SUCCESSIVE APPROXIMATIONS TO POSITIVE SOLUTIONS OF NONLINEAR DIFFERENCE EQUATIONS

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ABSTRACT. By means of successive approximations, existence of positive solutions are established for a class of nonlinear higher order neutral difference equations.

A number of methods have been employed for obtaining positive solutions to difference equations. Among these methods there are comparison principles [1], Schauder type fixed point theorem [2], Banach's contraction principle [3], methods of super- and lower-solutions [4], and others [5]. In this paper, we will employ the basic method of successive approximations to obtain a sharp theorem which provides a positive solution to the neutral type difference equation

$$\Delta^n \left(x_k - cx_{k-\delta}\right) + F(k, x_{k-\tau}) = 0, k = 0, 1, 2, ..., \tag{1}$$

where n is a positive integer, δ and τ are nonnegative integers, c is a nonnegative constant, and $F(n, x)$ is a real function defined for $n = 0, 1, 2, ...$ and $x \in R$ such that F is continuous in the second variable. Neutral type difference equations have been studied by a number of authors, and the existence of a positive solution (i.e. a positive sequence $\{x_k\}$ defined for $k \geq -\mu \equiv -\max\{\delta, \tau\}$, which satisfies (1)) is an important issue in some of their studies. In particular, it is known [6] that when $F(n, x) \equiv bx$, then (1) has a positive solution if, and only if, its "characteristic equation"

$$(\lambda - 1)^n - c\lambda^{-\delta}(\lambda - 1)^n + b\lambda^{-\tau} = 0 \tag{2}$$

has a positive root.

We intend to find sufficient conditions for the existence of positive solutions of (1) which are also necessary for the special case just mentioned. In these conditions, we will employ the factorial function $h^{[m]}(i)$ which is defined to be $h(i)h(i-1)\cdots h(i-m+1)$.

THEOREM 1. In addition to the assumptions imposed on equation (1), suppose further that n is odd and that there exists a number β such that $0 \leq F(n, x) \leq F(n, y) \leq \beta y$ for all $n = 0, 1, 2, ...$ and $0 \leq x \leq y$. If the inequality

$$c\lambda^{-\delta} + \frac{\beta\lambda^{-\tau}}{(1-\lambda)^n} \leq 1 \tag{3}$$

has a solution $\gamma \in (0, 1)$, then equation (1) has a nonnegative (but nontrivial) solution.

PROOF. Let Ω be the set of real sequences of the form $x = \{x_k\}_{k=-\mu}^{\infty}$. We now define an operator $T : \Omega \to \Omega$ as follows: for $x = \{x_k\}_{k=-\mu}^{\infty} \in \Omega$,

$$(Tx)_k = cx_{k-\delta} + \sum_{j=k}^{\infty} \frac{(j-k+n-1)^{[n-1]}}{(n-1)!} F(j, x_{j-\tau}), \ k = 0, 1, 2, ...,$$

and

$$(Tx)_k = \gamma^k, \ -\mu \leq k < 0.$$

Consider the following successive approximations: $w^{(0)} = \{\gamma^k\}_{k=-\mu}^{\infty}$, $w^{(j+1)} = Tw^{(j)}$ for $j = 0, 1, 2, ...$. By means of the assumptions on F and the constant c, it is easy to see that

$$0 \leq ... \leq w_k^{(2)} \leq w_k^{(1)} \leq w_k^{(0)}, \ k \geq -\mu.$$

Indeed, our assertion clearly holds when $-\mu \leq k < 0$. Next, note that the binomial series expansion of $(1-\gamma)^{-n}$ is given by

$$\frac{1}{(1-\gamma)^n} = \sum_{j=0}^{\infty} \frac{(j+n-1)^{[n-1]}}{(n-1)!} \gamma^j,$$

so that by (2), we have

$$1 \geq c\gamma^{-\delta} + \beta\gamma^{-\tau} \sum_{j=0}^{\infty} \frac{(j+n-1)^{[n-1]}}{(n-1)!} \gamma^j.$$

Therefore, when $k = 0, 1, 2, ...$,

$$0 \leq (T\gamma)_k \leq c\gamma^{k-\delta} + \sum_{j=k}^{\infty} \frac{(j-k+n-1)^{[n-1]}}{(n-1)!} \beta\gamma^{j-\tau}$$

$$= \gamma^k \left\{ c\gamma^{-\delta} + \beta\gamma^{-\tau} \sum_{j=0}^{\infty} \frac{(j+n-1)^{[n-1]}}{(n-1)!} \gamma^j \right\} \leq \gamma^k.$$

That is, $0 \leq w^{(1)} \leq w^{(0)}$. Assume by induction that $0 \leq w^{(i)} \leq w^{(i-1)}$ for $i = 1, ..., m$. Then

$$w_k^{(m+1)} = cw_{k-\delta}^{(m)} + \sum_{j=k}^{\infty} \frac{(j-k+n-1)^{[n-1]}}{(n-1)!} F(j, w_{j-\tau}^{(m)})$$

$$\leq cw_{k-\delta}^{(m-1)} + \sum_{j=k}^{\infty} \frac{(j-k+n-1)^{[n-1]}}{(n-1)!} F(j, w_{j-\tau}^{(m-1)}) = w_k^{(m)}, k \geq 0,$$

Our assertion is thus proved.

As a consequence , as $j \to \infty$, $w^{(j)}$ converges (pointwise) to some nonnegative sequence $w^* = \{w_k^*\}_{k=-\mu}^{\infty}$, and furthermore, by means of the Lebesque dominated convergence theorem, we may take limits on both sides of $w^{(j+1)} = Tw^{(j)}$ to obtain

$$w_k^* - cw_{k-\delta}^* = \sum_{j=k}^{\infty} \frac{(j-k+n-1)^{[n-1]}}{(n-1)!} F(j, w_{j-\tau}^*), k \geq 0.$$

Taking differences on both sides of the above equality, we see that w^* is a nonnegative solution of equation (1). The proof is complete.

We may further show that w^* is positive under additional conditions. Indeed, $w_k^* = \gamma^k > 0$ for $-\mu \leq k < 0$. Suppose to the contrary that $w_k^* > 0$ for $-\mu \leq k < v$ and $w_v^* = 0$, where $v \geq 0$. Then

$$0 = w_v^* = cw_{v-\delta}^* + \sum_{j=v}^{\infty} \frac{(j-v+n-1)^{[n-1]}}{(n-1)!} F(j, w_{j-\tau}^*).$$

If $\delta = 0$, then since $(j-v+n-1)^{(n-1)} > 0$ for $j \geq v$, we must have $F(j, w_{j-\tau}^*) = 0$ for $j \geq v$. In other words, if we impose the condition that $\tau > 0$ and $F(n, x) > 0$ for all $n = 0, 1, 2, ...$ and $x > 0$, then a contradiction will be reached. Similarly, if $\delta > 0$, we must have $c = 0$ or $F(j, w_{j-\tau}^*) = 0$ for $j \geq v$. Thus if we impose the condition that $c > 0$, or, $\tau > 0$ and $F(n, x) > 0$ for all $n = 0, 1, 2, ...$ and $x > 0$, another contradiction will be reached. We summarize these as follows.

THEOREM 1'. Under the assumptions of Theorem 1, assume further that (i) $\tau > 0$ and $F(n, x) > 0$ for all $n = 0, 1, ...$ and $x > 0$, or (ii) $\delta > 0$ and $c > 0$. If the inequality (2) has a solution $\gamma \in (0, 1)$, then (1) has a positive solution.

When n is even, we have the following dual Theorem.

THEOREM 2. In addition to the assumptions imposed on equation (1), suppose further that n is even and that there exists a number β such that $\beta y \leq F(n, y) \leq F(n, x) \leq 0$ for all $n = 0, 1, 2, ...$ and $0 \leq x \leq y$. If the inequality

$$c\lambda^{-\delta} - \frac{\beta\lambda^{-\tau}}{(\lambda-1)^n} \leq 1 \tag{4}$$

has a solution $\gamma \in (0, 1)$, then equation (1) has a nonnegative (but nontrivial) solution.

The proof is similar to that of Theorem 1 and will therefore be sketched. Let Ω be the set of real sequences of the form $x = \{x_k\}_{k=-\mu}^{\infty}$. We now define an operator $T : \Omega \to \Omega$ as follows: for $x = \{x_k\}_{k=-\mu}^{\infty} \in \Omega$,

$$(Tx)_k = cx_{k-\delta} - \sum_{j=k}^{\infty} \frac{(j-k+n-1)^{[n-1]}}{(n-1)!} F(j, x_{j-\tau}), \; k = 0, 1, 2, ...,$$

and

$$(Tx)_k = \gamma^k, \; -\mu \leq k < 0.$$

Define the following successive approximations: $w^{(0)} = \{\gamma^k\}_{k=-\mu}^{\infty}$, $w^{(j+1)} = Tw^{(j)}$ for $j = 0, 1, 2, ...$. By means of the assumptions on F and rewriting the condition (3) as

$$c\lambda^{-\delta} - \beta\lambda^{-\tau} \sum_{j=0}^{\infty} \frac{(j+n-1)^{[n-1]}}{(n-1)!}\lambda^j \leq 1,$$

it is readily seen that

$$0 \leq ... \leq w_k^{(2)} \leq w_k^{(1)} \leq w_k^{(0)}, \; k \geq -\mu.$$

Finally, by Lebesgue's dominated convergence theorem, the limiting sequence of $\{w^{(j)}\}_{j=0}^{\infty}$ will be the desired nonnegative solution.

Similar to Theorem 1', we may also verify the validity of the following result.

THEOREM 2'. Under the assumptions of Theorem 1, assume further that (i) $\tau > 0$ and $F(n, x) < 0$ for all $n = 0, 1, ...$ and $x > 0$, or (ii) $\delta > 0$ and $c > 0$. If the inequality (3) has a solution $\gamma \in (0, 1)$, then (1) has a positive solution.

While Theorem 2 can be considered as a dual of Theorem 1, there is another possibility when $0 \leq F(n, x) \leq F(n, y) \leq \beta y$.

THEOREM 3. In addition to the assumptions imposed on equation (1), suppose further that n is even, that $c > 0$, and that there exists a number β such that $0 \leq F(n, x) \leq F(n, y) \leq \beta y$ for all $n = 0, 1, 2, ...$ and $0 \leq x \leq y$. If the inequality

$$\frac{1}{c}\lambda^\delta + \frac{\beta}{c}\lambda^{\delta-\tau}\frac{1}{(1-\lambda)^n} \leq 1 \tag{5}$$

has a solution $\gamma \in (0, 1)$, then equation (1) has a nonnegative solution.

The proof is again similar to that of Theorem 1 and will therefore be sketched. Let Ω be the set of real sequences of the form $x = \{x_k\}_{k=-\mu}^\infty$. We define an operator $T : \Omega \to \Omega$ as follows: for $x = \{x_k\}_{k=-\mu}^\infty \in \Omega$,

$$(Tx)_k = \frac{1}{c}x_{k+\delta} + \frac{1}{c}\sum_{j=k+\delta}^\infty \frac{(j-k-\delta+n-1)^{[n-1]}}{(n-1)!}F(j, x_{j-\tau}), k \geq -\delta,$$

and when $\delta < \tau$,

$$(Tx)_k = \gamma^k, -\mu \leq k \leq -\delta - 1. \tag{6}$$

Define the following successive approximations: $w^{(0)} = \{\gamma^k\}_{k=-\mu}^\infty, w^{(j+1)} = Tw^{(j)}$ for $j = 0, 1, 2, ...$ By means of the assumption on F and by rewriting (4) as

$$\frac{1}{c}\lambda^\delta + \frac{\beta}{c}\lambda^{\delta-\tau}\sum_{j=0}^\infty \frac{(j+n-1)^{[n-1]}}{(n-1)!}\lambda^j \leq 1,$$

it is readily seen that $0 \leq ... \leq w_k^{(2)} \leq w_k^{(1)} \leq w_k^{(0)}$ for $k \geq -\mu$. Finally, by Lebesgue's dominated convergence theorem, the limiting sequence w^* of $\{w^{(j)}\}_{j=0}^\infty$ will satisfy the equation

$$w_t^* - cw_{t-\delta}^* + \sum_{j=t}^\infty \frac{(j-t+n-1)^{[n-1]}}{(n-1)!}F(j, w_{j-\tau}^*) = 0, \quad t \geq 0.$$

By taking differences on both sides of the above equation, we see that w^* is the desired nonnegative solution.

We remark that in case $\delta < \tau$, then in view of (5), the nonnegative solution w^* obtained in the above Theorem cannot be trivial. Indeed, $w_k^* = \gamma^k > 0$ for $-\tau \leq k \leq -\delta - 1$. Thus the same reasoning following Theorem can be applied again to conclude the following.

THEOREM 3'. Under the assumptions of Theorem 3, suppose further that either (i) $\delta < \tau$ and $F(n, x) > 0$ for all $n = 0, 1, ...$ and $x > 0$, or (ii) $0 < \delta < \tau$ and $c > 0$. If the inequality (4) has a solution γ in $(0, 1)$, then (1) has a positive solution.

We remark that the above procedures can be extended to suit more general equations of the form

$$\Delta^n(x_k - cx_{k-\delta}) + F(k, x_{k-\tau_1}, x_{k-\tau_2}, ..., x_{k-\tau_m}) = 0.$$

By replacing conditions in the above Theorems with appropriate ones such as

$$c\lambda^{-\delta} + \frac{1}{(1-\lambda)^n}\sum_{i=1}^m \beta_i\lambda^{-\tau_i} \leq 1,$$

etc. nonnegative and positive solutions to this equation can be found.

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