EXTREMAL PERIODIC SOLUTIONS FOR NONLINEAR PARABOLIC EQUATIONS WITH DISCONTINUITIES

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Received October 26, 1995; revised December 12, 1995

Abstract. In this paper we consider a nonlinear periodic boundary value problem with a discontinuous forcing term. Assuming that the partial differential operator satisfies the Leray-Lions conditions, that the discontinuous perturbation term is locally of bounded variation and that there exist an upper solution ϕ and a lower solution ψ such that $\psi \leq \phi$, we prove the existence of a maximal and a minimal periodic solution within the order interval $[\psi, \phi]$ of an appropriately defined multivalued problem. Our approach is based on a Jordan-type decomposition for the discontinuous perturbation term due to Stuart [21] and on a fixed point theorem for monotone maps in order structures.

1. Introduction. In a series of interesting papers [19], [20], [21] and Stuart-Toland [22] studied ordinary differential equations and semilinear elliptic boundary value problems involving discontinuous nonlinearities. It is well-known that such problems need not have a solution even under restrictive hypotheses. The paper of Stuart [19] contains some characteristic examples illustrating this. It is then a good idea to replace the original equation by a multivalued version of it. In [21] Stuart isolated a broad class of nonlinearities which lead to multivalued problems obtained by filling in only the downward jumps of the original function. So if all the jumps are upward (i.e. $f(r^-) \leq f(r^+)$ for every $r \in \mathbf{R}$) then the single-valued and multivalued versions of the problem produce the same set of solutions. In his main existence theorem Stuart [21] (Theorem 3.1) proved the existence of a maximal and a minimal solution located in the order interval determined by an upper and a lower solution. In [22] Stuart-Toland developed a variational method to deal with such problems based on the nonconvex duality theory of Toland [23]. We should also mention the relevant works of Rauch [18] and Chang [5] who also deal with semilinear elliptic systems involving discontinuities. Rauch [18] used mollification techniques to establish the existence of a solution between an upper and a lower solution for problems in which the discontinuous nonlinearity is not monotone and we only assume that $f(\cdot)$ ultimately increase (i.e. $\overline{\lim\limits_{t\to-\infty}} f(t) \leq \underline{\lim\limits_{t\to+\infty}} f(t)$). Chang [15] used critical point theory for nondifferentiable functionals to deal with such problems.

The study of analogous dynamic (parabolic) problems is lagging behind. Only recently some special semilinear initial-boundary value problems were considered by Carl-Heikkila [4] and Feireisl-Norbury [9].

In this paper using the discontinuities introduced by Stuart [21], we examine nonlinear periodic parabolic boundary value problems and with the help of an upper solution ϕ and a

Key words and phrases. Upper solution, lower solution, function of bounded variation, property (M), periodic solution, Sobolev space, monotone function, extremal fixed points, regular cone.

1991 AMS *Subject Classification.* 35K20 .

lower solution ψ, we establish the existence of a maximum and a minimum periodic solution located in the order interval $[\psi, \phi]$ (assuming $\psi \leq \phi$).

2. Mathematical preliminaries. Let $T = [0, b]$ and $Z \subseteq \mathbf{R}^n$ be a bounded domain with a C^1-boundary γ. We consider the following periodic parabolic boundary value problem:

$$\left\{ \begin{array}{l} \dfrac{\partial x}{\partial t} - \displaystyle\sum_{k=1}^{N} D_k a_k(t, z, Dx) = f(x(t, z)) \text{ in } T \times Z \\ x(0, z) = x(b, z) \text{ a.e. on } Z, x|_{T \times \gamma} = 0 \end{array} \right\} \tag{1}$$

where $D_k = \dfrac{\partial}{\partial z_k} k = 1, 2, \cdots, N$ and $D = (D_k)_{k=1}^N$.

Here $f : \mathbf{R} \to \mathbf{R}$ is a discontinuous nonlinear perturbation term. We impose the following conditions on the date of (1):

$H(a) : a_k : T \times Z \times \mathbf{R}^N \to \mathbf{R} k \in \{1, 2, \cdots, N\}$ are functions such that

(i) $(t, z) \to a_k(t, z, \xi)$ is measurable,

(ii) $\xi \to a_k(t, z, \xi)$ is continuous,

(iii) $|a_k(t, z, \xi)| \leq \beta_1(t, z) + c_1\|\xi\|^{p-1}$ a.e. on $T \times Z$ for every $\xi \in \mathbf{N}^N$ and with $\beta_1 \in L^q(T \times Z), c_1 > 0, 2 \leq p < \infty, \dfrac{1}{p} + \dfrac{1}{q} = 1$,

(iv) $\sum\limits_{k=1}^{N} (a_k(t, z, \xi) - a_k(t, z, \xi'))(\xi_k - \xi_k') > 0$ a.e. on $T \times Z$ for every $\xi, \xi' \in \mathbf{R}^N$ with $\xi \neq \xi'$, and

(v) $\sum\limits_{k=1}^{N} a_k(t, z, \xi)\xi_k \geq c_2\|\xi\|^p_{\mathbf{R}^N} - \beta_2(t, z)$ a.e. on $T \times Z$ with $c_2 > 0$ and $\beta_2 \in L^1(T \times Z)$.

Remark. These are the standard Leray-Lions conditions on the coefficient function $a_k(t, z, \xi)$; cf. Lions [15].

$H(f) : f : \mathbf{R} \to \mathbf{R}$ is a function of bounded variation on every compact interval in $\mathbf{R}$ and $f(r) \in \hat{f}(r)$ for every $f \in \mathbf{R}$, where $\hat{f}(r) = \text{conv}\{f(r^+), f(r^-)\}$ with $f(r^+) = \lim\limits_{\epsilon \to 0^+} f(r + \epsilon)$ and $f(r^-) = \lim\limits_{\epsilon \to 0^+} f(r - \epsilon)$.

The following decomposition property of $f(\cdot)$ will be crucial in our subsequent considerations and can be found in Stuart [21] (Lemma 2.1).

Lemma 2.1. *If $f : \mathbf{R} \to \mathbf{R}$ satisfies hypothesis $H(f)$ and I is a bounded open interval in $\mathbf{R}$, then there exist two nondecreasing functions $g : I \to \mathbf{R}$ and $h : I \to \mathbf{R}$ such that (a) $f(r) = g(r) - h(r)$ for every $r \in I$; (b) $g(\cdot)$ is continuous on $\{r \in I : f(r^-) \geq f(r^+)\}$; and (c) $h(\cdot)$ is continuous on $\{r \in I : f(r^-) \leq f(r^+)\}$.*

Remark. According to Lemma 1 a function $f : \mathbf{R} \to \mathbf{R}$ satisfying $H(f)$, when restricted to a bounded open interval I admits a decomposition as the difference of two nondecreasing functions $g(\cdot)$ and $h(\cdot)$ (Jordan decomposition), with $g(\cdot)$ continuous at those points where a downward jump occurs and $h(\cdot)$ continuous at those points where an upward jump occurs.

Lemma 1 leads us to a convenient expression for the multifunction $\hat{f}(r)$ at those points where a downward jump occurs (cf. Stuart [21], Lemma 2.2).

Lemma 2. *If $f : \mathbf{R} \to \mathbf{R}$ satisfies hypothesis $H(f)$, I is a bounded open interval and $f = g - h$ is the decomposition of $f(\cdot)$ established in Lemma 1, then $\hat{f}(r) = g(r) - \hat{h}(r)$ for every $r \in I$ for which we have $f(r^+) \leq f(r^-)$.*

So for a function $f:\mathbf{R}\to\mathbf{R}$ satisfying $H(f)$, we can define:

$$F(r)=\begin{cases}\{f(r)\} & \text{if } f(r^-)\le f(r^+)\\ \hat{f}(r)=[f(r^+),f(r^-)] & \text{if } f(r^+)\le f(r^-)\end{cases}$$

and observe that by virtue of Lemma 2, $F(r)=g(r)-\hat{f}(r)$ for every $r\in\mathbf{R}$. Then we replace problem (1) by the following multivalued version of it:

$$\left\{\begin{array}{l}\dfrac{\partial x}{\partial t}-\displaystyle\sum_{k=1}^{N}D_k a_k(t,z,Dx)\in F(x(t,z)) \quad \text{in} \quad T\times Z\\ x(0,z)=x(b,z) \quad \text{a.e. on} \quad Z, x|_{T\times\Gamma}=0\end{array}\right\} \tag{2}$$

It is this problem that we will studying in the sequel. Let $W^{1,p}(Z)$ be the usual Sobolev space and $W^{1,p}(Z)^*$ its dual. Then the spaces $W^{1,p}(Z)\subseteq L^2(Z)\subseteq W^{1,p}(Z)^*$ form an evolution triple with all embeddings being continuous, dense and compact (cf. Zeidler [24]). Also by $W_0^{1,p}(Z)$ we denote the subspace of $W^{1,p}(Z)$ whose elements have zero trace (i.e. $W_0^{1,p}(Z)=\ker\ \gamma_0$ with $\gamma_0(\cdot)$ being the trace operator). As usual by $W^{-1,q}(Z)$ we denote the dual of $W_0^{1,p}(Z)$. Then $W_0^{1,p}(Z)\subseteq L^2(Z)\subseteq W^{-1,q}(Z)$ is also an evolution triple with all embeddings being again continuous, dense and compact. Then we introduce the following function spaces:

$$\hat{W}_{pq}(T)=\{f\in L^p(T,W^{1,p}(Z)):\frac{\partial f}{\partial t}\in L^q(T,W^{1,p}(Z)^*)\}$$

and

$$W_{pq}(T)=\{f\in L^p(T,W_0^{1,p}(Z)):\frac{\partial f}{\partial t}\in L^q(T,W^{-1,q}(Z))\}.$$

Here the derivative $\dfrac{\partial f}{\partial t}$ is understood in the sense of vector-valued distributions. Both spaces endowed with the obvious norm $\|f\|_{pq}=\|f\|_p+\|f\|_q$, become Banach spaces which are separable reflexive due to the separability and reflexivity of the Lebesgue-Bochner spaces $L^p(T,W^{1,p}(Z)),L^q(T,W^{1,p}(Z)^*)$ and $L^p(T,W_0^{1,p}(Z)),L^q(T,W^{-1,q}(Z))$.

Moreover, we know that both $W_{pq}(T)$ and $\hat{W}_{pq}(T)$ embed continuously in $C(T,L^2(Z))$ and compactly in $L^p(T\times Z)$ (cf. Lions [15], theorem 5.1 p.58 and Zeidler [24], proposition 23.23 p.422 and p.450).

By virtue of hypothesis $H(a)$ we can define the semilinear Dirichlet form $a:L^p(T,W^{1,p}(Z))\times L^p(T,W^{1,p}(Z))\to\mathbf{R}$, by

$$a(x,y)=\int_0^b\int_Z\sum_{k=1}^{N}a_k(t,z,Dx)D_k y(t,z)dzdt,$$

where as we already said $D_k=\dfrac{\partial}{\partial z_k},k\in\{1,2,\cdots,N\}$ and $D=(D_k)_{k=1}^N$.

In what follows by $(\cdot,\cdot))$ we will denote the duality brackets between $L^p(T,W^{1,p}(Z))$ and $L^q(T,W^{1,p}(Z)^*)$ and also between $L^p(T,W_0^{1,p}(Z))$ and $L^q(T,W^{-1,q}(Z))$. Recall that if X is a reflexive Banach space (or even more generally if X^* has the Radon-Nikodym property) and $1\le p<\infty$, then $L^p(T,X)^*=L^q(T,X^*),\dfrac{1}{p}+\dfrac{1}{q}=1$ (cf. Diestel-Uhl [8], theorem 1, p.98).

Definition 3. A function $\varphi = \hat{W}_{pq}(T)$ is said to be an '5pper solution" of (1) if

$$((\frac{\partial\varphi}{\partial t}, u)) + a(\varphi, u) \geq \int_0^b \int_z f(\varphi(t,z))u(t,z)dzdt$$

for all $u \in L^p(T, W_0^{1,p}(Z)) \cap L^p(T \times Z)_+, \varphi(0,z) \geq \varphi(b,z)$ a.e. on Z and $\varphi|_{T\times\Gamma} \geq 0$.

Similarly a function $\psi \in \hat{W}_{pq}(T)$ is a "lower solution" to (1) if the inequalities in the above definition are reversed.

H_0: there exist an upper solution φ and a lower solution ψ such that $\psi \leq \varphi$ and $\psi, \varphi \in L^\infty(T \times Z)$.

Remark. We can drop the requirement that $\psi, \varphi \in L^\infty(T \times Z)$ at the expense of strengthening hypothesis $H(f)$ by assuming that $f(\cdot)$ is of bounded variation on all of $\mathbf{R}$. Moreover in this case we also need to assume that $g(\varphi(\cdot,\cdot)), g(\psi(\cdot,\cdot)), h(\varphi(\cdot,\cdot)), h(\psi(\cdot,\cdot))$ all belong in $L^q(T, L^2(Z))$. It should be pointed out that Deuel-Hess [7], Mokrane [16] and Boccardo-Murat-Puel [3] incorporate in their definition of upper and lower solutions the assumption that they belong in $L^\infty(T \times Z)$.

Let us now introduce the notion of a (weak) solution for problem (2).

Definition 4. A function $x \in W_{pq}(T)$ is said to be a solution of (2) if there exists a function $v \in L^q(T \times Z)$ such that $v(t,z) \in F(x(t,z))$ a.e. on $T \times Z$ and

$$((\frac{\partial x}{\partial t}, u)) + a(x, u) = \int_0^b \int_Z v(t,z)u(t,z)dzdt$$

for all $u \in L^p(T, W_0(Z))$.

The standard pointwise partial ordering on $L^p(T \times Z)$ (i.e. $x \leq y$ if and only if $y - x \in L^p(T \times Z)_+ = \{$ the set of all nonnegative elements in $L^p(T \times Z)\}$) induces a corresponding partial ordering in $\hat{W}_{pq}(T)$. So we can define $[\psi, \varphi] = \{y \in \hat{W}_{pq}(T) : \psi \leq y \leq \varphi\}$, the order interval determined by $\psi \leq \varphi$. We will be looking for the extremal solutions of (2) in $[\psi, \varphi]$. By this we mean the greatest solution x^* and the least solution x_* of (2) within the order interval $[\psi, \varphi]$. So if x is any solution of (2) in $[\psi, \varphi]$, we have $x_* \leq x \leq x^*$.

3. An auxiliary periodic problem. In this section with the help of a truncation and a penalization functions (cf. Deuel-Hess [6]), we introduce and solve an auxiliary problem which will be used in the sequel.

First we consider the truncation function. So given $x \in L^p(T, W^{1,p}(Z))$ we define its truncation $T(x)(\cdot,\cdot)$ as follows:

$$T(x)(t,z) = \begin{cases} \varphi(t,z) & \text{if} \quad \varphi(t,z) \leq x(t,z) \\ x(t,z) & \text{if} \quad \psi(t,z) \leq x(t,z) \leq x(t,z) \\ \psi(t,z) & \text{if} \quad x(t,z) \leq \psi(t,z) \end{cases}$$

Proposition 5. $T : L^p(T, W^{1,p}(Z)) \to L^p(T, W^{1,p}(Z))$ is continuous.

Proof. First observe that by virtue of Lemma 7.6, p.145 of Gilbarg-Trudinger [10] we have that for almost all $t \in T$ $\tau(x)(t,\cdot) \in W^{1,p}(Z)$ (indeed just note that given any two functions $x_1, x_2, \max(x_1,x_2) = (x_1 - x_2)^+ + x_2$ and $\min(x_1,x_2) = x_2 - (x_1 - x_2)^-$ and then apply the aforementioned result of Gilberg-Trudinger). Therefore $\tau(x) \in L^p(T, W^{1,p}(Z))$. Next let $x_n \to x$ in $L^p(T, W^{1,p}(Z))$. Then by passing to a subsequence if necessary we may assume that $x_n(t,z) \to x(t,z)$ a.e. on $T \times Z, D_k x_n(t,z) \to D_k x(t,z)$ a.e. on $T \times Z$

for every $k \in \{1, 2, \cdots, N\}$ and by virtue of Theorem 2.8.1, p.74 of Kufner-John-Fučik [14], we can find $\theta, \theta_k \in L^p(T \times Z) k \in \{1, 2, \cdots, N\}$ such that $|x_n(t,z)| \leq \theta(t,z)$ and $|D_k x_n(t,z)| \leq \theta_k(t,z)$ a.e. on $T \times Z$. Observe that $\tau(x_n)(t,z) \to \tau(x)(t,z)$ a.e. on $T \times Z$ and $|\tau(x_n)(t,z)| \leq \max\{\theta(t,z), |\varphi(t,z), |\psi(t,z)|\}$ a.e. on $T \times Z$. So via the dominated convergence theorem we get that $\tau(x_n) \to \tau(x)$ in $L^p(T \times Z)$. Also using once again Lemma 7.6, p.145 of Gilbarg-Trudiner [10] we see that for every $y \in L^p(T, W^{1,p}(Z))$ we have

$$D\tau(y)(t,z) = \begin{cases} D\varphi(t,z) & \text{if} \quad \varphi(t,z) \leq y(t,z) \\ Dy(t,z) & \text{if} \quad \psi(t,z) \leq y(t,z) \leq \varphi(t,z) \\ D\psi(t,z) & \text{if} \quad y(t,z) \leq \psi(t,z). \end{cases}$$

In the light of this we have that $D\tau(x_n)(t,z) \to D\tau(x)(t,z)$ a.e. on $T \times Z$ and moreover

$$|D_k \tau(x_n)(t,z)| \leq \theta_k(t,z) + |D_k \varphi(t,z)| + |D_k \psi(t,z)| \text{ a.e. on } T \times Z$$

for every $k \in \{1, 2, \cdots, N\}$. Thus by the dominated convergence theorem we have that $D\tau(x_n) \to D\tau(x)$ in $L^p(T \times Z)$ and so we finally conclude that $\tau(x_n) \to \tau(x)$ in $L^p(T, W^{1,p}(Z))$ establishing the continuity of $x \to \tau(x)$. $\square$

Also we introduce a penalty function $u : T \times Z \times \mathbf{R} \to \mathbf{R}$, defined by

$$u(t,z,x) = \begin{cases} (x - \varphi(t,z))^{p-1} & \text{if} \quad \varphi(t,z) \leq x \\ 0 & \text{if} \quad \psi(t,z) \leq x \leq \varphi(t,z) \\ -(\psi(t,z) - x)^{p-1} & \text{if} \quad x \leq \psi(t,z) \end{cases}$$

A straightforward elementary calculation reveals that the following is true about the penalty function:

Proposition 6. $u : T \times Z \times \mathbf{R}$ is a Caratheodory function (i.e. measurable in (t,z) and continuous in x), $|u(t,z,x)| \leq \beta_3(t,z) + c_3|x|^{p-1}$ a.e. on $T \times Z$ with $\beta_3 \leq L^q(T \times Z), c_3 > 0$ and

$$\begin{aligned} &\int_0^b \int_z u(t,z,x(t,z))x(t,z)dzdt \\ &\geq c_4 \|x\|^p_{L^p(T \times Z)} - c_5 \|x\|^{p-1}_{L^p(T \times Z)} \quad \text{for some} \quad c_4, c_5 > 0. \end{aligned}$$

Now let $K = \{y \in L^2(T \times Z) : \psi(t,z) \leq y(t,z) \leq \varphi(t,z) \text{ a.e. on } T \times Z\}$. So K is the order interval in $L^2(T \times Z)$ determined by the functions $\psi \leq \varphi$. Given $y \in K$ we consider the following periodic boundary value problem:

$$\left\{ \begin{array}{c} \dfrac{\partial x}{\partial t} - \displaystyle\sum_{k=1}^n D_k a_k(t,z,Dx) \in g(y(t,z)) - \hat{h}(\tau(x)(t,z)) - u(t,z,x(t,z)) \\ \text{in} \quad T \times Z \\ x(0,z) = x(b,z) \quad \text{a.e. on} \quad Z, x|_{T \times \Gamma} = 0 \end{array} \right\} \tag{3}$$

Proposition 7. If hypotheses $H(a), H_0$ and $H(f)$ hold, then problem (3) has unique solution $R(y)(\cdot,\cdot) \in W_{pq}(T)$.

Proof. In what follows we consider the evolution triple $X = W_0^{1,p}(Z), H = L^2(Z)$ and $X^* = W^{-1,q}(Z)$ (recall that all the embeddings are continuous, dense and compact). Let

$L : V \subset L^p(T, X) \to L^q(T, X^*)$ be defined by $L(x) = \dfrac{\partial x}{\partial t}$ for $x \in V = \{y \in W_{pq}(T) : y(0) = y(b)\}$ (as before $\dfrac{\partial x}{\partial t}$ is to be understood in the sense of vector-valued distributions). From Zeidler [24] (Proposition 32.10, p.855) we know that $L(\cdot)$ is maximal monotone.

Next let $G : L^p(T, X) \to 2^{L^q(T, X^*)}$ be defined by

$$
\begin{aligned}
((G(x), y)) = \{(a(x, y) &+ \int_0^b \int_z v(t, z)y(t, z)dzdt \\
&+ \int_0^b \int_z u(t, z, x(t, z))y(t, z)dzdt : \\
v(t, z) \in \hat{h}(\tau(x)(t, z)) \quad &\text{a.e. on} \quad T \times Z, v \in L^q(T, H) \subseteq L^q(T, X^*)\}.
\end{aligned}
$$

From Krasnoselskii's theorem we see that $G(\cdot)$ is a bounded set-valued operator with closed and convex values. Moreover since $\hat{h}(\cdot)$ is upper semicontinuous as a multifunction (cf. Klein-Thompson [13], p.75), we see at once that $G(\cdot)$ is upper semi-continuous from each finite-dimensional subspace of $W_{pq}(T)$ into $L^q(T, X^*)$. Also let $\{x_n\}_{n \geq 1} \subseteq W_{pq}(T), w_n \in G(x_n)$ and assume that $x_n \xrightarrow{w} x$ in $L^p(T, X), ((w_n, y)) \to ((w, y))$ for every $y \in W_{pq}(T)$ and $\overline{\lim}((w_n, x_n)) \leq ((w, x))$. Note that

$$((w_n, x_n)) = a(w_n, x_n) + ((v_n, x_n)) + ((U(x_n), x_n))$$

with $v_n \in L^q(t, H), v_n(t, z) \in \hat{h}(\tau(x_n)(t, z))$ a.e. on $T \times Z$ and $U(x_n)(t, z) = u(t, z, x_n(t, z))$. Since $|v_n(t, z)| \leq h(\varphi(t, z))$ a.e. on $T \times Z$, by passing to a subsequence if necessary, we may assume that $v_n \xrightarrow{w} v$ in $L^q(T \times Z)$ and $v(t, z) \in \hat{h}(\tau(x)(t, z))$ a.e. on $T \times Z$ (cf. Papageorgiou [17]). So $((v_n, x_n)) \to (v, x))$ with $v(t, z) \in \hat{h}(\tau(x)(t, z))$ a.e. on $T \times Z$. Also let $\hat{A} : L^p(T, X) \to L^q(T, X^*)$ be defined by

$$((\hat{A}(x), y)) = a(x, y) + ((U(x), y)) \quad \text{for every} \quad x, y \in L^p(T, X).$$

It is well-known (cf. Lions [15] or Berkovitz-Mustonen [2], Proposition 1, p.615) that because of hypothesis $H(a)$ and because of Proposition 6, $\hat{A}(\cdot)$ is pseudomonotone with respect to $W_{pq}(T)$, in particular then has property (M) with respect to $W_{pq}(T)$ (cf. Lions [15] pp.173 and 179).

Since $\overline{\lim}((w_n, x_n)) \leq ((w, x))$, we get $\overline{\lim}((\hat{A}(x_n), x_n)) \leq ((\hat{A}(x), x))$ and so by property (M) we conclude $\hat{A}(x_n) \xrightarrow{w} \hat{A}(x)$ in $L^q(T, X^*)$. Therefore $w = \hat{A}(x) + v$ with $v(t, z) \in \hat{h}(\tau(x), (t, z))$ a.e. on $T \times Z$ and so $w \in G(x)$. Thus we have checked that the set-valued operator $G(\cdot)$ has property (M) with respect to $W_{pq}(T)$ (cf. Gupta [11], Definition 1).

Next we claim that $G(\cdot)$ is coercive; i.e.

$$\lim_{\substack{w \in G(x) \\ \|x\|_{L^p(T,X)} \to \infty}} \frac{((w, x))}{\|x\|_{L^p(T,X)}} = +\infty.$$

To this end, because of hypothesis $H(a)(v)$ we have

$$
\begin{aligned}
\int_0^b \int_z \sum_{k=1}^N a_k(t, z, Dx) D_k x dz &\geq c_2 \int_0^b \int_z \|Dx(t, z)\|^p_{\mathbf{R}^N} dzdt \\
&= \hat{c}_2 \|x\|^p_{L^p(T,X)}, \hat{c}_2 > 0 \qquad (4)
\end{aligned}
$$

(recall that $\|Dx(t,\cdot)\|_{L^p(Z)}$ is an equivalent norm on $W_0^{1,p}(Z)$). Also from Proposition 6 we know that

$$\int_0^b \int_z u(t,z,x(t,z))x(t,z)dzdt \geq c_4\|x\|^p_{L^p(T\times Z)} - c_5\|x\|^{p-1}_{L^p(T\times Z)} \tag{5}$$

In addition for every $v \in L^p(T,H), v(t,z) \in \hat{h}(\tau(x)(t,z))$ a.e. on $T \times Z$, from Hölder's inequality we have that

$$|\int_0^b \int_z v(t,z)x(t,z)dzdt| \leq M_1\|x\|_{L^p(T\times Z)} \quad \text{for some} \quad M_1 > 0,$$

and so

$$\int_0^b \int_z v(t,z)x(t,z)dzdt \geq -M_1\|x\|_{L^p(T\times Z)} \tag{6}$$

Combining (4), (5) and (6) we get that for every $w \in G(x)$ we have

$$((w,x) \geq \hat{c}_2\|x\|^p_{L^p(T\times X)} + c_4\|x\|^p_{L^p(T\times Z)} - c_5\|x\|^{p-1}_{L^p(T\times Z)} - M_1\|x\|_{L^p(T\times Z)},$$

from which it follows easily that $G(\cdot)$ is coercive as claimed.

Now rewrite the periodic boundary value problem (3) in the following abstract operator equation form

$$L(x) + C(x) \ni -g(y) \tag{7}$$

Applying Theorem 1.2 p.319 of Lions [15] (see also Theorem 1 of Gupta [11]), we get that this problem has a solution $x \in W_{pq}(T)$. Since $\hat{A}(\cdot)$ is strictly monotone (see hypothesis $H(a)(iv)$ and recall the definition of the penalty function $u(t,z,x)$ and because $\hat{h}(\cdot)$ is a monotone multifunction (the function $h(\cdot)$ being monotone), we have that $G(\cdot)$ is strictly monotone and so we conclude that (7) (hence (3) too) has a unique solution $x = R(y) \in W_{pq}(T)$. Q.E.D.

4. Existence of extremal periodic solutions. In this section we establish the existence of extremal solutions for problem (2). Our approach is based on the following result essentially due to Amann [1] (Corollary 1.5) (see also Heikkila-Hu [12], Corollary 3.2):

Proposition 8. If $[x_0, y_0]$ is a nonempty order interval in a regularly ordered metric space, then every increasing map $R : [x_0, y_0] \to [x_0, y_0]$ has the least and the greatest fixed points.

Note that because of the dominated convergence theorem the positive cone in $L^2(T,H) = L^2(T \times Z), L^2(T\times Z)_+ = \{y \in L^2(T\times Z) : 0 \leq y(t,z) \text{ a.e. on } T \times Z\}$ is regular, i.e. every order bounded (hence pointwise bounded by an $L^2(T \times Z)$-function) sequence $\{y_n\}_{n\geq 1}$ in $L^2(T \times Z)_+$ converges in the $L^2(T\times Z)$-norm.

To apply Proposition 8 we take K as our order interval in $L^2(T \times Z)$ and as $R(\cdot)$ the single-valued map obtained in Proposition 7.

Proposition 9. If hypothesis $H(a), H(f)$ and H_0 hold, then $R(K) \subseteq K$.

Proof. Let $y \in K$ and set $x = R(y)$. From Gilbarg-Trudinger [10] as before we get that $(\psi - x)^+ \in W_{pq}(T) \cap L^p(T \times Z)_+$. Since $\psi(\cdot,\cdot)$ is a lower solution of (1) according to Definition 3 with $(\psi - x)^+$ as our test function, we have:

$$-((\frac{\partial\psi}{\partial t}, (\psi - x)^+)) - a(\psi, (\psi - x)^+) \geq -\int_0^b \int_Z f(\psi(t,z))(\psi - x)^+(t,z)dzdt$$
$$\psi(0,z) \leq \psi(b,z) \text{ a.e. on } Z. \tag{8}$$

Also since $x = R(y)$, we have for some $v \in L^q(T, H)$ with $v(t,z) \in \hat{h}(\tau(x)(t,z))$ a.e. on $T \times Z$:

$$((\frac{\partial x}{\partial t}, (\psi - x)^+)) + a(x, (\psi - x)^+) = \int_0^b \int_Z g(y(t,z))(\psi - x)^+ dzdt$$
$$- \int_0^b \int_Z v(t,z)(\psi - x)^+(t,z)dzdt - \int_0^b \int_Z u(t,z,x(t,z))(\psi - x)^+(t,z)dzdt$$
$$\geq \int_0^b \int_Z g(y(t,z))(\psi - x)^+(t,z)dzdt$$
$$- \int_0^b \int_Z h(\tau(x)(t,z)^-)dzdt - \int_0^b \int_Z u(t,z,x(t,z))(\psi - x)^+(t,z)dzdt \tag{9}$$

Adding inequalities (8) and (9) above and recalling that $f(r) = g(r) - h(r)$, we get that

$$((\frac{\partial (x - \psi)}{\partial t}, (\psi - x)^+))+$$
$$\int_0^b \int_Z \sum_{k=1}^N (a_k(t,z,Dx) - a_k(t,z,D\psi))D_k(\psi - x)^+(t,z)dzdt$$
$$\geq \int_0^b \int_Z (g(y(t,z)) - g(\psi(t,z)))(\psi - x)^+(t,z)dzdt$$
$$- \int_0^b \int_Z (h(\tau(x)(t,z)^-) - h(\psi(t,z)))(\psi - x)^+(t,z)dzdt$$
$$- \int_0^b \int_Z u(t,z,x(t,z))(\psi - x)^+(t,z)dzdt. \tag{10}$$

From the integration by parts formula for functions in $W_{pq}(T)$ (cf. Zeidler [24], Proposition 23.23, p.423), we have that

$$((\frac{\partial (x - \psi)}{\partial t}, (\psi - x)^+)) = -((\frac{\partial (x - \psi)^+}{\partial t}, (\psi - x)^+))$$
$$= -\frac{1}{2}\|\psi(b,\cdot) - x(b,\cdot))^+\|^2_{L^2(Z)^+} + \frac{1}{2}\|(\psi(0,\cdot) - x(0,\cdot))^+\|^2_{L^2(Z)}.$$

Note that $(\psi - x)(0,\cdot) = \psi(0,\cdot) - x(0,\cdot) \leq \psi(b,\cdot) - x(b,\cdot) = (\psi - x)(b,\cdot)$ in $L^2(Z)$ and so we have $(\psi - x)^+(0,\cdot) \leq (\psi - x)^+(b,\cdot)$, from which we deduce that $\|(\psi - x)^+(0)\|_{L^2(Z)} \leq \|(\psi - x)^+(b)\|_{L^2(Z)}$. Thus we get that

$$((\frac{\partial (x - \psi)}{\partial t}, (\psi - x)^+)) \leq 0 \tag{11}$$

Since

$$D_k(\psi - x)^+(t,z) = \begin{cases} D_k(\psi - x)(t,z) & \text{if } x(t,z) \leq \psi(t,z) \\ 0 & \text{if } \psi(t,z) \leq x(t,z) \end{cases}$$

(cf. Gilbarg-Trudinger [10], p.145), and using hypothesis $H(a)$ (iv), we get that

$$\int_0^b \int_Z \sum_{k=1}^N (a_k(t,z,Dx) - a_k(t,z,D\psi))D_k(\psi - x)^+(t,z)dzdt \leq 0 \tag{12}$$

Also because $y \in K$ and $g(\cdot)$ is nondecreasing (see Lemma 1), we have that

$$\int_0^b \int_Z (g(y(t,z)) - g(\psi(t,z)))(\psi - x)^+(t,z)dzdt \geq 0 \tag{13}$$

Finally becasue of the fact that $h(\cdot)$ too is nondecreasing (see Lemma 1), we have

$$\int_0^b \int_Z (h(\tau(x)(t,z)^-) - h(\psi(t,z)))(\psi - x)^+(t,z)dzdt \leq 0. \tag{14}$$

Using inequalities (11) $\to$ (14) in (10) we get

$$\int_0^b \int_Z u(t,z,x(t,z))(\psi - x)^+(t,z)dzdt \geq 0$$

hence

$$\int_0^b \int_Z -(\psi(t,z) - x(t,z))^{p-1}(\psi - x)^+(t,z)dzdt \geq 0$$

and so

$$\int \int_{\{y \geq x\}} [(\psi - x)(t,z)]^p dzdt = \int_0^b \int_Z [(\psi - x)^+(t,z)]^p dzdt = 0$$

from which we conclude that $\psi(t,z) \leq x(t,z)$ a.e. on $T \times Z$. In a similar manner we can show that $x(t,z) \leq \varphi(t,z)$ a.e. on $T \times Z$. Therefore we finally conclude that $R(K) \subseteq K$. Q.E.D.

Proposition 10. If hypotheses $H(a), H(f)$ and H_0 hold, then $R(\cdot)$ is nondecreasing on K.

Proof. Assume that $y_1, y_2 \in K, y_1(t,z) \leq y_2(t,z)$ a.e. on $T \times Z$ and set $x_1 = R(y_1), x_2 = R(y_2)$. We need to show that $x_1(t,z) \leq x_2(t,z)$ a.e. on $T \times Z$. As before let $(x_1 - x_2)^+ \in W_{pq}(T) \cap L^p(T \times Z)_+$ be the test function. Then we have:

$$\begin{aligned}((\frac{\partial x_1}{\partial t}, (x_1 - x_2)^+)) + a(x_1, (x_1 - x_2)^+) &= \int_0^b \int_Z g(y_1(t,z))(x_1 - x_2)^+(t,z)dzdt \\ &- \int_0^b \int_Z v_1(t,z)(x_1 - x_2)^+(t,z)dzdt \end{aligned} \tag{15}$$

and

$$\begin{aligned}&((\frac{\partial x_2}{\partial t}, (x_1 - x_2)^+)) - a(x_2, (x_1 - x_2)^+) \\ &= \int_0^b \int_Z -g(y_2(t,z))(x_1 - x_2)^+(t,z)dzdt \\ &\quad + \int_0^b \int_Z v_2(t,z)(x_1 - x_2)^+(t,z)dzdt \end{aligned} \tag{16}$$

with $v_1, v_2 \in L^q(T, L^2(Z))$ and $v_i(t,z) \in \hat{h}(x_1(t,z))$ a.e. on $T \times Z, i = 1,2$. Remark that because of Proposition 9, $\tau(x_i) = x_i$ and $u(t,z,x_i(t,z)) = 0$ for $i = 1,2$.

Adding (15) and (16) we get that

$$((\frac{\partial(x_1-x_2)}{\partial t},(x_1-x_2)^+))+a(x_1,(x_1-x_2)^+)-a(x_2,(x_1-x_2)^+)$$
$$=\int_0^b\int_Z(g(y_1(t,z))-g(y_2(t,z)))(x_1-x_2)^+(t,z)dzdt$$
$$+\int_0^b\int_Z(v_2(t,z)-v_1(t,z))(x_1-x_2)^+(t,z)dzdt \tag{17}$$

As in the proof of Proposition 9 we can get that

$$((\frac{\partial(x_1-x_2)}{\partial t},(x_1-x_2)^+))\geq 0 \tag{18}$$

and

$$a(x_1,(x_1-x_2)^+)-a(x_2,(x_1-x_2)^+)$$
$$=\int_0^b\int_Z\sum_{k=1}^N(a_k(t,z,Dx_1)-a_k(t,z,Dx_2))D_k(x_1-x_2)^+(t,z)dzdt\geq 0 \tag{19}$$

On the other hand exploiting the monotonicity of $g(\cdot)$ and $\hat{h}(\cdot)$ (see Lemma 1), we get

$$\int_0^b\int_Z(g(y_1(t,z))-g(y_2(t,z)))(x_1-x_2)^+(t,z)dzdt\leq 0 \tag{20}$$

and

$$\int_0^b\int_Z(v_2(t,z)-v_1(t,z))(x_1-x_2)^+(t,z)dzdt\leq 0 \tag{21}$$

Combining (17) $\to$ (21) above, we readily see that $\lambda\{(t,z)\in T\times Z : x_1(t,z)>x_2(t,z)\}=0$, with $\lambda(\cdot)$ being the Lebesgue measure on $T\times Z$. Therefore $x_1\leq x_2$ and so we have proved that $R(\cdot)$ is nondecreasing. Q.E.D.

Proposition 9 and 10 permit the application of Proposition 8. Note that a fixed point of $R(\cdot)$ is a solution of (2) and vice versa of course. Moreover $L^2(T\times Z)_+$ is regular. So we get:

Theorem 11. *If hypotheses $H(a),H(f)$ and H_0 hold, then problem (2) has a greatest solution x^* and a least solution x_* (extremal solutions) in $K=[\psi,\varphi]$.*

If the discontinuous perturbation $f(\cdot)$ has only upward jumps, then the extremal solutions of Theorem 11, also are extremal solutions for the original single-valued problem (1).

Corollary 12. If hypotheses $H(a),H(f),H_0$ hold and $f(r^-)\leq f(r^+)$ for every $r\in[-\|\psi\|_\infty,\|\varphi\|_\infty]$, then problem (1) has a greatest solution x^* and a least solution x_* (extremal solutions) in $K=[\psi,\varphi]$.

Remark. In the terminology of Stuart [21] (used there in the context of semilinear elliptic systems), a solution of problem (1) is called "solution of type I", while a solution of problem (2) is called "solution of type II". While clearly a solution of type I (i.e. of problem (1)) is always a solution of type II (i.e. of problem (2)), the converse need not be true. Stuart [19] produced some nice examples of ordinary differential equations in $\mathbf{R}$, illustrating this. This then justifies the passage to the multivalued problem (2). Corollary (12) tell us that when only upward jumps occur then the two solution sets are equal and nonempty.

Acknowledgement. The authors wish to thank the referee for his (her) corrections and remarks.

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