SIMPLIFIED PROOF OF CHARACTERIZATION OF CHAOTIC ORDER VIA SPECHT'S RATIO

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Abstract. Simplified proof of a characterization of chaotic order via Specht's ratio is given by using our two previous results.

1. Statement of result. In [1], the determinant $\triangle_x(T) = \exp < (\log T)x, x >$ has been considered for positive invertible operator T at a unit vector x. We shall give a simplified proof of the following result in [3] by using Corollary in [2].

Theorem 1. *Let A and B be positive and invertible operators on a Hilbert space H satisfying $MI \geq B \geq mI > 0$. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $M_h(p)A^p \geq B^p$ *holds for all* $p > 0$, *where* $h = \frac{M}{m} > 1$ *and*

$$M_h(p) = \frac{h^{\frac{p}{h^p-1}}}{e\log(h^{\frac{p}{h^p-1}})}.$$

2. Proof of the result. First of all we cite the following result in [2, Corollary].

Theorem A. *Let T be positive and invertible operator on a Hilbert space H satisfying $MI \geq T \geq mI > 0$. Then the ratio of (Tx, x) to the determinant for T at x is not greater than Specht's ratio;*

$$(T^p x, x) \leq M_h(p)\triangle_x(T^p)$$

for all real numbers p, where $h = \dfrac{M}{m}$ *and* $M_h(p) = \dfrac{h^{\frac{p}{h^p-1}}}{e\log(h^{\frac{p}{h^p-1}})}$.

We cite the following Lemma [1] and we shall give a proof slight different from one in [1].

Lemma 2. *For positive invertible operator A and unit vector x, $(A^t x, x)^{\frac{1}{t}}$ is increasing function of $t > 0$ and $\lim\limits_{t\to 0}(A^t x, x)^{\frac{1}{t}} = \triangle_x(A)$. Especially $(Ax, x) \geq \triangle_x(A)$ holds.*

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Proof. By Hölder-McCarthy inequality, $(A^t x, x) \geq (Ax, x)^t$ for $t \geq 1$, that is, $(A^q x, x)^{\frac{1}{q}} \geq (A^p x, x)^{\frac{1}{p}}$ for $q \geq p > 0$ and $(A^t x, x)^{\frac{1}{t}}$ is increasing function of $t > 0$.

$$\lim_{t \to 0} \frac{\log(A^t x, x)}{t} = \lim_{t \to 0} \frac{\log(A^t x, x) - \log(A^0 x, x)}{t - 0} = \left[\frac{d}{dt} \log(A^t x, x)\right]_{t=0} =< (\log A)x, x > .$$

Lemma 3 ([3]). *Let $M_h(p)$ be the same as defined in Theorem 1. Then* $\lim_{p \to +0} \{M_h(p)\}^{\frac{1}{p}} = 1$.

Proof of Theorem 1. (i) $\Longrightarrow$ (ii).

$$\begin{aligned} (B^p x, x) &\leq M_h(p) \triangle_x(B^p) && \text{by Theorem A} \\ &\leq M_h(p) \triangle_x(A^p) && \text{by definition of } \triangle_x(T) \text{ and } \log A \geq \log B \\ &\leq M_h(p)(A^p x, x) && \text{by the latter half of Lemma 2.} \end{aligned}$$

(ii) $\Longrightarrow$ (i). For the sake of convenience, we cite the proof in [3] as follows. Taking logarithm of both sides of (ii), we have $\log(\{M_h(p)\}^{\frac{1}{p}} A) \geq \log B$, so we have (i) since $\lim_{p \to +0} \{M_h(p)\}^{\frac{1}{p}} = 1$ by Lemma 3. Whence the proof of Theorem 1 is complete.

References

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