WEAK CONTINUITY FORMS, GRAPH CONDITIONS, AND APPLICATIONS

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Abstract. This paper introduces several weak continuity forms and graph conditions and provides some applications of these forms and conditions. The main tools which are utilized for their definitions are the θ-closure and u-closure operators. Applications include (1) a characterization of quasi Urysohn-closed (QUC) topological spaces analogous to that of the original definition of Urysohn-closed spaces, i. e. that Urysohn-closed spaces are those spaces which are closed subspaces of the Urysohn spaces in which they are embedded, and (2) a weakening of the continuity condition to improve the result that a continuous image of a Urysohn-closed space is Urysohn-closed. Certain subsets of a space are defined as quasi Urysohn-closed (QUC) relative to the space and investigated. Among the discoveries about these subsets is the fact that for a Urysohn space there is a class of functions such that each closed subset of the space is QUC if and only if each member of the class which maps the space into a Urysohn space is a closed function. Parallels for Urysohn-closed spaces of theorems for functionally compact and C-compact spaces are provided. In the final section of the paper the u-closure operator is utilized to isolate a "second category type" property of topological spaces. It is proved that QUC spaces have this property, and the property is employed to establish several generalizations of the Uniform Boundedness Principle from analysis.

Introduction. Applications of notions which generalize continuity of mappings and mappings with closed graphs between topological spaces have received widespread and continual attention over a span of many years (see [1], [6], [7], [11], [12], [13], [14], [17] , [18], [19], [24], [25]). The purpose of this paper is to introduce several weak continuity and graph notions and to provide applications of these notions. The main tools which are utilized to define these notions are the $\theta-$closure operator, $[\ ]_\theta$, introduced along with the $\theta-$adherence of a filterbase by Veličko [26] and used by him and other researchers to study generalizations of compact spaces such as H-closed, and minimal Hausdorff spaces (see [5], [6], [13], [14]), and the u-closure operator, $[\ ]_u$, which was introduced by Joseph [19] and comes naturally from the notion of u-adherence of a filterbase, introduced by Herrington [12], and employed by him and others to study Urysohn-closed and minimal Urysohn spaces in terms of arbitrary filterbases, nets, weak continuity forms and graph conditions (see [8], [12], [20]). If X is a space and $A \subset X$ these operators are selfmaps of 2^X (where 2^X is the power set of X), $[A]_\theta$ ($[A]_u$) represents the value of $[\ ]_\theta$ ($[\ ]_u$) at A and is called the $\theta-$closure (u-closure) of A. Unexpected connections between set-valued functions, compactness and these operators, such as the following, have been discovered [5], [8].

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(1°) *The following statements are equivalent for a space X :*

(1) The space X is compact.

(2) For each upper semi-continuous (u.s.c.) set-valued function η on X the set-valued function defined on X by $[\eta(x)]_\theta$ assumes a maximal value with respect to set inclusion.

(3) For each u.s.c. set-valued function η on X the set-valued function defined on X by $[\eta(x)]_u$ assumes a maximal value with respect to set inclusion.

The results of this paper are organized into four sections. In §1 the new weak continuity forms and some characterization theorems are presented. Relationships between these forms, and these forms and several forms which have been studied by earlier researchers are investigated and a collection of pertinent examples is provided. For instance, it is shown that while continuity implies the property referred to in this paper as u-continuity, Fomin's $\theta-$continuity does not. It is also shown that continuity at a point does not force u-continuity at that point.

It is known that $[\ ]_u$ is not a Kuratowski operator. Another feature of this section is an investigation of when this operation is Kuratowski. Necessary and sufficient conditions for $[\ ]_u$ to be Kuratowski and some consequences of $[\ ]_u$ being such an operator are established.

In §2 the graph conditions are introduced and results to be applied in §3 and §4 are established. In §3 the results in §1 and §2 are applied to produce several characterizatons of QUC spaces analogous to the original definition and well-known characterizations of Urysohn-closed spaces. These include

(2°) *The following statements are equivalent for a space X :*

(1) The space X is QUC.

(2) If X is a dense (dense open) subspace of a space Y, the equality $Y = \bigcup_X [x]_u$ holds.

(3) Each u-continuous (continuous) function g on X satisfies the inclusion

$$\overline{g(A)} \subset \bigcup_{[A]_u} [g(x)]_u$$

for each $A \subset X$.

(4) Each filterbase on X with at most one u-adherent point is u-convergent.

A number of by-products of (2°) such as new characterizations of Urysohn-closed spaces, and a generalization of the well-known fact that a continuous image of a Urysohn-closed space is Urysohn-closed, are obtained in §3. Analogues for Urysohn spaces of theorems for functionally compact and C-compact spaces are also provided in that section. These are analogues of the results for functionally compact spaces stated in (3°), [21]. The results in (4°), analogues of known results for C-compact spaces [2], [24] are also proved. If Ω is a family of sets the intersection of the members of Ω will be denoted by $\mathcal{I}(\Omega)$. If Ω is a filterbase on a space, $[\Omega]_\theta$ ($[\Omega]_u$) denotes the $\theta-$adherence (u-adherence) of Ω.

(3°) *The following statements are equivalent for a space X :*

(1) The space X is functionally compact.

(2) Each filterbase Ω on X satisfying $[\Omega]_\theta = \mathcal{I}(\Omega)$ converges to $\mathcal{I}(\Omega)$.

(3) For each space Y, each strongly-closed set-valued function $\lambda \subset Y \times X$ is u.s.c.

(4) Functions on X with strongly-closed inverses are closed functions.

It should be noted that A. Bella proved the equivalence of (1) and (3) of Theorem 2.4 in [1], and that the equivalence was independently proved by the authors prior to its publication by Bella.

Analogues of theorems for C-compact spaces are proved in the form of

(4°) *The following statements are equivalent for a Urysohn space X :*

(1) Each closed subset of X is QUC.

(2) For each Urysohn space Y each $g : X \to Y$ with a u-strongly-subclosed inverse is a closed function.

(3) For each Urysohn space Y each u-strongly-subclosed set-valued function $F \subset Y \times X$ is u.s.c.

The paper is completed in §4, with formulations and proofs of generalizations of the Uniform Boundedness Principle from classical analysis. To illustrate the type of uniform boundedness results which are established, the classical theorem is set in the following framework. A family $\mathcal{F}$ of functions from a space X to a space Y is *uniformly bounded on $A \subset X$ by $\Delta \subset 2^Y$* if there is a $C \in \Delta$ such that $g(A) \subset C$ for each $g \in \mathcal{F}$; let $\mathcal{UB}[\mathcal{F}, X, Y, \Delta] = \{A \subset X : \mathcal{F}$ is uniformly bounded on A by $\Delta\}$. Using this terminology the classical Uniform Boundedness Principle is stated as follows (see [19]): Let X be a complete metric space, let R be the Euclidean line, let $\mathcal{F}$ be a family of real-valued continuous functions on X and let Δ be a nonempty countable family of compact subsets of the reals. If $\mathcal{UB}[\mathcal{F}, X, R, \Delta]$ contains the collection of singletons then $\mathcal{UB}[\mathcal{F}, X, R, \Delta]$ has a nonempty open subset as an element.

One of the theorems proved in this vein is (5°). Crucial to the proof of this theorem is the fact (Theorem 4.6) that a QUC space X cannot be written as the union of a countable number of sets of the form $[A]_u$ each having empty interior in X, a result of "second category type".

(5°) *Let X be a QUC space, let Y be any space, let Δ be a nonempty countable family of QUC relative to Y subsets and let $\mathcal{F}$ be a family of functions from X to Y with u-subclosed graphs such that $\mathcal{UB}[\mathcal{F}, X, Y, \Delta]$ contains the collection of singletons of X. Then the following properties hold:*

(1) There is a nonempty open $V \subset X$ such that $[V]_u \in \mathcal{UB}[\mathcal{F}, X, Y, \Delta]$.

(2) For each $g \in \mathcal{F}$ the restriction of g to $[V]_u$ is u-continuous for the open subset V of part (1).

(3) Each $g \in \mathcal{F}$ is u-continuous at each $x \in V$ of part (1) such that V contains a closed neighborhood of x.

(6°) *Let X be a QHC (quasi H-closed) space, let Y be a space, let Δ be a nonempty countable family of QUC relative to Y subsets and let $\mathcal{F}$ be a family of functions from X to Y with (θ,u)-subclosed graphs such that $\mathcal{UB}[\mathcal{F}, X, Y, \Delta]$ contains the collection of singletons of X. Then*

(1) There is a nonempty open V of X such that $\overline{V} \in \mathcal{UB}[\mathcal{F}, X, Y, \Delta]$.

(2) For each $g \in \mathcal{F}$ the restriction of g to $\overline{V}$ of part (1) is (θ, u)-continuous.

(3) Each $g \in \mathcal{F}$ is (θ,u)-continuous at each point of V of part (1).

(4) There is a W open in X such that each $g \in \mathcal{F}$ is (θ,u)-continuous at each $x \in W$ and $\overline{W} = X$.

Several theorems of this type are given in §4.

1. The weak continuity forms. In this section the new weak continuity forms are introduced. The relationships between these forms and the relationships between these forms and some forms which have been investigated are delineated. Let X be a space, $A \subset X$, $x \in X$, and let Ω be a filterbase on X; let $\Sigma(A)$ represent the collection of open neighborhoods of A, let $\Gamma(A)$ be the collection of closed neighborhoods of A and let $\Lambda(A) = \bigcup_{\Gamma(A)} \Sigma(V)$ (if $A = \{x\}$ the notation $\Sigma(x)$, $\Gamma(x)$, or $\Lambda(x)$ will be used). The closure of A is denoted by $\overline{A}$ and the adherence of Ω by $\mathcal{A}(\Omega)$. Recall [18] that the *$\theta-$closure of* A is $\bigcap_{\Gamma(A)} W$ and that the *u-closure of* A is $\bigcap_{\Lambda(A)} \overline{W}$ [8]. The subset A is *$\theta-$closed (u-closed)* if $A = [A]_\theta$ $(A = [A]_u)$. A space X is *Urysohn* if each singleton is u-closed. A *Urysohn filterbase* on a space X is an open filterbase (i.e one consisting of open subsets) Ω such that whenever $x \in X - \mathcal{A}(\Omega)$ some $V \in \Sigma(x)$ and $F \in \Omega$ satisfy $\overline{V} \cap \overline{F} = \emptyset$. It is known that a Urysohn space is Urysohn-closed if and only if each Urysohn filterbase on the space has nonempty adherence. An arbitrary space is *Quasi Urysohn-closed (QUC)* if each Urysohn filterbase on the space has nonempty intersection. The *θ-adherence (u-adherence) of a filterbase* Ω on a space is $\bigcap_\Omega [F]_\theta$ [26] ($\bigcap_\Omega [F]_u$ [12]). Herrington [12] has shown that a Urysohn space is Urysohn-closed if and only if each filterbase on the space has nonempty u-adherence. It is easy to see from his proof that a space is QUC if and only if each filterbase on the space has nonempty u-adherence. For a space X, it is known and not difficult to establish that $[V]_u = [\overline{V}]_\theta$ for an open set V, that $[A]_u = \bigcap_{\Sigma(A)} [V]_u$ and that, for $x, y \in X$, the relation $x \in [y]_u$ whenever $y \in [x]_u$ holds. If $int(A)$ represents the interior of A, A is called *regular-closed (regular-open)* if $\overline{int(A)} = A$ $(int(\overline{A}) = A)$.

For spaces X, Y Fomin [9] (Levine [22]) has called a function $g : X \to Y$ *θ-continuous (weakly-continuous) at* $x \in X$ if for each $W \in \Sigma(g(x))$ some $V \in \Sigma(x)$ satisfies $g(\overline{V}) \subset \overline{W}$ $(g(V) \subset \overline{W})$ and has called g *θ-continuous (weakly-continuous)* if g is $\theta-$continuous (weakly-continuous) at each $x \in X$. The function $g : X \to Y$ is *u-continuous at* $x \in X$ if for each $W \in \Lambda(g(x))$ some $V \in \Lambda(x)$ satisfies $g(\overline{V}) \subset \overline{W}$ and g is *u-continuous* if g is u-continuous at each $x \in X$. Our first theorem is a characterization theorem and our second result shows that continuous functions are u-continuous. A filterbase Ω on a space X *u-converges to* $x \in X$ $(\Omega \underset{u}{\to} x)$ if for each $W \in \Lambda(x)$ some $F \in \Omega$ satisfies $F \subset \overline{W}$. A net η in X *u-converges to* $x \in X$ $(\eta \underset{u}{\to} x)$ if η is ultimately in $\overline{W}$ for each $W \in \Lambda(x)$.

Theorem 1.1. *The following statements are equivalent for spaces X, Y and $g : X \to Y$:*

(1) The function g is u-continuous.

(2) For each $x \in X$ each filterbase Ω on X satisfying $\Omega \underset{u}{\to} x$ also satisfies $g(\Omega) \underset{u}{\to} g(x)$.

(3) For each $x \in X$ each net η on X satisfying $\eta \underset{u}{\to} x$ also satisfies $g \circ \eta \underset{u}{\to} g(x)$.

(4) Each filterbase Ω on X satisfies $g([\Omega]_u) \subset [g(\Omega)]_u$.

(5) Each $A \subset X$ satisfies $g([A]_u) \subset [g(A)]_u$.

(6) Each $B \subset Y$ satisfies $[g^{-1}(B)]_u \subset g^{-1}([B]_u)$.

(7) Each filterbase Ω on $g(X)$ satisfies $[g^{-1}(\Omega)]_u \subset g^{-1}([\Omega]_u)$.

(8) Each $B \subset Y$ satisfies $[g^{-1}(int([B]_\theta))]_u \subset g^{-1}([B]_u)$.

(9) Each open $W \subset Y$ satisfies $[g^{-1}(int(cl(W)))]_u \subset g^{-1}([W]_u)$.

(10) Each regular-closed $R \subset Y$ satisfies $[g^{-1}(int(R))]_u \subset g^{-1}([R]_\theta)$.

(11) Each open $W \subset Y$ satisfies $[g^{-1}(W)]_u \subset g^{-1}([W]_u)$.

Proof. Only the proofs of implications $(\mathbf{7}) \Rightarrow (\mathbf{8})$, $(\mathbf{10}) \Rightarrow (\mathbf{11}) \Rightarrow (\mathbf{1})$ will be given since the proofs of the other implications necessary to complete a cycle of implications beginning at (1) and proceeding numerically are either obvious or entirely similar to the arguments used to prove analogous statements for continuity.

(7) $\Rightarrow$ **(8)**. Let $B \subset Y$ satisfy $g(X) \cap int([B]_\theta) \neq \emptyset$. Then $[g^{-1}(int([B]_\theta))]_u$

$$\subset g^{-1}([int([B]_\theta)]_u) = g^{-1}([\overline{int([B]_\theta)}]_\theta) \subset g^{-1}([\ [B]_\theta]_\theta) \subset g^{-1}([B]_u)$$

since $[B]_\theta$ is closed and $[\ [B]_\theta]_\theta \subset [B]_u$. Hence (8) holds.

(10) $\Rightarrow$ **(11)**. Let $W \subset Y$ be open. Then $\overline{W}$ is regular-closed in Y and consequently

$$[g^{-1}(W)]_u \subset [g^{-1}(int(\overline{W})]_u \subset g^{-1}([\overline{W}]_\theta) = g^{-1}([W]_u).$$

Thus (11) holds.

(11) $\Rightarrow$ **(1)**. Let $x \in X$ and let $W \in \Lambda(g(x))$. Then $g(x) \notin [g(X - g^{-1}(\overline{W}))]_u$. Therefore $g(x) \notin g([X - g^{-1}(\overline{W}))]_u$, $x \notin [X - g^{-1}(\overline{W})]_u$ and some $V \in \Lambda(x)$ satisfies $g(\overline{V}) \subset \overline{W}$. Hence (1) holds. $\square$

Corollary 1.2 establishes that continuous functions are u-continuous while Example 1.3 exhibits that continuity at a point need not imply u-continuity at the point.

Corollary 1.2. *Continuous functions are u-continuous.*

Proof. Suppose $g : X \to Y$ is continuous, let $A \subset X$ and let $W \in \Lambda(g(A))$. Some $Q \in \Sigma(g(A))$ satisfies $A \subset g^{-1}(Q) \subset \overline{g^{-1}(Q)} \subset g^{-1}(\overline{Q}) \subset g^{-1}(W)$. Since g is continuous it follows that $[A]_u \subset \overline{g^{-1}(W)}$ and that $g([A]_u) \subset g(\overline{g^{-1}(W)}) \subset \overline{W}$. So $g([A]_u) \subset [g(A)]_u$ and g is u-continuous by equivalence (5) of Theorem 1.1. $\square$

Example 1.3. Let $X = \{0\} \cup (1, \infty)$ and let $\{D(k) : k = 1, 2, 3, 4, 5\}$ be a partition of $(1, \infty)$ into subsets dense in $(1, \infty)$ in the usual topology and suppose $N - \{1\} \subset D(1)$ where N is the set of positive integers. Let $\mathcal{B}$ be a base of open intervals for the usual topology on $(1, \infty)$ with end points in $D(1)$. Declare W to be open in X if W satisfies the following properties: If $k = 1, 3$ or 5 and $x \in W \cap D(k)$ some $B \in \mathcal{B}$ satisfies $x \in B \cap D(k) \subset W$; if $k = 2$ or $k = 4$ and $x \in W \cap D(k)$ some $B \in \mathcal{B}$ satisfies $x \in B \cap (D(k-1) \cup D(k) \cup D(k+1)) \subset W$; if $0 \in W$ then some $m \in N$ satisfies $\bigcup_{n \geq m}(2n, 2n+1) \cap D(1) \subset W$. Let Y have the same description as X except that $N - \{1\} \subset D(5)$. Let $g : X \to Y$ be the identity function. Then g is clearly continuous at 0. To see that g is not u-continuous at 0 let $W = \{0\} \cup \bigcup_N((2n, 2n+1) \cap D(1))$ and $V = W \cup \bigcup_N((2n, 2n+1) \cap D(2))$. Then $\overline{W} \subset V$ and $W, V \in \Sigma(0)$ in Y so $V \in \Lambda(0)$. If $E \subset N$ is the set of multiples of 2 then $E \cap \overline{A} \neq \emptyset$ is satisfied for each $A \in \Sigma(0)$ in X and that $E \cap \overline{V} = \emptyset$ in Y.

Next a space introduced by Herrlich [15] is used to show that u-continuity does not force weak-continuity.

Example 1.4. This space $\mathcal{H}$ is described as follows: Let $\{D(k) : k = 1, 2, 3\}$ be a partition of $[0, 1]$ into dense subsets in the usual topology. Let $\mathcal{H} = [0, 1]$ supplied with the following sets V as open sets. If $x \in V \cap D(k)$, $k = 1, 2$ there exists a, b satisfying $x \in (a, b) \cap D(k) \subset V$; if $x \in V \cap D(3)$ there exist a, b with $x \in (a, b) \cap \mathcal{H} \subset V$. It may be assumed without loss of generality that $0, 1 \in D(3)$ and that all open intervals (a, b) used to describe our open sets satisfy $a, b \in D(3)$. It is straightforward to show that if $a, b \in \mathcal{H}$ and $a < b$ then $\overline{(a, b) \cap D(1)} = [a, b] - D(2)$, $\overline{(a, b) \cap D(2)} = [a, b] - D(1)$, $\overline{(a, b) \cap D(3)} = [a, b]$ and $[(a, b)]_u = [a, b]$. Let $[0, 1]$ have the usual topology, let $g : [0, 1] \to \mathcal{H}$ be the identity function and let $v \in D(1)$. Then $D(1) \in \Sigma(v)$ in $\mathcal{H}$ and $\overline{D(1)} = [0, 1] - D(2)$, so no $V \in \Sigma(v)$ in $[0, 1]$ satisfies $g(V) \subset \overline{D(1)}$, and g is not weakly-continuous. Now if $v \in [0, 1]$ and $W \in \Lambda(v)$ in $\mathcal{H}$

choose $a, b \in [0,1]$ for which $v \in [a,b] \subset \overline{W}$ in $\mathcal{H}$. Since $[0,1]$ is regular there is a $V \in \Lambda(v)$ in $[0,1]$ for which $\overline{V} \subset [a,b]$. So g is u-continuous.

The next theorem offers two properties of functions between spaces, the first of which is a consequence of either $\theta-$continuity or u-continuity and the second of which is an implication of $\theta-$continuity but not of u-continuity.

Theorem 1.5. *Let X, Y be spaces and let $g : X \to Y$ be $\theta-$continuous. Then*
(1) Each open $A \subset X$ satisfies $g([A]_u) \subset [g(A)]_u$.
(2) For each $x \in X$ and $W \in \Sigma(g(x))$, some $V \in \Sigma(x)$ satisfies $g([V]_u) \subset [W]_u$.

Proof. To prove (1) note that for open $A \subset X$,

$$g([A]_u) = g([\overline{A}]_\theta) \subset [\ [g(A)]_\theta]_\theta \subset [g(A)]_u.$$

As for the proof of (2) if $W \in \Sigma(g(x))$, some $V \in \Sigma(x)$ satisfies

$$g([V]_u) = g([\overline{V}]_\theta) \subset [g(\overline{V})]_\theta \subset [\overline{W}]_\theta. \quad \square$$

Example 1.6 verifies that a u-continuous function might fail (2) of Theorem 1.5.

Example 1.6. Let $X = Y = \{0\} \cup (1, \infty)$. A set V is open in X if V satisfies the following properties: If $0 \in V$ there is a sequence ν_n in $(0,1)$ and a $k \in N$ such that $\bigcup_{n \geq k}(2n - \nu_n, 2n + \nu_n) \subset V$; if $x \in V - \{0\}$ there is an open interval I such that $x \in I \subset V$. A set V is open in Y if V satisfies the following properties: If $0 \in V$ there is a $k \in N$ such that $\bigcup_{n \geq k}(2n, 2n+1) \subset V$; if $x \in V - \{0\}$ there is an open interval I such that $x \in I \subset V$. Let $g : X \to Y$ be the identity function. It is not difficult to see that g is u-continuous, and that $W = \{0\} \cup \bigcup_{n \geq 1}(2n, 2n+1)$ is open in Y and that no $V \in \Sigma(0)$ in X satisfies $g([V]_u) \subset [W]_u$.

Since the function in Example 1.4 satisfies Theorem 1.5(2) it follows that $\theta-$continuity is not implied by Theorem 1.5(2) and Example 1.7 shows that $\theta-$continuity does not imply u-continuity.

Example 1.7. Let Ψ be the set of irrationals between 0 and 1. Let $\{\mathcal{S}(n,\pi) : (n,\pi) \in N \times \Psi\}$ be a family of pairwise disjoint countably infinite sets so that $\mathcal{S}(n,\pi) \cap [0,\infty) = \emptyset$ for each $(n,\pi) \in N \times \Psi$. Let $\{D(k) : k = 1,2,3\}$ be a partition of $(1,\infty)$ into dense sets in the usual topology. Let $\mathcal{B}$ be a base of open intervals for the usual topology on $(1,\infty)$ with end points in $D(1)$ and suppose that $N - \{1\} \subset D(1)$. Let E be the set of even positive integers and let $X = \{0\} \cup (1,\infty) \cup \Psi \cup \bigcup_{N \times \Psi} \mathcal{S}(n,\pi)$, with open subsets V defined as follows: $V \subset \bigcup_{N \times \Psi} \mathcal{S}(n,\pi)$; if $x \in V \cap (D(k) - E)$, $k = 1,3$, some $B \in \mathcal{B}$ satisfies $x \in B \cap D(k) \subset V$; if $x \in V \cap D(2)$ some $B \in \mathcal{B}$ satisfies $x \in B \subset V$; if $0 \in V$ some $m \in N$ satisfies $\{0\} \cup \bigcup_{n \geq m}((2n, 2n+1) \cap D(1)) \subset V$; if $2n \in V$, then $2n \in B \cap D(1) \subset V$ for some $B \in \mathcal{B}$ and all but finitely many elements of all but finitely many of the $\mathcal{S}(n,\pi)$ belong to V; if $\pi \in V \cap \Psi$ then all but finitely many elements of all but finitely many of the $\mathcal{S}(n,\pi)$ belong to V. Let $Y = \{-1, 0\} \cup (1,\infty)$. A subset V is open in Y if V meets the following criteria: If $x \in V \cap (D(k) - E)$, $k = 1,3$, some $B \in \mathcal{B}$ satisfies $x \in B \cap D(k) \subset V$; if $x \in V \cap D(2)$ some $B \in \mathcal{B}$ satisfies $x \in B \subset V$; if $0 \in V$ some $m \in N$ satisfies $\{0\} \cup \bigcup_{n \geq m}((2n, 2n+1) \cap D(1)) \subset V$; if $-1 \in V$ there is an $m \in N$ such that all but finitely many elements of $D(3) \cap (2n-1, 2n)$ belong to V for each $n \geq m$. For each $n \in N$ let n_k be a strictly increasing sequence in $(2n-1, 2n) \cap D(2)$ such that $n_k \to 2n$ in the usual topology and for each $(n,\pi) \in N \times \Psi$ denote $\mathcal{S}(n,\pi)$ by $\{n(\pi,k) : k \in N\}$. Define

$g : X \to Y$ by $g(x) = x$ for $x \in \{0\} \cup (1, \infty)$, $g(n(\pi, k)) = n_k$ for $(n, \pi) \in N \times \Psi$ and $n(\pi, k) \in \mathcal{S}(n, \pi)$; $g(\pi) = -1$ for $\pi \in \Psi$. Straightforward arguments will verify that g is $\theta-$continuous. It will now be shown that g is not u-continuous at 0. Let $V = \{0\} \cup \bigcup_{n \geq 1}((2n, 2n+1) \cap D(1))$. Then $V \in \Sigma(0)$ and $\overline{V} = \{0\} \cup \bigcup_{n \geq 1}[[2n, 2n+1] \cap (D(1) \cup D(2))]$ and so if $W = \{0\} \cup \bigcup_{n \geq 1}[(2n - 1/2, 2n + 3/4) \cap (D(1) \cup D(2))] \cup \bigcup_{n \geq 1}[2n, 2n + 1]$ then $W \in \Sigma(\overline{V})$ and $-1 \notin \overline{W}$. If $A \in \Lambda(0)$ in X, there is an $m \in N$ such that $\{2n : n \geq m\} \subset A$ and hence at most countably many $\pi \in \Psi$ satisfy $\mathcal{S}(n, \pi) \cap A = \emptyset$ for some $n \geq m$. Thus $\Psi \cap \overline{A} \neq \emptyset$ and $g(\overline{A}) \not\subset \overline{W}$.

Example 1.8 shows that a weakly-continuous function need not satisfy the property of Theorem 1.5(2) and Example 1.9 exihibits that $\theta-$continuity at a point does not force the property described in (2) of that theorem at that point. In these two examples subspaces of a space denoted by $\mathcal{J}$ are used: Let $W = N \cup \{0\}$ and let $\{p_k : k \in W\}$ be a strictly increasing sequence of primes. For $(j, k, m) \in N \times W \times N$ let $J(j, k, m) = \{(j + p_k^{-n}, m) : n \in N\}$; let

$$\mathcal{J} = (W \times N) \cup \bigcup_{(j,k) \in N \times W} J(j, k, 1) \cup \bigcup_{(j,k,m) \in N \times N \times N} J(j, k, m) \cup \{(0, 0),\ (1, 0)\}$$

with the topology generated by the aggregate of basic open sets listed below:

(1) Subsets of $\mathcal{J} - ((W \times N)) \cup \{(0, 0),\ (1, 0)\}$.

(2) Subsets of the form $\{(0, 0)\} \cup \bigcup_{j \geq j_0} J(j, 0, 1)$.

(3) Subsets of the form $\{(1, 0)\} \cup \bigcup_{m \geq m_0} J(j, k, m)$.

(4) Subsets of the form $\{(0, m)\} \cup \bigcup_{j \geq j_0, k \geq k_0} J(j, k, m)$.

(5) Relative open subsets from the plane in $\{(k, 1) : k \in N\} \cup \bigcup_{(j,k) \in N \times W} J(j, k, 1)$.

(6) For $m > 1$ and $k \in N$ all sets of the form $A \cup B$ where A is relative open in the plane about (k, m) and B is of the form $\bigcup_{j \geq j_0} J(j, k, m - 1)$.

Example 1.8. Let

$$\mathcal{K} = (N \times \{1, 2\}) \cup \bigcup_{(j,k) \in N \times W} J(j, k, 1) \cup \bigcup_{(j,k) \in N \times N} J(j, k, 2) \cup \{(0, 0),\ (1, 3)\}$$

with the topology generated by the following open set base:

(1) Relative open sets from $\mathcal{J}$ on $\{(0, 0)\} \cup N \times \{1\} \cup \bigcup_{(j,k) \in N \times W} J(j, k, 1)$ and about $(1, 3)$,

(2) all sets of the form $(V \cap \{(x, 2) : (x, 2) \in \mathcal{J}\} \cup \{j + p_k^{-n}, 1) : 1 \leq k \leq m, n \in N, j > j_0\}$, where $(m, 2) \in N \times \{2\}$ and $V \in \Sigma((m, 2))$ in $\mathcal{J}$, and

(3) all subsets of $\bigcup_{(j,k) \in N \times N} J(j, k, 2)$.

Define $g : \mathcal{K} \to \mathcal{K}$ by $g(0, 0) = g(1, 3) = (1, 3)$, $g(x, 2) = (< x >, 2)$ for all $x \geq 2, g((1 + p_k^{-n}, 2) = (p_k^n, 1)$ for all $k, n \in N$, $g(j + p_1^{-n}, 1) = (j + p_1^n, 1)$ for all $j, n \in N$, $g(1, 2) = (0, 0)$ and $g(x, 1) = (< x >, 2)$ otherwise (the notation $< x >$ represents the greatest integer in x). Choose $Q \in \Sigma(g(0, 0))$ such that $(0, 0) \notin [\overline{Q}]_\theta$. Since $(1, 2) \in [\overline{V}]_\theta$ for all $V \in \Sigma((0, 0))$ it follows that no such V satisfies $g([\overline{V}]_\theta) \subset [\overline{Q}]_\theta$ and that g does not satisfy (2) of Theorem 1.5. Let $Q \in \Sigma(g(0, 0))$. Choose $m_0 \in N$ such that $\{(m, 2) : m \geq m_0\} \subset \overline{Q}$. Let $V = \{(j + p_0^{-1}, 1) : j > m_0\}$. Then $g(V) \subset \overline{Q}$ and g is weakly-continuous at $(0, 0)$. Let $Q \in \Sigma(g(1, 3))$. Choose $m_0 \in N$ such that $\{(m, 2) : m \geq m_0\} \subset \overline{Q}$. Let $V = \{(j + p_1^{-n}, 2) : j > m_0\}$. Then $g(V) \subset \overline{Q}$ and g is weakly-continuous at $(1, 3)$. Let $j \in N - \{1\}$ and let $Q \in \Sigma(g(j, 2))$ be basic open. Then $g(Q) \subset \overline{Q}$ and g is weakly-continuous at $(j, 2)$. Let $Q \in \Sigma(g(1, 2))$. Choose

$m_0 \in N$ such that $\{(m,1) : m \geq m_0\} \subset \overline{Q}$. Choose $n_0 \in N$ such that $p_k^{n_0} > m_0$ for all $k \in N$, and choose $j_0 \in N$ such that $j_0 \geq m_0$. Let $A = \{(1+p_k^{-n}, 2) : k \in N, n > n_0\}$, $B = \{(j + p_1^{-n}, 1) : j > n_0\}$ and let $V = A \cup B$. Then $g(V) \subset \overline{Q}$, so g is weakly-continuous at $(1,2)$. Let $Q \in \Sigma(g(j,1))$ where $j \in N$. Choose $n_0 \in N$ such that $p_1^{n_0} > m_0$ and let $V = \{(j + p_k^{-n}, 1) : n > n_0\}$. Then $g(V) = g(\{(j + p_1^{-n}, 1) : n > n_0\} \cup g(\{j + p_k^{-n}, 1) : n > n_0, k \neq 1\} = \{(j + p_1^{n}, 1) : n > n_0\} \cup \{(j,2)\} \subset \overline{Q}$. Hence g is weakly-continuous at $(j,1)$. Since g is obviously weakly-continuous at all other points, g is weakly-continuous.

Example 1.9. Let $\mathcal{K} = \{(0,0),(1,2)\} \cup (N \times 1) \cup \bigcup_{(j,k) \in N \times \{0,1\}} J(j,k,1)$ with the subspace topology from $\mathcal{J}$. Define $g : \mathcal{K} \to \mathcal{J}$ by $g(x) = (0,0)$ if $x \neq (1,2)$ and $g(1,2) = (1,1)$. Let $Q \in \Sigma(g(0,0))$. Choose $V \in \Sigma((0,0))$ such that $(1,2) \notin \overline{V}$. Then $g(\overline{V}) \subset \overline{Q}$. Hence g is $\theta-$continuous at $(0,0)$. Choose $Q \in \Sigma(g(0,0))$ such that $(1,1) \notin [\overline{Q}]_\theta$. Since $(1,2) \in [\overline{V}]_\theta$ for all $V \in \Sigma((0,0))$ it follows that no such V satisfies $g([\overline{V}]_\theta) \subset [\overline{Q}]_\theta$ and that g does not satisfy (2) of Theorem 1.5 at $(0,0)$.

In [20] Joseph introduced the notion of u-weakly-continuous function. This notion, similar to that of u-continuous function, was employed to give several characterizations of Urysohn-closed and minimal Urysohn spaces. A function $g : X \to Y$ is *u-weakly-continuous at* $x \in X$ if for $W \in \Lambda(g(x))$ in Y some $V \in \Sigma(x)$ satisfies $g(V) \subset \overline{W}$ and is *u-weakly-continuous* if g is u-weakly-continuous at each $x \in X$. Define $g : X \to Y$ to be *(θ, u)-continuous at* $x \in X$ if for $W \in \Lambda(g(x))$ in Y some $V \in \Sigma(x)$ satisfies $g(\overline{V}) \subset \overline{W}$ and is *(θ, u)-continuous* if g is (θ, u)-continuous at each $x \in X$. It is readily observed that weakly-continuous functions are u-weakly-continuous and that $\theta-$continuous functions are (θ, u)-continuous. The following two characterization theorems are stated without proof since the proofs are analogous to the proof of Theorem 1.1.

Theorem 1.10. *The following statements are equivalent for the spaces X, Y and $g : X \to Y$:*

(1) The function g is u-weakly-continuous.

(2) For each $x \in X$ each filterbase Ω on X satisfying $\Omega \to x$ also satisfies $g(\Omega) \underset{u}{\to} g(x)$.

(3) For each $x \in X$ each net η on X satisfying $\eta \to x$ also satisfies $g \circ \eta \underset{u}{\to} g(x)$.

(4) Each filterbase Ω on X satisfies $g(\mathcal{A}(\Omega)) \subset [g(\Omega)]_u$.

(5) Each $A \subset X$ satisfies $g(\overline{A}) \subset [g(A)]_u$.

(6) Each $B \subset Y$ satisfies $\overline{g^{-1}(B))} \subset g^{-1}([B]_u$.

(7) Each filterbase Ω on $g(X)$ satisfies $\mathcal{A}(g^{-1}(\Omega)) \subset g^{-1}([\Omega]_u)$.

(8) Each $B \subset Y$ satisfies $\overline{g^{-1}(int([B]_\theta))} \subset g^{-1}([B]_u)$.

(9) Each open $W \subset Y$ satisfies $\overline{g^{-1}(int(cl(W)))} \subset g^{-1}([W]_u)$.

(10) Each regular-closed $R \subset Y$ satisfies $\overline{g^{-1}(int(R))} \subset g^{-1}([R]_\theta)$.

(11) Each open $W \subset Y$ satisfies $\overline{g^{-1}(W)} \subset g^{-1}([W]_u)$.

Theorem 1.11. *The following statements are equivalent for the spaces X, Y and $g : X \to Y$:*

(1) The function g is (θ, u)-continuous.

(2) For each $x \in X$ each filterbase Ω on X satisfying $\Omega \underset{\theta}{\to} x$ also satisfies $g(\Omega) \underset{u}{\to} g(x)$.

(3) For each $x \in X$ each net η on X satisfying $\eta \underset{\theta}{\to} x$ also satisfies $g \circ \eta \underset{u}{\to} g(x)$.

(4) Each filterbase Ω on X satisfies $g([\Omega]_\theta) \subset [g(\Omega)]_u$.

(5) Each $A \subset X$ satisfies $g([A]_\theta) \subset [g(A)]_u$.

(6) Each $B \subset Y$ satisfies $[g^{-1}(B))]_\theta \subset g^{-1}([B]_u$.

(7) Each filterbase Ω on $g(X)$ satisfies $[g^{-1}(\Omega)]_\theta \subset g^{-1}([\Omega]_u)$.

(8) Each $B \subset Y$ satisfies $[g^{-1}(int([B]_\theta))]_\theta \subset g^{-1}([B]_u)$.

(9) Each open $W \subset Y$ satisfies $[g^{-1}(int(cl(W)))]_\theta \subset g^{-1}([W]_u)$.

(10) Each regular-closed $R \subset Y$ satisfies $[g^{-1}(int(R))]_\theta \subset g^{-1}([R]_\theta)$.

(11) Each open $W \subset Y$ satisfies $[g^{-1}(W)]_\theta \subset g^{-1}([W]_u)$.

Sufficient conditions for (θ, u)-continuity and u-weak-continuity are now given and examples are provided to show that they are not necessary. Only the proof of Theorem 1.12(1) is given since the proof of (2) is similar.

Theorem 1.12. *Let X, Y be spaces and let $g : X \to Y$.*

(1) If for each $x \in X$ and $W \in \Sigma(g(x))$ some $V \in \Sigma(x)$ satisfies $g(\overline{V}) \subset [W]_u$ then g is $(\theta$, u)-continuous.

(2) If for each $x \in X$ and $W \in \Sigma(g(x))$ some $V \in \Sigma(x)$ satisfies $g(V) \subset [W]_u$ then g is u-weakly-continuous.

Proof of (1). Let $x \in X$, $W \in \Lambda(g(x))$. Choose $Q \in \Sigma(g(x))$ satisfies $\overline{Q} \subset W$. So some $V \in \Sigma(x)$ satisfies $g(\overline{V}) \subset [Q]_u \subset \overline{W}$. □

Corollary 1.13. *Weakly-continuous functions are $(\theta$, u)-continuous.*

Proof. Let $x \in X$, $W \in \Sigma(g(x))$. Choose $V \in \Sigma(x)$ such that $g(V) \subset \overline{W}$. Then $g(\overline{V}) \subset [g(V)]_\theta \subset [W]_u$. □

While it is clear that the property (2) in Theorem 1.5 implies the hypothesis in Theorem 1.12(1) our Example 1.14 shows that they are not equivalent and that the hypothesis in Theorem 1.12(1) does not force weak-continuity.

Example 1.14. Let $\mathcal{K}$ be the space in Example 1.9 and define $g : \mathcal{K} \to \mathcal{J}$ by $g(0,0) = (1,2)$, $g(1,2) = (0,2)$ and $g(x,1) = (< x >, 2)$ otherwise. Choose $Q \in \Sigma(g(0,0))$ such that $(0,2) \notin [Q]_u$. For each $V \in \Sigma((0,0))$, $(1,2) \in [V]_u$. Thus no such V satisfies $g([V]_u) \subset [Q]_u$ and g does not satisfy property (2) in Theorem 1.5. It is immediate that g is $(\theta$, u)-continuous at each $x \in \mathcal{K} - (\{(j,1) : j \in N\} \cup \{(0,0),\ (1,2)\})$. If $Q \in \Sigma(g(j_0,1))$ then $V = \{(j_0,1)\} \cup \{(j_0 + p_k^{-n}, 1) : n \in N,\ k = 0,1\}$ is open and $g(\overline{V}) = \{(j_0,2)\} \subset Q$, so g is $(\theta$, u)-continuous at $(j_0,1)$. Let $Q \in \Sigma(g(1,2))$. Choose $j_0,\ k_0 \in N$ such that $(j + p_k^{-n}, 2) \in Q$ for all $j \geq j_0$, $k \geq k_0$ and $n \in N$. Let $V = \{(1,2)\} \cup \{(j + p_1^{-n}, 1) : j \geq j_0\}$. Then $V \in \Sigma((1,2))$ in $\mathcal{K}$ and $g(\overline{V}) \subset \overline{Q}$. Finally let $Q \in \Sigma(g(0,0))$. Choose $j_0 \in N$ such that $\{(j + p_1^{-n}, 1) : j \geq j_0\} \subset Q$. Then $(j,2) \in [Q]_u$ for all $j \in N$. It is now obvious that any basic $V \in \Sigma((0,0))$ in $\mathcal{K}$ satisfies $g(\overline{V}) \subset [Q]_u$. This completes the demostration that g satisfies the hypothesis of Theorem 1.12(1).

Example 1.15 shows that a function satisfying the hypothesis of Theorem 1.12(2) need not satisfy the hypothesis of Theorem 1.12(1).

Example 1.15. Let $\mathcal{L} = N$ with the topology $\{A \subset N : A \cap \{1,2\} = \emptyset$ or $N - A$ is finite $\}$. Define $g : \mathcal{L} \to \mathcal{J}$ by $g(1) = (0,0)$, $g(2) = (1,3)$, $g(n) = (n,2)$ otherwise. Choose $Q \in \Sigma(g(2))$ such that $(0,0) \notin [\overline{Q}]_\theta$. For each $V \in \Sigma(2)$ in $\mathcal{L}$, $1 \in \overline{V}$. Hence no such V satisfies $g(\overline{V}) \subset [Q]_u$ and g does not satisfy the hypothesis of Theorem 1.12(1). Now if $Q \in \Sigma(g(1))$, $(n,2) \in [Q]_u$ for all $n \in N$ and $N - \{2\} \in \Sigma(1) \in \mathcal{L}$ satisfies $g(N - \{2\}) \subset [Q]_u$. If $Q \in \Sigma(g(2))$ choose $n_0 \in N$ such that $(n,2) \in \overline{Q}$ for all $n \geq n_0$. Let $V = \{n \in N : n \geq$

$n_0\} \cup \{2\}$. Then $V \in \Sigma(2)$ in $\mathcal{L}$ and $g(V) \subset \overline{Q}$. Since $\mathcal{L}$ is discrete on $N - \{1, 2\}$, g satisfies the hypothesis of Theorem 1.12(2).

The function constructed in Example 1.4 is not weakly-continuous but is u-continuous and afortiori (θ, u)-continuous and u-weakly-continuous. The function in Example 1.7 is (θ, u)-continuous but fails to be u-continuous. A function which is u-weakly-continuous but not (θ, u)-continuous is provided by Example 1.16.

Example 1.16. Let $X = Y = (0, \infty)$ and let $\{D(k) : k = 1, 2, 3, 4, 5, 6\}$ be a partition of X into subsets dense in the usual topology. Let $\mathcal{B}$ be a base of open intervals for the usual topology on X with endpoints in $D(3)$. First assume $N \subset D(5)$ and define $V \subset X$ to be open if V satisfies the properties: If $x \in V \cap D(k)$, $k = 1, 3, 5$ there is a $B \in \mathcal{B}$ satisfying $x \in B \cap D(k) \subset V$; if $x \in V \cap D(k)$, $k = 2, 4$ there is a $B \in \mathcal{B}$ satisfying $x \in B \cap (D(k-1) \cup D(k) \cup D(k+1)) \subset V$; if $x \in V \cap D(6)$ some $B \in \mathcal{B}$ satisfies $x \in B \cap (D(5) \cup D(6)) \subset V$. Now assume $N \subset D(1)$ and define $W \subset Y$ to be open if W satisfies the following properties: If $x \in D(1)$ some $B \in \mathcal{B}$ satisfies $x \in B \cap (D(1) \cup D(2)) \subset W$; if $x \in W \cap D(k)$, $k = 2, 4, 6$ there is a $B \in \mathcal{B}$ satisfying $x \in B \cap D(k) \subset W$; if $x \in W \cap D(k)$, $k = 3, 5$ there is a $B \in \mathcal{B}$ satisfying $x \in B \cap (D(k-1) \cup D(k) \cup D(k+1)) \subset W$. The identity function from X to Y is u-weakly-continuous but not (θ, u)-continuous.

The function g defined in Example 1.3 fails to be (θ, u)-continuous at 0 and it is obvious that continuity at a point implies weak-continuity at the point, which in turn implies u-weak-continuity at the point. It is also evident that $\theta-$continuity at a point implies (θ, u)-continuity at the point. The following two implication diagrams summarize relationships between the various weak continuity forms which we have discussed. None of the implications reverse. The first diagram gives the implications when the property is satisfied globally while the second deals with the situation when the property is satisfied at a point but not necessarily at all points.

Diagram 1.17.

<u>ON THE SPACE</u>

$$
\begin{tikzcd}
 & \text{continuity} \arrow[dl] \arrow[dr] & \\
\text{u-continuity} \arrow[d] & & \theta-\text{continuity} \arrow[d] \\
(\theta,\text{u})\text{-continuity} \arrow[d] & & \text{weak-continuity} \arrow[ll] \\
\text{u-weak-continuity} & &
\end{tikzcd}
$$

Diagram 1.18.

<u>AT A POINT</u>

$$
\begin{tikzcd}
\text{continuity} \arrow[dr] & & \theta-\text{continuity} \arrow[dl] \arrow[dr] & & \text{u-continuity} \arrow[dl] \\
 & \text{weak-continuity} \arrow[d] & & (\theta,\ \text{u})\text{-continuity} & \\
 & \text{u-weak-continuity} & & &
\end{tikzcd}
$$

It is known that the operator $[\ \ ]_u$ need not satisfiy $[\,[A]_u]_u = [A]_u$ for all $A \subset X$ [8] although it may be readily shown that the operator satisfies the other Kuratowski closure axioms. It will be shown that when $[\ \ ]_u$ is a Kuratowski closure operator many of the generalizations in this paper may be studied in terms of continuity with appropriate changes of topologies. Conditions which are equivalent to $[\ \ ]_\theta$ being a Kuratowski closure operator are known [10], [16]. Such conditions for $[\ \ ]_u$ will now be produced. To this end, on a space consider several subcollections of the closed subsets of a topology $\mathcal{T}$ on X. Let $\mathcal{T}_s$ be the topology on X generated by the regular-closed subsets. This topology known as the semiregularization of the topology on X has played an important role in the literature, for instance in the study of minimal Hausdorff spaces. Let $\mathcal{T}_\theta$ denote the topology generated by the $\theta-$closed subsets. In [16] a space X is *almost regular* if for each regular-closed subset F and $x \in X - F$ there are disjoint subsets $A \in \Sigma(F)$ and $B \in \Sigma(x)$. Herrman [16] has noted that a space X is almost regular if and only if $\mathcal{T}_\theta = \mathcal{T}_s$ or equivalently, $[\ \ ]_\theta$ is a Kuratowski closure operator. Hamlett [10] has also established the latter equivalence. Since $[A]_u = [\overline{A}]_\theta$ for open A it follows that $[A]_u = [A]_\theta$ for all $A \subset X$ when $[\ \ ]_\theta$ is a Kuratowski closure operator, so in this case $[\ \ ]_u$ is such an operator. Example 1.19 exhibits that $[\ \ ]_u$ may be a Kuratowski closure operator when $[\ \ ]_\theta$ is not.

Example 1.19. Let $\mathcal{H}$ be a space in Example 1.4. It is known that $\mathcal{H}$ is Urysohn-closed, semiregular and not compact [15]. It follows from the characterization above that $[\ \ ]_\theta$ is a Kuratowski closure operator on a semiregular space if and only if the space is regular. Hence $[\ \ ]_\theta$ is not such an operator on $\mathcal{H}$. To show that $[\ \ ]_u$ is a Kuratowski closure operator it will be enough to prove that $[\overline{V}]_u \subset [\overline{V}]_\theta$ for all V open in $\mathcal{H}$. Let V be open in $\mathcal{H}$ and let $x \in D(1) - [\overline{V}]_\theta$. Choose an interval (a, b) such that $x \in A = (a, b) \cap D(1)$ and $\overline{A} \cap \overline{V} = \emptyset$. If $[A]_u \cap \overline{V} = \emptyset$ it is easy to see that $x \in X - [\overline{V}]_u$. Since $\overline{A} = [a, b] - D(2)$, it will suffice to show that $\overline{V} \cap D(2) \cap [a, b] = \emptyset$. Assume the contrary and let $y \in \overline{V} \cap D(2) \cap [a, b]$. Since $y \in \overline{V} \cap D(2) \cap (a, b)$, choose an open interval (c, d) such that $y \in (c, d) \cap D(2) \cap (a, b)$ and $z \in (c, d) \cap D(2) \subset V$. Then there exists an interval (e, f) with $z \in (e, f) \cap D(2) \subset V$ and $(e, f) \subset [a, b]$. Then $\overline{(c, d) \cap D(2)} = [e, f] - D(1) \subset V$, which contradicts $([a, b] - D(2)) \cap \overline{V} = \emptyset$. Thus, for V open in $\mathcal{H}$, if $x \in D(1) - [\overline{V}]_\theta$ then $x \in X - [\overline{V}]_u$. The proof for $x \in D(2) - [\overline{V}]_\theta$ is similar and the proof for $x \in D(3) - [\overline{V}]_\theta$ is clear. The fact that $[\ \ ]_u$ is a Kuratowski closure operator will now follow from Theorem 1.21 below.

Now let $\mathcal{T}_k$ denote the topology on X generated by the $\theta-$closures of regular-closed subsets and let $\mathcal{T}_u$ denote the topology generated by the u-closed subsets. The topology $\mathcal{T}_k$ plays a role for $\mathcal{T}_u$ similar to that of $\mathcal{T}_s$ for $\mathcal{T}_\theta$.

Lemma 1.20. *For a space X,*

(1) The topology $\mathcal{T}_u \subset T_k$.

(2) If a net $\eta \underset{\mathcal{T}_k}{\rightarrow} x$ then $\eta \underset{u}{\rightarrow} x$.

(3) If a filterbase $\Omega \underset{\mathcal{T}_k}{\rightarrow} x$ then $\Omega \underset{u}{\rightarrow} x$.

Proof. Let A be a basic $\mathcal{T}_u-$closed subset of X and suppose $x \in X - A$. Choose $V \in \Sigma(A)$ such that $x \notin [\overline{V}]_\theta$, a basic $\mathcal{T}_k-$closed subset of X and the proof of (1) is complete. For the verification of (2) assume $\eta \underset{\mathcal{T}_k}{\rightarrow} x$ and let $B \in \Lambda(x)$. Choose $A \in \Sigma(x)$ satisfying $x \in A \subset X - [X - B]_\theta \subset B$. If B is regular-open in X then η is ultimately in $X - [X - B]_\theta$. Hence, for any $B \in \Lambda(x)$, η is ultimately in $\overline{B} = \overline{(int(B))}$. The proof of (3) is omitted. □

Theorem 1.21. *The following statements are equivalent for a space X.*

(1) For all regular-closed subsets $A \subset X$, $[A]_\theta = [A]_u$.

(2) For all $A \subset X,\ [\ [A]_u]_u = [A]_u$.

(3) For all regular-closed $A \subset X,\ [\ [A]_u]_u = [A]_u$.

(4) The topologies $\mathcal{T}_k$ *and* $\mathcal{T}_u$ *are the same.*

(5) If $A \subset X$ *is regular-closed in* X *and* $x \in X - [A]_\theta$, *then there exist* $P \in \Sigma(x),\ Q \in \Sigma([A]_\theta)$ *such that* $\overline{P} \cap \overline{Q} = \emptyset$.

Proof. $(\mathbf{1}) \Rightarrow (\mathbf{2})$. If $A \subset X$ and $x \notin [A]_u$ choose $V \in \Sigma(A)$ such that $x \notin [\overline{V}]_u$. There exists $B \in \Lambda(\overline{V})$ with $x \notin \overline{B}$. Repeated application of (1) yields $[\ [A]_u]_u \subset \overline{B}$ and therefore $[\ [A]_u]_u = [A]_u$.

$(\mathbf{2}) \Rightarrow (\mathbf{3})$. For any regular-closed $A \subset X,\ [\ [A]_u]_u = [A]_u$ since $[V]_u = [\overline{V}]_\theta$ for open V.

$(\mathbf{3}) \Rightarrow (\mathbf{4})$. Since $\mathcal{T}_u \subset \mathcal{T}_k$ always, the proof is completed by noting that (3) forces $\mathcal{T}_k \subset T_u$.

$(\mathbf{4}) \Rightarrow (\mathbf{5})$. Suppose A is a regular-closed subset of X with $x \in X - [A]_\theta$. Since $\mathcal{T}_k = \mathcal{T}_u$ choose F such that $[A]_\theta \subset F = [F]_u,\ x \notin F$. It follows that there exist $P \in \Sigma(x),\ Q \in \Lambda(x)$ satisfying $x \in P,\ [A]_\theta \subset X - \overline{Q},\ \overline{P} \cap \overline{X - \overline{Q}} = \emptyset$.

$(\mathbf{5}) \Rightarrow (\mathbf{1})$. From (5) $[A]_u \subset [A]_\theta$ for regular-closed A. □

Corollary 1.22 follows immediately from Lemma 1.20 and Theorem 1.21.

Corollary 1.22. *For a space* X, *if* $[\ \ ]_u$ *is a Kuratowski closure operator then whenever a net* $\eta \underset{u}{\rightarrow} x,\ \eta$ *is ultimately in each* $W \in \Lambda(x)$.

Theorem 1.23 follows easily from Theorem 1.1(6) and the fact that $\mathcal{T}_u \subset \mathcal{T}_\theta \subset \mathcal{T}$.

Theorem 1.23. *Let* $g : (X, \mathcal{T}) \to (Y, \mathcal{Q})$ *be u-continuous. Then*

(1) $g : (X, \mathcal{T}_u) \to (Y, \mathcal{Q}_u)$ *is continuous,*

(2) $g : (X, \mathcal{T}_\theta) \to (Y, \mathcal{Q}_u)$ *is continuous, and*

(3) $g : (X, \mathcal{T}) \to (Y, \mathcal{Q}_u)$ *is continuous.*

Theorems 1.24 and 1.25 are consequences of Theorems 1.10(6) and 1.11(6), respectively, and the fact that $\mathcal{T}_\theta \subset \mathcal{T}$.

Theorem 1.24. *If* $g : (X, \mathcal{T}) \to (Y, \mathcal{Q})$ *is u-weakly-continuous then* $g : (X, \mathcal{T}) \to (Y, \mathcal{Q}_u)$ *is continuous.*

Theorem 1.25. *If* $g : (X, \mathcal{T}) \to (Y, \mathcal{Q})$ *is* $(\theta,$ *u)-continuous then*

(1) $g : (X, \mathcal{T}_\theta) \to (Y, \mathcal{Q}_u)$ *is continuous, and*

(2) $g : (X, \mathcal{T}) \to (Y, \mathcal{Q}_u)$ *is continuous.*

When $[\ \ ]_u$ is a Kuratowski closure operator converses of some of the above theorems may be obtained.

Theorem 1.26. *Let* $(X, \mathcal{T}),\ (Y, \mathcal{Q})$ *be spaces and suppose the operator* $[\ \ ]_u$ *is a Kuratowski closure operator. Then* $g : (X, \mathcal{T}) \to (Y, \mathcal{Q})$ *is u-continuous if* $g : (X, \mathcal{T}_u) \to (Y, \mathcal{Q}_u)$ *is continuous*

Proof. If $A \subset Y$ then $[A]_u$ is u-closed and $g^{-1}([A]_u)$ is u-closed. It follows that $[g^{-1}(A)]_u \subset g^{-1}([A]_u)$ and g is u-continuous by Theorem 1.1(6). □

The next two results are given without proof.

Theorem 1.27. *Let* $(X, \mathcal{T}),\ (Y, \mathcal{Q})$ *be spaces and suppose the operator* $[\ \ ]_u$ *induced by* $\mathcal{Q}$ *is a Kuratowski closure operator. Then* $g : (X, \mathcal{T}) \to (Y, \mathcal{Q})$ *is u-weakly-continuous if* $g : (X, \mathcal{T}) \to (Y, \mathcal{Q}_u)$ *is continuous*

Theorem 1.28. *Let $(X,\mathcal{T})$, $(Y,\mathcal{Q})$ be spaces and suppose the operator $[\ \ ]_u$ induced by $\mathcal{Q}$ is a Kuratowski closure operator. Then $g : (X,\mathcal{T}) \to (Y,\mathcal{Q})$ is $(\theta,\ u)$-continuous if $g : (X,\mathcal{T}_\theta) \to (Y,\mathcal{Q}_u)$ is continuous*

Example 1.29 presents spaces $(X,\mathcal{T})$, $(Y,\mathcal{Q})$ and a function g such that $g : (X,\mathcal{T}_u) \to (Y,\mathcal{Q}_u)$ is continuous while $g : (X,\mathcal{T}) \to (Y,\mathcal{Q})$ is not u-continuous.

Example 1.29. Let $\mathcal{K}$, $\mathcal{J}$, g be the spaces and function in Example 1.14. Choose $Q \in \Lambda(g(0,0))$ such that $(0,2) \notin \overline{Q}$. For each $V \in \Lambda(0,0)$, $(1,2) \notin \overline{V}$, so no such V satisfies $g(\overline{V}) \subset \overline{Q}$. Hence g is not u-continuous at $(0,0)$. Now let A be u-closed in $\mathcal{J}$ and let $(x,y) \in [g^{-1}(A)]_u$. If $(x,y) \in \mathcal{K} - (\{(0,0),\ (1,2)\} \cup \{(n,1) : n \in N\})$ it is easily seen that $(x,y) \in g^{-1}(A)$. If $(x,y) = (0,0)$ then $J(j,0,1) \subset g^{-1}(A)$ for infinitely many j, so $(j,2) \in A$ for infinitely many j and $(1,2) \in [A]_u = A$. Hence $(0,0) \in g^{-1}(A)$. If $(1,2) \in [g^{-1}(A)]_u$ we see that $(j,2) \in A$ for infinitely many j so that $(0,2) \in [A]_u = A$ and $(1,2) \in g^{-1}(A)$. Finally if $n \in N$ and $(n,1) \in [g^{-1}(A)]_u$ then $(n,1) \in \overline{g^{-1}(A)} = g^{-1}(A)$. Therefore $g^{-1}(A)$ is u-closed.

Example 1.30 exhibits spaces $(X,\mathcal{T})$, $(Y,\mathcal{Q})$ and a function g such that $g : (X,\mathcal{T}_\theta) \to (Y,\mathcal{Q}_u)$ is continuous while $g : (X,\mathcal{T}_u) \to (Y,\mathcal{Q}_u)$ is not continuous.

Example 1.30. Let $\mathcal{K}$, g be the space and function in Example 1.8. Since g is weakly-continuous it follows from Corollary 1.13 and Theorem 1.25(1) that $g^{-1}(A)$ is $\theta-$closed whenever A is u-closed. However $A = \{(1,3)\}$ is u-closed in $\mathcal{K}$ but $g^{-1}(A) = \{(1,3),\ (0,0)\}$ is not u-closed.

Example 1.31 produces spaces $(X,\mathcal{T})$, $(Y,\mathcal{Q})$ and a function g such that $g : (X,\mathcal{T}) \to (Y,\mathcal{Q}_u)$ is continuous while $g : (X,\mathcal{T}_\theta) \to (Y,\mathcal{Q}_u)$ is not continuous.

Example 1.31. Let $\mathcal{L}$, g be the space and function defined in Example 1.15. Since g is u-weakly-continuous it follows from Theorem 1.10 that $g^{-1}(A)$ is closed in $\mathcal{L}$ when A is u-closed in $\mathcal{J}$. On the other hand, $A = (W \times (N - \{1,2\})) \cup \{(0,2)\}$ is u-closed in $\mathcal{J}$ and $g^{-1}(A) = \{2\}$, which is not $\theta-$closed in $\mathcal{L}$.

Example 1.32. Spaces $(X,\mathcal{T})$, $(Y,\mathcal{Q})$ and a function g are given such that $g : (X,\mathcal{T}) \to (Y,\mathcal{Q}_\theta)$ is continuous while $g : (X,\mathcal{T}) \to (Y,\mathcal{Q})$ is not u-weakly-continuous. Let $g : \mathcal{J} \to \mathcal{J}$ be defined by $g(0,0) = (1,0) = g(1,0)$, $g(0,1) = (0,3)$, $g(x,m) = (< x >, m+1)$ otherwise. Let $A = \{(x,1) : (x,1) \in \mathcal{J}\} - (W \times \{0,1\})$. Then $(0,1) \in \overline{A}$ but $(0,3) \notin [g(A)]_u$, so g is not u-weakly-continuous. Let A be $\theta-$closed in $\mathcal{J}$ and let $(x,y) \in [g^{-1}(A)]_\theta$. If $(x,y) = (1,0)$ then $A \cap \{(j,m) : j \in W,\ m \geq m_0\} \neq \emptyset$ is satisfied for all $m_0 \in N$. So $(1,0) \in [A]_\theta = A$ and $(1,0) \in g^{-1}(A)$. If $(x,y) = (0,0)$ then $A \cap \{(j,2) : j \geq j_0\} \neq \emptyset$ is satisfied for all $j_0 \in N$. Hence $(1,0) \in [A]_\theta = A$ and $(0,0) \in g^{-1}(A)$. Suppose $x = 0$ and $y \in N$; then $(0,m) \in A$ for all $m > y$ and $(0,y) \in g^{-1}(A)$. Now assume $x, y \in N$. If $g^{-1}(A) \cap \overline{\bigcup_W J(x,k,y)} \neq \emptyset$ then $(x,y+1) \in A$ and $(x,y) \in g^{-1}(A)$; if $g^{-1}(A) \cap \overline{\bigcup_{j \geq j_0} J(j,k,y-1)} \neq \emptyset$ is satisfied for each $j_0 \in N$ then $A \cap \{(j,y) : j \geq j_0\} \neq \emptyset$ is satisfied for each $j_0 \in N$. Hence $(x,y+1) \in [A]_\theta = A$ and $(x,y) \in g^{-1}(A)$. Clearly if $(x,y) \notin (W \times N) \cup \{(0,0),\ (1,0)\}$ then $(x,y) \in g^{-1}(A)$. Thus if $(X,\mathcal{T}) = \mathcal{J}$, $g : (X,\mathcal{T}_\theta) \to (X,\mathcal{T}_\theta)$ is continuous.

2. THE GRAPH CONDITIONS. In this section the operators $[\ \ ]_\theta$ and $[\ \ ]_u$ are used to define conditions on graphs and to establish results which will be used in §3 and §4. If X is a space let $\mathcal{S}(x) = \{V - \{x\} : V \in \Sigma(x)\}$. If X, Y are spaces, a set-valued function $F \subset X \times Y$ is *subclosed (strongly-subclosed) [u-strongly- subclosed]* if

$$\mathcal{A}(F(\mathcal{S}(x)))([F(\mathcal{S}(x))]_\theta)[\ [F(\mathcal{S}(x))]_u] \subset F(x)$$

for each $x \in X$ for which $\mathcal{S}(x)$ is a filterbase on X; the set-valued function F is *closed (strongly-closed) [u-strongly- closed]* if $\mathcal{A}(F(\Sigma(x)))([F(\Sigma(x))]_\theta)[\,[F(\Sigma(x))]_u] \subset F(x)$ for each $x \in X$. It can be established that for spaces X, Y, $F \subset X \times Y$ is closed (strongly-closed) [u-strongly-closed] if F is subclosed (strongly-subclosed) [u-strongly-subclosed] and $F(x)$ is closed ($\theta-$closed) [u-closed] in Y for each $x \in X$ (see [20]). A set-valued function $F \subset X \times Y$ with domain X is said to be *upper semi-continuous at* $x \in X$ if for each $W \in \Sigma(F(x))$ there exists $V \in \Sigma(x)$ such that $F(V) \subset W$, and is *upper semi-continuous* if it is upper semi-continuous at each $x \in X$. If X is a space and $x \in X$ let $\mathcal{G}(x) = \{A - \{x\} : A \in \Gamma(x)\}$ and $\mathcal{L}(x) = \{\overline{A} - \{x\} : A \in \Lambda(x)\}$, and, for a space Y, call $F \subset X \times Y$ *(θ, u)-subclosed (u-subclosed)* if $[F(\mathcal{G}(x))]_u$ $([F(\mathcal{L}(x))]_u) \subset F(x)$ for each $x \in X$ for which $\mathcal{G}(x)$ $(\mathcal{L}(x))$ is a filterbase on X. If X, Y are spaces and $g : X \to Y$ is a function, P is one of the properties defined above, and g satisfies property P, the the terminology "g has a P graph" will be used. It will be said that "g has a P inverse" if $g^{-1} \subset g(X) \times X$ has property P. Proposition 2.1 (Proposition 2.4) for $\theta-$continuous (u-continuous) functions into Hausdorff (Urysohn) spaces enables us to see immediately that such functions have strongly-closed (u-strongly-closed) inverses. Only the proof of Proposition 2.4 is given.

Proposition 2.1. *A function $g : X \to Y$ is $\theta-$continuous if and only if*

$$\bigcap_{\Sigma(A)} [g^{-1}(V)]_\theta \subset g^{-1}([A]_\theta)$$

for each $A \subset Y$.

It is easy to see from Proposition 2.1 that

$$\bigcap_{\Sigma(A)} [g^{-1}(V)]_\theta = g^{-1}(A)$$

for each $\theta-$closed $A \subset Y$. In particular we have the next corollary.

Corollary 2.2. *If Y is a Hausdorff space and $g : X \to Y$ is $\theta-$continuous then*

$$\bigcap_{\Sigma(y)} [g^{-1}(V)]_\theta = g^{-1}(y)$$

for each $y \in Y$.

Corollary 2.3. *If Y is Hausdorff and $g : X \to Y$ is $\theta-$continuous then g has a strongly-closed inverse.*

Proposition 2.4. *A function $g : X \to Y$ is u-continuous if and only if*

$$\bigcap_{\Sigma(A)} [g^{-1}(V)]_u \subset g^{-1}([A]_u)$$

for each $A \subset Y$.

Proof. The sufficiency comes by applying equivalence (6) of Theorem 1.1 since

$$[g^{-1}(A)]_u \subset \bigcap_{\Sigma(A)} [g^{-1}(V)]_u \subset g^{-1}([A]_u)$$

for each $A \subset Y$. The necessity is also straightforward after we observe from equivalence (11) of Theorem 1.1 that

$$\bigcap_{\Sigma(A)} [g^{-1}(V)]_u \subset \bigcap_{\Sigma(A)} g^{-1}([V]_u) \subset g^{-1}([A]_u)$$

for each $A \subset Y$. $\square$

Corollary 2.5. *If Y is a Urysohn space and $g : X \to Y$ is u-continuous then*

$$\bigcap_{\Sigma(y)} [g^{-1}(V)]_u = g^{-1}(y)$$

for each $y \in Y$.

Corollary 2.6. *If Y is Urysohn and $g : X \to Y$ is u-continuous then g has a u-strongly-closed inverse.*

Corollary 2.7. *If Y is Hausdorff (Urysohn) and $g : X \to Y$ is continuous then g has a strongly-closed (u-strongly-closed) inverse.*

A subset A of a space X is called *Quasi H-closed (QHC) relative to the space* X if each filterbase Ω on A satisfies $A \cap [\Omega]_\theta \neq \emptyset$. For Hausdorff spaces these are the H-sets of Veličko [26]. Subsets of a similar type for Urysohn spaces will be utilized in the sequel.

Definition 2.8. A subset A of a space X is *Quasi Urysohn-closed (QUC) relative to the space* X if each filterbase Ω on A satisfies $A \cap [\Omega]_u \neq \emptyset$.

Theorems 2.9–2.11 will be applied in later sections. Only 2.11 is proved since the proofs of the other two are similar to the proof of 2.11.

Theorem 2.9. *Let X, Y be spaces and let $g : X \to Y$ have a u-strongly-subclosed graph. Then $g^{-1}(A)$ is closed in X for each QUC relative to Y subset A.*

Theorem 2.10. *Let X, Y be spaces and let $g : X \to Y$ have a $(\theta$,u)-subclosed graph. Then $g^{-1}(A)$ is $\theta-$closed in X for each QUC relative to Y subset A.*

Theorem 2.11. *Let X, Y be spaces and let $g : X \to Y$ have a u-subclosed graph. Then $g^{-1}(A)$ is u-closed in X for each QUC relative to Y subset A.*

Proof. For $v \in [g^{-1}(A) - \{v\}]_u$, there is a filterbase Ω on $g^{-1}(A) - \{v\}$ such that $\Omega \underset{u}{\to} v$. Then $g(\Omega)$ is a filterbase on A and, since g has a u-subclosed graph, it follows that $\emptyset \neq A \cap [g(\Omega)]_u \subset \{g(v)\}$ and $v \in g^{-1}(A)$. $\square$

Theorems 2.12–2.14 will also be instrumental in the next sections. Only 2.14 is proved.

Theorem 2.12. *Let X, Y be spaces and let A be QUC relative to Y. If $g : X \to Y$ has a u-strongly-subclosed graph and $g(X) \subset A$ then g is u-weakly-continuous.*

Theorem 2.13. *Let X, Y be spaces and let A be QUC relative to Y. If $g : X \to Y$ has a $(\theta$, u)-subclosed graph and $g(X) \subset A$ then g is $(\theta$,u)-continuous.*

Theorem 2.14. *Let X, Y be spaces and let A be QUC relative to Y. If $g : X \to Y$ has a u-subclosed graph and $g(X) \subset A$ then g is u-continuous.*

Proof. Let $v \in X$. If $\overline{V} = \{v\}$ for some $V \in \Lambda(v)$ then g is u-continuous at v. Otherwise $\Omega = \{\overline{V} - \{v\} : V \in \Lambda(v)\}$ is a filterbase on X, and $[g(\Omega)]_u \subset \{g(v)\}$ since g has a u-subclosed graph. Let $W \in \Lambda(g(v))$. There exists $F \in \Omega$ such that $g(F) \subset \overline{W}$. If not then $\Upsilon = \{g(F) - \overline{W} : F \in \Omega\}$ is a filterbase on A, and $g(v) \notin [\Upsilon]_u \subset [g(\Omega)]_u$. $\square$

3. The weak continuity forms, graph conditions, old and new compactness generalizations. In this section the results in §1 and §2 are applied to produce several characterizations of QUC spaces analogous to the original definition and well-known characterizations of Urysohn-closed spaces. In addition analogues of theorems for functionally compact and C-compact spaces are provided for Urysohn-closed spaces.

Theorem 3.1. *The following statements are equivalent for a space X.*

(1) The space X is QUC.

(2)The inclusion $\mathcal{A}(g(\Omega)) \subset \bigcup_{[\Omega]_u}[g(x)]_u$ is satisfied for all u-continuous functions g and filterbases Ω on X.

(3) All u-continuous functions g on X and $A \subset X$ satisfy $\overline{g(A)} \subset \bigcup_{[A]_u}[g(x)]_u$.

(4) Same as (3) except that A is open.

(5) Same as (3) except that $A = X$ and the conclusion is $\overline{g(X)} \subset \bigcup_X [g(x)]_u$.

(6) The inclusion $\overline{X} \subset \bigcup_X [x]_u$ holds in any space which contains X as a subspace.

(7) The equality $Y = \bigcup_X [x]_u$ holds in any space Y which contains X as a dense subspace.

(8) The equality $Y = \bigcup_X [x]_u$ holds in any space Y which contains X as a dense open subspace.

(9) Each filterbbase on X with at most one u-adherent point u-converges.

Proof. $(\mathbf{2}) \Rightarrow (\mathbf{3}) \Rightarrow (\mathbf{4}) \Rightarrow (\mathbf{5}) \Rightarrow (\mathbf{6}) \Rightarrow (\mathbf{7}) \Rightarrow (\mathbf{8})$, and $(\mathbf{9}) \Rightarrow (\mathbf{1})$ are all obvious.

$(\mathbf{1}) \Rightarrow (\mathbf{2})$. Let Y be a space, $g : X \to Y$ be u-continuous, let Ω be a filterbase on X and $y \in \mathcal{A}(g(\Omega))$. Then $\Omega_1 = \{g^{-1}(V) \cap F : V \in \Sigma(y),\ F \in \Omega\}$ is a filterbase on X. Since g is u-continuous it follows that $g([\Omega_1]_u) \subset [g(\Omega_1)]_u$. Since $\emptyset \neq g([\Omega_1]_u) \subset [\Sigma(y)]_u = [y]_u$, there exists $x \in [\Omega]_u$ satisfying $g(x) \in [y]_u$; this implies $y \in [g(x)]_u$ and hence $y \in \bigcup_{[\Omega]_u}[g(x)]_u$ and (2) holds.

$(\mathbf{8}) \Rightarrow (\mathbf{9})$. Let Ω be a filterbase on X with at most one u-adherent point, say x_0. If $\Omega \underset{u}{\not\to} x_0$, then there is a $W \in \Lambda(x_0)$ such that $\Omega_1 = \{V - \overline{W} : V \in \bigcup_\Omega \Sigma(F)\}$ is an open filterbase on X. Choose $\infty \notin X$ and let $Y = X \cup \{\infty\}$ endowed with the topology generated by the open set base composed of the topology of X and sets of the form $H \cup \{\infty\}$, where $H \in \Omega_1$. Then X is a dense open subspace of Y. Hence $Y = \bigcup_X [x]_u$ and there is an $x \in X$ such that $\infty \in [x]_u$ in Y. This implies that $x \in [\infty]_u$ and that $x = x_0$. A contradiction is reached since $\Lambda(x_0)$ in Y is the same as $\Lambda(x_0)$ in X and $\overline{W} \cap (H \cup \{\infty\}) = \emptyset$ for all $H \in \Omega_1$. Hence (9) holds. □

Corollary 3.2 offers additional chacterizations of QUC spaces and Corollary 3.3 offers new characterizations of Urysohn-closed spaces.

Corollary 3.2. *A space X is QUC if an only if any of the statements obtained by replacing " u-continuous " by " continuous " in Theorem 3.1(2), (3), (4), or (5) holds.*

Corollary 3.3. *The following statements are equivalent for a Urysohn space X.*

(1) The space X is Urysohn-closed.

(2) The inclusion $\mathcal{A}(g(\Omega)) \subset g([\Omega]_u)$ is satisfied for all u-continuous functions g and filterbases Ω on X.

(3) All u-continuous functions g on X and $A \subset X$ satisfy the inclusion $\overline{g(A)} \subset g([A]_u)$.

(4) Same as (3) except that A is open.

(5) Each u-continuous (continuous) function g from X to a Urysohn space maps u-closed subsets onto closed subsets.

(6) Each u-closed subset of X is closed in any Urysohn space in which X is embedded.

(7) Each filterbbase on X with at most one u-adherent point u-converges.

Theorem 3.4 is an improvement of the result of Herrlich [15] that the continuous image of a Urysohn-closed space is Urysohn-closed.

Theorem 3.4. *The u-continuous image of a QUC space is QUC.*

Proof. Let X be QUC and let $g : X \to Y$, $f : g(X) \to Z$ be u-continuous. Then $f \circ g$ is u-continuous. Hence, since X is QUC, $\overline{f(g(X))} = \overline{f \circ g(X)} \subset \bigcup_X [f \circ g(x)]_u = \bigcup_{g(X)} [f(y)]_u$. □

Definition 3.5. A function g from a subset A of a space X to a space Y will be called *u-continuous on A relative to X* if $g(A \cap [\Omega]_u) \subset [g(\Omega)]_u$ for each filterbase Ω on A.

It is readily seen that if g is u-continuous on X then g is u-continuous on A relative to X for each $A \subset X$, and that g is u-continuous on X relative to X if an only if g is u-continuous. Previous proofs may be modified to produce proofs of the next two theorems,

Theorem 3.6. *The following statements are equivalent for a a subset A of a space X.*

(1) The subset A is QUC relative to X.

(2) The inclusion $\mathcal{A}(g(\Omega)) \subset \bigcup_{A \cap [\Omega]_u} [g(x)]_u$ is satisfied for all u-continuous relative to X functions g on A and filterbases Ω on A.

(3) All u-continuous relative to X functions g on A and $Q \subset A$ satisfy the inclusion $\overline{g(Q)} \subset \bigcup_{A \cap [Q]_u} [g(x)]_u$.

(4) Same as (3) except that Q is open in A.

(5) Same as (3) except that $A = Q$ and the conclusion is $\overline{g(A)} \subset \bigcup_A [g(x)]_u$.

(6) Each filterbbase on A with at most one u-adherent point u-converges to a point in A.

Theorem 3.7. *The following statements are equivalent for a a subset A of a Urysohn space X.*

(1) The subset A is QUC relative to X.

(2) The inclusion $\mathcal{A}(g(\Omega)) \subset g(A \cap [\Omega]_u)$ is satisfied for all u-continuous relative to X functions g on A and filterbases Ω on A.

(3) All u-continuous relative to X functions g on A and $Q \subset A$ satisfy the inclusion $\overline{g(Q)} \subset g(A \cap [Q]_u)$.

(4) Same as (3) except that Q is open in A.

(5) Same as (3) except that $A = Q$ and the conclusion is $\overline{g(A)} \subset g(A)$.

(6) If $g(A)$ is a u-continuous on A relative to X image of A in a Urysohn space, then $g(A)$ is closed.

Corollary 3.8. *A QUC relative subset to X subset of a Urysohn space X is closed in X.*

It is not difficult to prove that if $A \subset X$, and $g : X \to Y$ is u-continuous on A relative to X, and $f : g(A) \to Z$ is u-continuous on $g(A)$ relative to Y then $f \circ g$ is u-continuous on A relative to X. Hence the next result.

Corollary 3.9. *If A is QUC relative to X and $g : X \to Y$ is u-continuous on A relative to X then $g(A)$ is QUC relative to Y.*

The following three corollaries are consequences of the results above.

Corollary 3.10. *Let Y be a Urysohn space and let $g : X \to Y$ be u-continuous. If A is QUC relative to X then $g(\overline{A}) = g(A)$.*

Proof. It is shown in [8] that $\overline{A} \subset \bigcup_A [x]_u$ in X. Thus $g(\overline{A}) \subset \bigcup_A [g(x)]_u$ and since Y is Urysohn, the set on the right side of the inclusion is $g(A)$. $\square$

A subset of a space is called *u-rigid* in [8] if $A \cap [\Omega]_u \neq \emptyset$ is satisfied for each filterbase Ω on X for which $F \cap \overline{V} \neq \emptyset$ is satisfied for each $V \in \Lambda(A)$, $F \in \Omega$. It is shown in [8] that $[A]_u = \bigcup_A [x]_u$ for each u-rigid subset A of the space. This equality leads to Corollary 3.11.

Corollary 3.11. *Let Y be a Urysohn space and let $g : X \to Y$ be u-continuous. Then $g([A]_u) = g(A)$ for each u-rigid subset A of X.*

Recall that a Hausdorff space is *functionally compact* if each open filterbase on the space satisfying $\mathcal{A}(\Omega) = \mathcal{I}(\Omega)$ is an open set base for $\mathcal{I}(\Omega)$. Theorem 3.12, a characterization theorem for functionally compact spaces, is proved in [21].

Theorem 3.12. *The following statements are equivalent for a space X :*

(1) The space X is functionally compact.

(2) Each filterbase Ω on X satisfying $[\Omega]_\theta = \mathcal{I}(\Omega)$ converges to $\mathcal{I}(\Omega)$.

(3) For each Hausdorff space Y each strongly-closed set-valued function $\lambda \subset Y \times X$ is u. s. c.

(4) For each Hausdorff space Y each function from Y to X with a strongly-closed inverse is a closed function.

Corollary 3.13. *[3] If a space X is functionally compact and Y is Hausdorff and $g : X \to Y$ is $\theta-$continuous then g is a closed function.*

Proof. From Theorem 3.12(3) and Corollary 3.7. $\square$

For Urysohn spaces the following parallel of the class of functionally compact spaces is given.

Definition 3.14. A Urysohn space is *u-functionally compact* if each Urysohn filterbase Ω on the space which satisfies $\mathcal{A}(\Omega) = \mathcal{I}(\Omega)$ is an open set base for $\mathcal{I}(\Omega)$.

Herrington [12] showed that a Urysohn filterbase Ω on a space satisfies $\mathcal{A}(\Omega) = [\Omega]_u$. In fact it is quite easy to see that the following converse holds. The proof is omitted.

Theorem 3.15. *If Ω is an open filterbase on a space satisfying $\mathcal{A}(\Omega) = [\Omega]_u$ then Ω is a Urysohn filterbase.*

Corollary 3.16. *If Ω in an open filterbase on a space X and $[\Omega]_u = \emptyset$ then Ω is a Urysohn filterbase.*

The following parallel of Theorem 3.12 may now be established.

Theorem 3.17. *The following statements are equivalent for a Urysohn space X :*

(1) The space X is u-functionally compact.

(2) Each open filterbase Ω on X satisfying $[\Omega]_u = \mathcal{I}(\Omega)$ converges to $\mathcal{I}(\Omega)$.

(3) Each filterbase Ω on X satisfying $[\Omega]_u = \mathcal{I}(\Omega)$ converges to $\mathcal{I}(\Omega)$.

(4) For each Urysohn space Y each u-strongly-closed set-valued function $\lambda \subset Y \times X$ is u. s. c.

(5) For each Urysohn space Y each function from Y to X with a u-strongly-closed inverse is a closed function.

(6) For each Urysohn space Y each u-continuous function from X to Y is a closed function.

(7) For each Urysohn space Y each continuous function from X to Y is a closed function.

Proof. $(\mathbf{1}) \Rightarrow (\mathbf{2})$. Follows from Theorem 3.15.

$(\mathbf{2}) \Rightarrow (\mathbf{3})$. Follows from the fact that $[\Omega]_u = [\bigcup_\Omega \Sigma(F)]_u$.

$(\mathbf{3}) \Rightarrow (\mathbf{4})$. Note that $[\lambda(\Sigma(y))]_u = \lambda(y)$ for each $y \in Y$ and hence λ is u. s. c.

$(\mathbf{4}) \Rightarrow (\mathbf{5})$. Assuming (4), X is Urysohn-closed from a result due to Joseph [18]. Thus $[\Omega]_u \neq \emptyset$ is satisfied for each filterbase Ω on X. Consequently if $g : X \to Y$ satisfies the hypothesis in (5) and $v \in \overline{g(X)}$, then $g^{-1}(\Sigma(v))$ is a filterbase on X and $\emptyset \neq [g^{-1}(\Sigma(v))]_u = g^{-1}(v)$. So $v \in g(X)$ and $g(X)$ is closed in X. Now $g^{-1} \subset g(X) \times X$ is u. s. c. from (4) so g is a closed function [23] and (5) holds.

$(\mathbf{5}) \Rightarrow (\mathbf{6})$. From Corollary 2.6, u-continuous functions into Urysohn spaces have u-strongly-closed inverses.

$(\mathbf{6}) \Rightarrow (\mathbf{7})$. Continuous functions are u-continuous.

$(\mathbf{7}) \Rightarrow (\mathbf{1})$. The space X is Urysohn-closed and if Ω is an open filterbase on X satisfying $\mathcal{A}(\Omega) = I(\Omega)$ then Ω is a Urysohn filterbase and $\mathcal{A}(\Omega) \neq \emptyset$ [20]. The proof may be completed as in the proof of the sufficiency of Theorem 3 in [4]. $\square$

A Hausdorff space is *C-compact* if each closed subset of the space is an H-set. Nayar [24] has recently proved that a Hausdorff space X is C-compact if an only if for each Hausdorff space Y each $g : X \to Y$ with a strongly-subclosed inverse is a closed function. If a space X is Urysohn we call a subset of X a *U-set* if it is QUC relative to X. We define a Urysohn space to be *u-C-compact* if each closed subset of the space is a U-set, and prove the following analogue of Nayar's result and the main result in [2].

Theorem 3.18. *The following statements are equivalent for a Urysohn space X :*

(1) The space X is u-C-compact.

(2) For each Urysohn space Y each $g : X \to Y$ with a u-strongly-subclosed inverse is a closed function.

(3) For each Urysohn space Y each u-strongly-subclosed set-valued function $F \subset Y \times X$ is u.s.c.

Proof. $(\mathbf{1}) \Rightarrow (\mathbf{2})$. Let Y be Urysohn, $A \subset X$ be closed, and $g : X \to Y$ have a u-strongly-subclosed inverse. If y is a limit point of $g(A)$ then $\Omega = \{g^{-1}(W - \{y\}) \cap A : W \in \Sigma(y)\}$ is a filterbase on A and hence $\emptyset \neq A \cap [\Omega]_u \subset A \cap g^{-1}(y)$. So $g(A)$ is closed.

$(\mathbf{2}) \Rightarrow (\mathbf{1})$. Suppose A is a closed subset of X and that Ω is a filterbase on A such that $A \cap [\Omega]_u = \emptyset$. Since continuous functions into Urysohn spaces have u-strongy-subclosed inverses, it follows that X is Urysohn-closed and hence that $A \neq X$. Choose $v \in A$ and define $g : X \to Y$ by $g(x) = x$ if $x \in A$, $g(x) = v$ if $x \in X - A$, where $Y = X$ with the topology $\{V \subset X : v \in X - V$ or some $F \in \Omega$ satisfies $F \subset V\}$. Then Y is Urysohn and $g^{-1} \subset g(X) \times X$ is u-strongly-subclosed since $\mathcal{S}(y)$ is a filterbase on Y only if $v = y$, and $[g^{-1}(\mathcal{S}(v))]_u \subset [\Omega]_u \subset X - A \subset g^{-1}(v)$. Choose $H \in \Omega$ and $W \in \Sigma(v)$ with $\overline{W} \cap H = \emptyset$ in X. It follows that $g(H) = H \subset A - W = g(A - W)$, so $v \in \overline{A - W} - (A - W)$ in Y. Since $A - W$ is closed in X the function g is not a closed function.

$(\mathbf{1}) \Rightarrow (\mathbf{3})$. Let Y be Urysohn and $F \subset Y \times X$ be u-strongly-subclosed. If $y \in Y$ and some $W \in \Sigma(F(y))$ satisfies $\emptyset \neq F(V) - W$ for all $V \in \Sigma(y)$ then $\mathcal{S}(y)$ is a filterbase on Y and $\emptyset \neq [F(\mathcal{S}(y))]_u - W \subset F(y) - W$ since each closed subset of X is QUC and F is u-strongly-subclosed. This is a contradiction.

(3) $\Rightarrow$ **(1)**. Suppose A is a closed subset of X and that Ω is a filterbase on A such that $A \cap [\Omega]_u = \emptyset$. It follows from Theorem 3.17(4) that X is Urysohn-closed and hence that $A \neq X$. Choose $v \in A$, and let $Y = X$ with the topology $\{V \subset X : v \in X - V$ or some $H \in \Omega$ satisfies $H \subset V\}$. Then Y is Urysohn. Define $F \subset Y \times X$ by $F = \{(x, x) : x \neq v\} \cup \{(v, x) : x \in X - A\}$. To see that F is u-strongly-subclosed only v need be checked. Now $\Omega \subset \mathcal{S}(v)$ in Y so $[F(\mathcal{S}(v))]_u \subset [F(\Omega)]_u = [\Omega]_u \subset Y - A = F(v)$. However F is not u.s.c. since $Y - A \in \Sigma(F(v))$ in X and no $V \in \Sigma(v)$ satisfies $F(V) \subset Y - A$. $\square$

4. Generalizations of the Uniform Boundedness Principle. In this final section results of previous sections are utilized to provide generalizations of the Uniform Boundedness Principle. Initially several generalizations of the Uniform Boundedness Principle are established when the range space Y is arbitrary and Δ is a nonempty countable collection of QUC relative to Y subsets. In establishing these generalizations it will be evident that the following generalization of the Uniform Boundedness Principle is valid (recall that a topological space is a *Baire space* if the intersection of each sequence of dense open subsets of the space is dense in the space [17]). A subset Q of a topological space X is *of second category in* X if Q is not the union of a countable number of nowhere dense subsets of X. It is known that a topological X is a Baire space if an only if each nonempty open subset of the space is of second category in X [17]: Let X be a Baire space, let Δ be a nonempty countable family of compact subsets of a space Y, and let $\mathcal{F}$ be a family of functions from X to Y with subclosed graphs such that $\mathcal{UB}[\mathcal{F}, X, Y, \Delta]$ contains the collection of singletons of X. Then there is a nonempty open subset V of X satisfying (1) $V \in \mathcal{UB}[\mathcal{F}, X, Y, \Delta]$ and (2) each $g \in \mathcal{F}$ is continuous at each $x \in V$. Moreover, there is a W open in X such that each $g \in \mathcal{F}$ is continuous at each $x \in W$ and $\overline{W} = X$.

Theorem 4.1 is the first main result in this section. The proof is omitted since the proofs of subsequent theorems will indicate the method of proof.

Theorem 4.1. *Let X be a Baire space, let Y be a space, let Δ be a nonempty countable family of QUC relative to Y subsets and let $\mathcal{F}$ be a family of functions from X to Y with u-strongly-subclosed graphs such that $\mathcal{UB}[\mathcal{F}, X, Y, \Delta]$ contains the collection of singletons of X. Then*

(1) There is a nonempty open V of X such that $\overline{V} \in \mathcal{UB}[\mathcal{F}, X, Y, \Delta]$.

(2) For each $g \in \mathcal{F}$ the restriction of g to V of part (1) is u-weakly-continuous.

(3) Each $g \in \mathcal{F}$ is u-weakly-continuous at each point of V of part (1).

(4) There is a W open in X such that each $g \in \mathcal{F}$ is u-weakly-continuous at each $x \in W$ and $\overline{W} = X$.

Definition 4.2. A subset A is of the $\theta-$*second category* in X if A is not contained in the union of a countable family, Θ, of subsets of X such that $int([F]_\theta) = \emptyset$ for each $F \in \Theta$; A is of the $\theta-$*second category* if A is of the $\theta-$second category in A. A space X is $\theta-$*Baire* if $\overline{W}$ is of the $\theta-$second category for each nonempty open subset of X.

In [19] it is shown that Baire spaces and QHC spaces are $\theta-$Baire. Theorem 4.3 is the second main result in this section.

Theorem 4.3. *Let X be a $\theta-$Baire space, let Y be a space, let Δ be a nonempty countable family of QUC relative to Y subsets and let $\mathcal{F}$ be a family of functions from X to Y with (θ,u)-subclosed graphs such that $\mathcal{UB}[\mathcal{F}, X, Y, \Delta]$ contains the collection of singletons of X. Then*

(1) There is a nonempty open V of X such that $[V]_u \in \mathcal{UB}[\mathcal{F}, X, Y, \Delta]$.

(2) For each $g \in \mathcal{F}$ the restriction of g to $[V]_u$ of part (1) is $(\theta$, u)-continuous.

(3) Each $g \in \mathcal{F}$ is $(\theta$,u)-continuous at each point of V of part (1).

(4) There is a W open in X such that each $g \in \mathcal{F}$ is $(\theta$,u)-continuous at each $x \in W$ and $\overline{W} = X$.

Proof. **(1)** For each $C \in \Delta$ let $I(C) = \bigcap g^{-1}(C)$. Then $I(C)$ is $\theta-$closed since it is the intersection of sets which are $\theta-$closed from Theorem 2.10. If $x \in X$ there is a $C(x) \in \Delta$ such that $g(x) \in C(x)$ for all $g \in \mathcal{F}$. Hence $\{I(C) : C \in \Delta\}$ is a cover of X by a countable family of $\theta-$closed sets. Since X is of the $\theta-$ second category there is a $C_0 \in \Delta$ such that $int(I(C_0)) \neq \emptyset$. Let $V = int(I(C_0))$. Evidently, $[V]_u \subset I(C_0)$. Hence for each $g \in \mathcal{F},\ g([V]_u) \subset g(I(C_0)) \subset g(g^{-1}(C_0)) \subset C_0$.

(2) Let V be an open set of the type guaranteed in part (1) and let $g \in \mathcal{F}$. Since C_0 is QUC relative to Y and the restriction, $g_{[V]_u}$, of g to $[V]_u$ has a $(\theta$,u)-subclosed graph, then $g_{[V]_u}$ is $(\theta$,u)-continuous from Theorem 2.13.

(3) Let V be an open set of the type guaranteed in part (1), let $x \in V$, let $g \in \mathcal{F}$, and let $W \in \Lambda(g(x))$. Choose $Q \in \Sigma(x)$ such that $g_{[V]_u}(\overline{Q \cap [V]_u}) \subset \overline{W}$. Then $Q \cap V \in \Sigma(x)$ and $\overline{Q \cap V} \subset Q \cap [V]_u$, so $g(\overline{Q \cap V}) \subset \overline{W}$.

(4) Let W be the union of all open V such that each $g \in \mathcal{F}$ is $(\theta$,u)-continuous at each $x \in V$. Then each $g \in \mathcal{F}$ is $(\theta$,u)-continuous at each $x \in W$. Let A be a nonempty open subset of the $\theta-$Baire space X. Then $\overline{A}$ is of the $\theta-$second category and $\mathcal{F}_{\overline{A}} = \{g_{\overline{A}} : g \in \mathcal{F}\}$ satisfies the conditions imposed on $\mathcal{F}$ in the hypotheses (relative to X). Hence, from Parts (1) and (2), choose a nonempty relatively open B of $\overline{A}$ with each $h \in \mathcal{F}_{\overline{A}}$ $(\theta$,u)-continuous at each point of the nonempty subset $A \cap H$ where $B = H \cap \overline{A}$ and H is open in X. It follows as in the proof of part (3) that each $g \in \mathcal{F}$ is $(\theta$,u)-continuous at each $x \in A \cap H$. Hence $A \cap H \subset W,\ A \cap W \neq \emptyset$, and $\overline{W} = X$. $\square$

The next several results are preliminary to the final main generalization of the Uniform Boundedness Principle, Theorem 4.9.

Theorem 4.4. *The following statements are equivalent for a space X and $A \subset X$.*

(1) The relation $int([A]_u) = \emptyset$ holds.

(2) If V is a nonempty open subset of X and $W \in \Lambda(V)$ there is an $x \in V$ and $Q \in \Lambda(x)$ such that $Q \subset W$ and $A \cap \overline{Q} = \emptyset$.

(3) For each nonempty open V in X there is a nonempty open $Q \subset V$ such that $Q \cap [A]_u = \emptyset$.

Proof. $(\mathbf{3}) \Rightarrow (\mathbf{1})$. Obvious.

$(\mathbf{1}) \Rightarrow (\mathbf{2})$. If (1) holds and V is a nonempty open subset of X then $V - [A]_u \neq \emptyset$. Let $x \in V - [A]_u, P \in \Lambda(x)$ such that $A \cap \overline{P} = \emptyset$. For $W \in \Lambda(x),\ Q = W \cap P$ satisfies $Q \in \Lambda(x),\ Q \subset W$, and $A \cap \overline{Q} = \emptyset$. Hence (2) holds.

$(\mathbf{2}) \Rightarrow (\mathbf{3})$. Assume (2) and let V be a nonempty open subset of X. Then $X \in \Lambda(V)$; from (2) choose $x \in V$ and $Q \in \Lambda(x)$ satisfying $A \cap \overline{Q} = \emptyset$. Let $T \in \Sigma(x)$ with $Q \in \Lambda(T)$. Then $X - \overline{T} \in \Lambda(A)$ and, consequently, $V \cap T \cap [A]_u \subset T \cap \overline{X - \overline{T}} = \emptyset$. $\square$

Theorem 4.6. *If X is a space and $A \subset X$ satisfies $int([A]_u) = \emptyset$ then for each nonempty open $V \subset X$ there is a nonempty open $Q \subset V$ satisfying $A \cap [Q]_u = \emptyset$.*

Proof. If $int([A]_u) = \emptyset$ and V is a nonempty open subset of X choose $W \in \Lambda(A)$ such that $V - \overline{W} \neq \emptyset$. Then $Q = V - \overline{W}$ satisfies $Q \subset V$ and $A \cap [Q]_u = \emptyset$. $\square$

Theorem 4.6 utilizes $[\ \ ]_u$ to show that QUC spaces satisfy a property of " second category type ".

Theorem 4.6. *If X is a QUC space and V is a nonempty open subset of X then $[V]_u$ is not contained in the union of a countable family, Ω, of subsets of X such that $int([F]_u) = \emptyset$ for all $F \in \Omega$.*

Proof. Let $\Omega = \{F_n : n \in N\}$. Using Theorem 4.5, a decreasing sequence V_n of nonempty subsets of X is constructed inductively such that $V_1 \subset V$ and $F_n \cap [V_n]_u = \emptyset$. Since X is QUC choose $x \in \bigcap_N [V_n]_u$. It follows that $x \in [V]_u - \bigcup_N F_n$. $\square$

Corollary 4.7. *If X is QUC and V is a nonempty open subset of X then $[V]_u$ is not contained in the union of a countable family of u-closed subsets with empty interiors.*

Corollary 4.8. *If X is QUC and $W \in \Lambda(A)$ for some nonempty subset A then $\overline{W}$ is not contained in the union of a countable family, Ω, of subsets of X such that $int([F]_u) = \emptyset$ for all $F \in \Omega$.*

Proof. There exists $V \in \Sigma(A)$ such that $[V]_u \subset \overline{W}$. $\square$

Theorem 4.9. *Let X be a QUC space, let Y be a space, let Δ be a nonempty countable family of QUC relative to Y subsets and let $\mathcal{F}$ be a family of functions from X to Y with u-subclosed graphs such that $\mathcal{UB}[\mathcal{F}, X, Y, \Delta]$ contains the collection of singletons of X. Then*

(1) There is a nonempty open V of X such that $[V]_u \in \mathcal{UB}[\mathcal{F}, X, Y, \Delta]$.

(2) For each $g \in \mathcal{F}$ the restriction of g to $[V]_u$ for V guaranteed in part (1) is u-continuous.

(3) For each V satisfying (1), each $g \in \mathcal{F}$ is u-continuous at each point x of V with $V \in \Lambda(x)$.

Proof. The proofs of (1) and (2) may be produced by the use of Theorems 2.11, 2.14, 4.6, the fact that the intersection of u-closed subsets is u-closed, the readily established fact that restrictions of functions with u-subclosed graphs have u-subclosed graphs, and techniques similar to those used in the proof of Theorem 4.3. For the proof of (3) let $g \in \mathcal{F}$, $W \in \Lambda(x)$ satisfy (1), and let $Q \in \Lambda(g(x))$. From (2) choose $A \in \Sigma(x)$ in X such that $A \cap \overline{W} \in \Lambda(x)$ in $\overline{W}$ and $g(\overline{A \cap \overline{W}}) \subset \overline{Q}$. Choose $B \in \Sigma(x)$ in X such that $B \cap \overline{W} \subset \overline{B \cap \overline{W}} \subset A \cap \overline{W}$ and $P \in \Sigma(x)$ in X such that $\overline{P} \subset W$. Then $P \cap B \cap W \in \Sigma(x)$ in X and $\overline{P \cap B \cap W} \subset \overline{P} \cap \overline{B \cap W} \subset W \cap A$. Hence $W \cap A \in \Lambda(x)$ in X and $g(\overline{A \cap W}) = g(\overline{A \cap \overline{W}}) \subset \overline{Q}$. $\square$

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