THE NON-EXISTENCE OF A POSITIVE SOLUTION FOR SOME NONLINER ELLIPTIC PROBLEMS IN UNBOUNDED DOMAINS*

ALEXANDRU LEONTE AND NICOLAE TARFULEA

Received May 12, 1995; revised March 28, 1996

ABSTRACT. In this note we prove the non-existence of a positive solution for the nonlinear problem: $-\Delta_p u = \lambda u^q + W(x)u^r$ in Ω, $u = 0$ on $\partial\Omega$, where $\lambda > 0$, $0 \le q < r \le \frac{N+2}{N-2}$, $1 < p \le \frac{2N}{N-2}$, Ω is an unbounded domain in $\mathbf{R}^N$ $(N \ge 3)$ with some properties and $W \in W^{1,\infty}(\Omega)$.

1. Introduction. Let $\Omega \subset \mathbf{R}^N$ $(N \ge 3)$ be an unbounded domain with smooth boundary and with the property
$(*)$ there exists $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbf{R}^N$ such that $n(x) \cdot \alpha \ge 0 (\ne 0)$ on $\partial\Omega$, where $n(x) = (n_1(x), \cdots, n_N(x))$ denotes the unit outward normal to $\partial\Omega$ at the point x.

For $\lambda > 0$, $0 \le q < r \le \frac{N+2}{N-2}$, $1 < p \le \frac{2N}{N-2}$ and $W(\cdot) \in W^{1,\infty}(\Omega)$ we consider the following problem:

$$(1_\lambda) \qquad \begin{cases} -\Delta_p u \equiv -div(|\nabla u|^{p-2}\nabla u) = \lambda u^q + W(x)u^r & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Our result in this note, under the above assumptions, is the following:

Theorem 1. If $\sum_{i=1}^N \alpha_i \frac{\partial W}{\partial x_i} \equiv 0$ then the problem (1_λ) does not admit solutions in $W_0^{2,1}(\Omega) \bigcap W_0^{2,2^*}(\Omega)$, where $2^* = \frac{2N}{N-2}$.

There is a large literature on problem (1_λ). After initial works of Pohozaev [5] and Yamabe [6] there has been, in the last times, a great number of contributions to the study of that kind of problems (see for example: [1],…,[4] and others). The review, even partial, of their results is out of the scope of this note. We would like nevertheless to point out the following facts. The proof of the above theorem uses only elementary tools. Similar ideas are used in [3] for proving various existence and non-existence results for certain semilinear elliptic problems in unbounded domains. As it turns out, the condition $(*)$ about the domain Ω is the same as the condition (3) in [3].

2. Proof of the Theorem 1. Denote by B_R the ball of radius R and center 0 in $\mathbf{R}^N$. Suppose that there exists $\lambda > 0$ such that the problem (1_λ) admits solutions.

* Partially supported by a CNCSU-Grant n. 32/95

Multiplying (1_λ) by $\frac{\partial u}{\partial x_i}$, and integrating by parts on $\Omega\bigcap B_R$, we obtain that

$$-\int_{\partial(\Omega\bigcap B_R)}\frac{\partial u}{\partial n}\frac{\partial u}{\partial x_i}|\nabla u|^{p-2}ds+\int_{\Omega\bigcap B_R}|\nabla u|^{p-2}\nabla u\nabla(\frac{\partial u}{\partial x_i})dx=$$

$$=\int_{\Omega\bigcap B_R}\lambda u^q\frac{\partial u}{\partial x_i}+W(x)u^r\frac{\partial u}{\partial x_i}dx=$$

$$=\int_{\Omega\bigcap B_R}\frac{\partial}{\partial x_i}(\frac{\lambda}{q+1}u^{q+1}+W(x)\frac{u^{r+1}}{r+1})dx-\int_{\Omega\bigcap B_R}\frac{\partial W}{\partial x_i}\frac{u^{r+1}}{r+1}dx$$

Hence:

$$-\int_{\partial(\Omega\bigcap B_R)}\frac{\partial u}{\partial n}\frac{\partial u}{\partial x_i}|\nabla u|^{p-2}ds+\int_{\Omega\bigcap B_R}\frac{\partial W}{\partial x_i}\frac{u^{r+1}}{r+1}dx=$$

$$=\int_{\Omega\bigcap B_R}\frac{\partial}{\partial x_i}(\frac{\lambda}{q+1}u^{q+1}+W(x)\frac{u^{r+1}}{r+1}-\frac{1}{p}|\nabla u|^p)dx=$$

$$=\int_{\partial(\Omega\bigcap B_R)}n_i(x)(\frac{\lambda}{q+1}u^{q+1}+W(x)\frac{u^{r+1}}{r+1}-\frac{1}{p}|\nabla u|^p)ds$$

From this and from the fact that $u=0$ on $\partial\Omega$, we have that:

$$-\frac{p-1}{p}\int_{\partial\Omega\bigcap B_R}n_i(x)|\nabla u|^pds+\int_{\Omega\bigcap B_R}\frac{\partial W}{\partial x_i}\frac{u^{r+1}}{r+1}dx=$$

$$=\int_{\Omega\bigcap\partial B_R}n_i(x)(\frac{\lambda}{q+1}u^{q+1}+W(x)\frac{u^{r+1}}{r+1}-\frac{1}{p}|\nabla u|^p)+\frac{\partial u}{\partial n}\frac{\partial u}{\partial x_i}|\nabla u|^{p-2}ds$$

Now observe that:

$$|\int_{\Omega\bigcap\partial B_R}n_i(x)(\frac{\lambda}{q+1}u^{q+1}+W(x)\frac{u^{r+1}}{r+1}-\frac{1}{p}|\nabla u|^p)+\frac{\partial u}{\partial n}\frac{\partial u}{\partial x_i}|\nabla u|^{p-2}ds|\leq$$

$$\leq\int_{\Omega\bigcap\partial B_R}\frac{\lambda}{q+1}u^{q+1}+\|W\|_\infty\frac{u^{r+1}}{r+1}+\frac{p+1}{p}|\nabla u|^pds$$

Since $u\in W_0^{2,1}(\Omega)\bigcap W_0^{2,2^*}(\Omega)$ we have that:

$$\frac{\lambda}{q+1}u^{q+1}+\|W\|_\infty\frac{u^{r+1}}{r+1}+\frac{p+1}{p}|\nabla u|^p\in L^1(\Omega)$$

Then we obtain that there exists a sequence $(R_n)_{n\geq1}$ such that $R_n\to\infty$, as $n\to\infty$, and:

$$\int_{\Omega\bigcap\partial B_{R_n}}(\frac{\lambda}{q+1}u^{q+1}+\|W\|_\infty\frac{u^{r+1}}{r+1}+\frac{p+1}{p}|\nabla u|^p)ds\to0,$$

as $n\to\infty$.

Hence:

$$\lim_{n\to\infty}\int_{\partial\Omega\bigcap B_{R_n}}n_i(x)|\nabla u|^pds=\frac{p}{p-1}\lim_{n\to\infty}\int_{\Omega\bigcap B_{R_n}}\frac{\partial W}{\partial x_i}\frac{u^{r+1}}{r+1}dx$$

Or equivalently:

$$\int_{\partial\Omega}n_i(x)|\nabla u|^pds=\frac{p}{p-1}\int_\Omega\frac{\partial W}{\partial x_i}\frac{u^{r+1}}{r+1}dx$$

From this we conclude that:

$$\int_{\partial\Omega}(n(x)\cdot\alpha)|\nabla u|^p ds = \frac{p}{p-1}\int_\Omega (\sum_{i=1}^N \alpha_i \frac{\partial W}{\partial x_i}) \frac{u^{r+1}}{r+1} dx = 0$$

Obviously this implies that:

$$(n(x)\cdot\alpha)|\nabla u|^p \equiv 0 \quad \text{on } \partial\Omega.$$

Since $n(x)\cdot\alpha \geq 0$ ($\neq 0$) it is easy to conclude that there exists a non-empty ball B, sufficiently small, such that $\nabla u = 0$ on $\partial\Omega\bigcap B$ and

$$-\Delta_p u = \lambda u^q + W(x)u^r > 0 \;\text{ on } \Omega\cap B, \quad \text{since } q < r.$$

Since, by the well known Strong Maximum Principle for the p-Laplacian, we have that $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega\bigcap B$, we obtain a contradiction with the fact that $\nabla u = 0$ on $\partial\Omega\bigcap B$. This contradiction means that (1_λ) cannot have solution for all $\lambda > 0$.

References

1. A. Ambrosetti, H. Brezis and G. Cerami, Combined Effects of Concave and Convex Nonlinearities in Some Elliptic Problems, J.Funct.Anal. 122 (1994), 519-543.
2. L. Boccardo, M. Escobedo and I. Peral, A Dirichlet Problem Involving Critical Exponent, Nonlinear Anal. Theory Methods Appl., to appear.
3. M.J. Esteban and P.L. Lions, Existence and non-existence results for semilinear elliptic problems in unbounded domain, Proc.Royal Edinburgh Soc. 93A (1982), 1-14.
4. J.P. Garcia Azorero and J. Peral Alonso, Existence and non-existence for the p-laplacian: Nonlinear eigenvalues, Commun.in P.D.E. 12 (1987), 1389-1430.
5. S.I. Pohozaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Soviet Math.Dokl. 5 (1965), 1408-1411.
6. H. Yamabe, On a deformation of Riemannian structures on compact manifolds, Osaka Math.J. 12 (1960), 21-37.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CRAIOVA, 1100 CRAIOVA, ROMANIA