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SOME RESULTS OF SELF–MAPS IN BCK–ALGEBRAS

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Abstract. In this paper we consider the left and right maps in BCK-algebras and obtain certain interesting results.

K. H. Dar, B. Ahmad and M. Kondo noticed the right and left maps in BCK-algebras and got some results. In this paper we shall further discuss these maps and their properties.

Definition 1. Let X be a BCK-algebra and a be a given element in X. Then the map L_a from X into itself is called a left map on X if $L_a(x) = a * x$ for all x in X. The right map, denoted by R_a, is defined by a similar way — $R_a(x) = x * a$ for all x in X.

Noticing that $x * (x * (x * y)) = x * y$ holds for all x, y in X, we can easily verify that the following two propositions are true.

Proposition 2. Let X be a BCK-algebra. Then

(1) L_a^2 is idempotent, i.e., $L_a^2 \cdot L_a^2 = L_a^2$ where $L_a^2(x) = L_a(L_a(x)) = a * (a * x)$;
(2) For every natural number n,

$$L_a^n = \begin{cases} L_a, & \text{if } n \text{ is old,} \\ L_a^2, & \text{if } n \text{ is even;} \end{cases}$$

(3) $L_a^2(x) * L_a(y) = L_a^2(y) * L_a(x)$;
(4) $L_a^2(x) * y = L_a(y) * L_a(x) = L_a^2(x) * L_a^2(y)$;
(5) The set of fixed points of L_a^2, i.e., $\mathrm{Fix}(L_a^2) = \{x \in X \mid L_a^2(x) = x\}$, is equal to the image set, $\mathrm{Im}(L_a) = \{L_a(x) \mid x \in X\}$, of L_a;
(6) L_a^2 is isotonic, i.e., $x \leq y$ implies $L_a^2(x) \leq L_a^2(y)$;
(7) $L_a^2(x) = 0$ if and only if $R_x(a) = a$.

Proposition 3. Let X be a BCK-algebra. Then the following conditions are equivalent: for every x, y in X,

(8) L_a^2 is an endomorphism;
(9) $L_a^2(x * y) = L_a^2(x) * y$;
(10) $L_a^2(x * y) = L_a(y) * L_a(x)$.

We know that L_a is endomorphism iff $a = 0$ (see [3]). However the conclusion is false for L_a^2, for example, if X is involutory (i.e., X contains the greatest element 1 and $1*(1*x) = x$ for all x in X) then L_1^2 is an endomorphism. Besides it is also given that L_a is surjective iff L_a is injective, or iff L_a^2 is the identical map (see [3] and [4]), and from this we also have the following

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Theorem 4. Let a be a fixed element in a BCK-algebra X. Then the following conditions are equivalent:

(a) L_a^2 is injective;
(b) L_a^2 is identical;
(c) L_a^2 is surjective.

Proof. (a) $\Longrightarrow$ (b) By (1), $L_a^2(L_a^2(x)) = L_a^2(x)$ for all x in X, and then $L_a^2(x) = x$ by L_a^2 being injective. Hence L_a^2 is identical.

(b) $\Longrightarrow$ (c) It is trivial.

(c) $\Longrightarrow$ (a) Suppose that $L_a^2(x) = L_a^2(y)$. Since L_a^2 is surjective, there exists an element z in X such that $L_a^2(z) = x$ and so by (1) and (4)

$$x * y = L_a^2(z) * y = L_a^2(L_a^2(z)) * y = L_a^2(x) * y = L_a^2(x) * L_a^2(y) = 0.$$

Similarily we can prove that $y * x = 0$. Hence $x = y$ and L_a^2 is injective. $\square$

Noticing that $L_a^2(x) \leq a$ and $L_a^2(a) = a$, we have $\mathrm{Im}(L_a^2) \subseteq [0, a]$ $(= \{x \in X \mid 0 \leq x \leq a\})$ and so if L_a^2 is surjective, thus L_a surjective, then there is $X \subseteq [0, a]$ and this shows that X is bounded and a is the greatest element 1, moreover, by Theorem 4, $L_a^2(x) = x$. We have proved the following

Proposition 5. Let X be a BCK-algebra. Then

(a) There is at most one surjective left map on X;
(b) X is involutory if and only if there exists a surjective left map on X.

Theorem 6. Let X be a BCK-algebra. Then, for any a in X, the kernel, denoted by $\mathrm{Ker}(L_a^2)$, of L_a^2 is an ideal of X.

Proof. Because $L_a^2(0) = 0$, $0 \in \mathrm{Ker}(L_a^2)$. If x, $y * x \in \mathrm{Ker}(L_a^2)$ then by (4),

$$\begin{aligned} L_a^2(y) &= L_a^2(y) * L_a^2(y * x) = L_a^2(y) * (y * x) = (L_a^2(y) * L_a^2(x)) * (y * x) \\ &= (L_a^2(y) * x) * (y * x) \leq L_a^2(y) * y = L_a(y) * L_a(y) = 0. \end{aligned}$$

Hence $y \in \mathrm{Ker}(L_a^2)$, as shown.

We remark that the conclusion of Theorem 6 was found due to H. Jiang in [5] in 1988 by another way.

Let us consider the right maps. The power R_a^n of a right map R_a is ordinary composition of maps. We denote the set of all fixed point of R_a^n by $\mathrm{Fix}(R_a^n)$. The proof of the following proposition 7 is trivial and omitted.

Proposition 7. Let R_a be a right map on a BCK-algebra X and n a natural number. Then for any x, $y \in X$,

(11) $R_a^n(x * y) = R_a^n(x) * y$;
(12) $x \leq y$ implies $R_a^n(x) \leq R_a^n(y)$;
(13) $x \geq R_a(x) \geq R_a^2(x) \geq \cdots$;
(14) $R_a^n(x) * R_a^n(y) \leq x * y$;
(15) $\mathrm{Im}(R_a) \supseteq \mathrm{Im}(R_a^2) \supseteq \cdots$;
(16) $\mathrm{Fix}(R_a) \subseteq \mathrm{Fix}(R_a^2) \subseteq \cdots$;
(17) $\mathrm{Ker}(R_a) \subseteq \mathrm{Ker}(R_a^2) \subseteq \cdots$;
(18) $\mathrm{Fix}(R_a^n) \subseteq \mathrm{Im}(R_a^n)$;
(19) $\mathrm{Fix}(R_a^n) \cap \mathrm{Ker}(R_a^n) = 0$.

Theorem 8. Let R_a be a right map on a BCK-algebra X. Then R_a^n is an endomorphism if and only if $R_a^n = R_a^{n+1}$.

Proof. If R_a^n is an endomorphism then for any $x \in X$,

$$R_a^{n+1}(x) = R_a^n(R_a(x)) = R_a(x * a) = R_a^n(x) * R_a^n(a) = R_a^n(x) * 0 = R_a^n(x).$$

Hence $R_a^n = R_a^{n+1}$. Conversely if $R_a^n = R_a^{n+1}$ then clearly we have $R_a^n = R_a^{n+m}$ and so for any $x, y \in X$, by (14) and (11),

$$R_a^n(x) * R_a^n(y) = R_a^n(R_a^n(x)) * R_a^n(y) \leq R_a^n(x) * y = R_a^n(x * y).$$

Next by (11) and (13),

$$R_a^n(x * y) = R_a^n(x) * y \leq R_a^n(x) * R_a^n(y).$$

Hence $R_a^n(x * y) = R_a^n(x) * R_a^n(y)$ and R_a^n is an endomorphism.

Theorem 9. $\mathrm{Im}(R_a^n)$, $\mathrm{Fix}(R_a^n)$ and $\mathrm{Ker}(R_a^n)$ are subalgebras of X but may not necessarily be ideals of X.

Proof. $\mathrm{Im}(R_a^n)$, $\mathrm{Fix}(R_a^n)$ and $\mathrm{Ker}(R_a^n)$ are nonempty subset of X, for, 0 is in them, respectively. If $x, y \in \mathrm{Im}(R_a^n)$, there exists an element z in X such that $R_a^n(z) = x$ so by (11), $x * y = R_a^n(z) * y = R_a^n(z * y) \in \mathrm{Im}(R_a^n)$.

If $x, y \in \mathrm{Fix}(R_a^n)$ then $R_a^n(x * y) = R_a^n(x) * y = x * y$ and so $x * y \in \mathrm{Fix}(R_a^n)$.

If $x, y \in \mathrm{Ker}(R_a^n)$ then $R_a^n(x * y) = R_a^n(x) * y = 0 * y = 0$, that is, $x * y \in \mathrm{Ker}(R_a^n)$.

Summarizing the above facts, we have proved the first part of this theorem.

The set $X = \{0, 1, 2, \cdots\}$ with the operation $*$ defined by

$$x * y = \begin{cases} 0, & \text{if } x \leq y, \\ x, & \text{if } x > y \end{cases}$$

forms a positive implicative BCK-algebra. It is easy to verify that the sets, $\mathrm{Im}(R_a)$ and $\mathrm{Fix}(R_a)$, are equal to the set $\{0\} \cup \{a+1, a+2, \cdots\}$ for all $a \in X$, and clearly these are not ideals of X whenever $a \neq 0$.

Moreover the same set $X = \{0, 1, 2, \cdots\}$ with another operation $*$ defined by

$$x * y = \max\{0, x - y\}$$

forms a commutative BCK-algebra. Put $a = 1$ then $\mathrm{Ker}(R_a^n) = \{0, 1, 2, \cdots, n\}$ and this is not an ideal of X. □

A BCK-algebra X is called n-fold positive implicative if $x * y^n = x * y^{n+1}$ for any x, $y \in X$, where $x * y^n = (\cdots((x * y) * y) * \cdots) * y$, ($y$ occurs n times). An n-fold positive implicative BCK-algebra can be characterized by its right maps and this fact is stated in the following theorem. The proof is trivial and omitted.

Theorem 10. Let X be a BCK-algebra. Then the following are equivalent: for any $a \in X$,

(a) X is n-fold positive implicative;
(b) $\mathrm{Im}(R_a^n) = \mathrm{Fix}(R_a^n)$;
(c) $R_a^n = R_a^{n+1}$;
(d) R_a^n is an endomorphism on X.

Let us consider the right maps on a BCK-algebra X with the condition (S). Using simple calculation, we see that the product of arbitrary two right maps on a BCK-algebra X is a right map if and only if X satisties the condition (S), and at the same time we also have, for all x, $y \in X$,

(20) $R_{x\circ y} = R_x R_y = R_y R_x$

where X satisfies the condition (S).

Proposition 11. Let X be a BCK-algebra with the condition (S) and

$$X_m = \{x \in X \mid R_x^m \text{ is an endomorphism on } X\}.$$

Then X_m is a commutative semigroup with respect to $\circ$.

Proof. As it is well known, $(X, \circ)$ is a commutative semigroup and so it suffices to prove that $(X_m, \circ)$ is a sub-semigroup of $(X, \circ)$. In fact, X_m is a nonempty subset of X, for, 0 is in X_m. Next if x, $y \in X_m$ then by (20) and Theorem 8,

$$R_{x\circ y}^{m+1} = (R_x R_y)^{m+1} = R_x^{m+1} R_y^{m+1} = R_x^m R_y^m = (R_x R_y)^m = R_{x\circ y}^m.$$

Now Theorem 8 implies that $R_{x\circ y}^m$ is an endomorphism on X so $x \circ y \in X_m$, as shown.

Finally let us investigate the relationships between $\mathrm{Ker}(L_a^2)$ and $\mathrm{Fix}(R_a)$. For convenience we give a lemma and omit the proof.

Lemma 12. Let A, B be two ideals of a BCK-algebra X. Then $A \cap B = \{0\}$ if and only if for any $a \in A$ and $b \in B$, $L_a^2(b) = L_b^2(a) = 0$.

Theorem 13. Let a be an element in a BCK-algebra X. If $\mathrm{Fix}(R_a)$ is an ideal of X then $\mathrm{Fix}(R_a) \subseteq \mathrm{Ker}(L_a^2)$ but the inverse containing relation may not necessarily hold.

Proof. We denote I_a to be the ideal of X generated by $\{a\}$. It is easy to verify that $I_a \cap \mathrm{Fix}(R_a) = \{0\}$ and then for any $x \in \mathrm{Fix}(R_a)$ since $\mathrm{Fix}(R_a)$ is an ideal of X and $a \in I_a$, we have $L_a^2(x) = 0$ by Lemma 12, that is, $x \in \mathrm{Ker}(L_a^2)$. This shows that $\mathrm{Fix}(R_a) \subseteq \mathrm{Ker}(L_a^2)$. Next put $X = \{0, a, b, c\}$ and define the operation $*$ as follows:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Then X is a BCK-algebra (H. Jiang calls this to be B_{4-1-6}) and $\mathrm{Ker}(L_b^2) = \{0, a, c\}$, $\mathrm{Fix}(R_b) = \{0, c\}$. It is easy to see that $\mathrm{Fix}(R_b)$ is an ideal of X but $\mathrm{Ker}(L_b^2) \nsubseteq \mathrm{Fix}(R_b)$.

Theorem 14. Let X be a BCK-algebra. Then the following are equivalent: for all a, $b \in X$,

(a) $\mathrm{Fix}(R_a)$ is an ideal of X;
(b) $\mathrm{Fix}(R_a) = \mathrm{Ker}(L_a^2)$;
(c) $R_a(b) = b$ implies $R_b(a) = a$.

Proof. (a) $\Longrightarrow$ (b) Since $\mathrm{Fix}(R_a)$ is an ideal of X, by Theorem 13, $\mathrm{Fix}(R_a) \subseteq \mathrm{Ker}(L_a^2)$. Next we assert that $I_a \cap \mathrm{Ker}(L_a^2) = \{0\}$. If it is false then there exists a nonzero element b in $I_a \cap \mathrm{Ker}(L_a^2)$. Then by b in $\mathrm{Ker}(L_a^2)$, we have $L_a^2(b) = 0$ and so $R_b(a) = a$ by (7), that is, $a \in \mathrm{Fix}(R_b)$ and consequently $I_a \subseteq \mathrm{Fix}(R_b)$ by $\mathrm{Fix}(R_b)$ an ideal of X. Now by $b \in I_a$, we get $b \in \mathrm{Fix}(R_b)$ and $b = R_b(b) = 0$, a contradiction with $b \neq 0$, as asserted. Hence for all x in $\mathrm{Ker}(L_a^2)$, by Theorem 6 and Lemma 12, we get $L_x^2(a) = 0$, that is, $R_a(x) = x$ by (7), proving $\mathrm{Ker}(L_a^2) \subseteq \mathrm{Fix}(R_a)$. Hence (b) holds.

(b) $\Longrightarrow$ (c) $R_x(y) = y$ implies $y \in \mathrm{Fix}(R_x) = \mathrm{Ker}(L_x^2)$ and so $L_x^2(y) = 0$. Hence by (7), $R_y(x) = x$.

(c) $\Longrightarrow$ (a) If $x \in \mathrm{Fix}(R_a)$, i.e, $R_a(x) = x$ then by (c), $R_x(a) = a$, i.e., $L_a^2(x) = 0$, which means $x \in \mathrm{Ker}(L_a^2)$. Conversely if $L_a^2(x) = 0$ then $R_x(a) = a$ and so $R_a(x) = x$ by (c). Hence $x \in \mathrm{Fix}(R_a)$. This proves that $\mathrm{Fix}(R_a) = \mathrm{Ker}(L_a^2)$ and $\mathrm{Fix}(R_a)$ is an ideal of X by Theorem 6.

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