrary intersection of I-α-compact sets are I-α-compact sets.

Define an operator $\beta : \Psi \longrightarrow P(Y)$ as follows:

$$\beta(V) = \begin{cases} \alpha(V) & \text{if V} \in \Gamma \\ \alpha(X \setminus C) \cup \{\infty\} & \text{if } V = X \setminus C \cup \{\infty\} \text{where } C \text{ is I-}\alpha\text{-compact.} \end{cases} \tag{1}$$

Let us show that (Y, Ψ) is I-β-compact. Let $\{U_i : i \in \Lambda\}$ be a Ψ-open cover of Y. This collection must have at least one element of type $X \setminus C \cup \{\infty\}$ where C is I-α-compact., say $U_{i_0} = X \setminus C \cup \{\infty\}$. Let $V_i = U_i \cap X$ for $i \neq i_0$. Then $\{V_i : i \in \Lambda\}$ is a Γ-open cover of C . Since C is I-α-compact, there exists a finite subcollection $\{i_1, \dots, i_n\}$ of Λ, such that $C \setminus U_{j=1}^{n} \alpha(V_{i_j}) \in I$. Now $Y \setminus (U_{j=1}^{n} \alpha(V_{i_j}) \cup \beta(Y/C)) \subset C \setminus U_{j=1}^{k} \alpha(V_{i_j})$.

In consequence $Y \setminus (U_{j=1}^{k} \alpha(V_{i_j}) \cup \beta(Y/C)) \in I$. Therefore (Y, Ψ, J, β) is I-β-compact. To show that (Y, Ψ) is β-T_2 use the theorem 4 in [4] and the fact that α is additive.The other propieties follows easily.

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1. Universidad de Andes, Departamento de Matemáticas, Facultad Ciencias, Mérida-Venezuela.
email:vielma@ciens.ula.ve
2. Universidad de Oriente, Departamento de Matemáticas, Cumaná, Venezuela.
email:erosas@cumana.sucre.udo.edu.ve