PERFORMANCE ANALYSIS OF A TWO-QUEUE MODEL WITH A BERNOULLI-THRESHOLD SERVICE SCHEDULE

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ABSTRACT. In this paper, we propose a Bernoulli-Threshold service schedule for a queueing system consisting of two-parallel queues and a single server. A threshold $N(0 < N)$ is set up in one of the two queues, say, the second queue. When the queue length of the second queue is less than or equal to the threshold N, the server serves two queues with a Bernoulli service schedule, otherwise only customers of the second queue are served until its queue length is back to the threshold N. The sever takes switching times in its transition from one queue to the other. For the queueing model, we carry out the performance analysis and derive the generating functions of the joint stationary queue-length distributions at service completion instants. We also determine the Laplace-Stieltjes transforms of waiting time distributions for both queues, and obtain their mean waiting times.

1. Introduction

Polling systems used for modelling distributed multiqueue systems sharing a single scarce resource (i.e., server) such as a communication channel or a processor, have received a considerable amount of attention in the recent literature. Important examples of such distributed multiqueue systems are local area networks (LAN), high-speed Asynchronous Transfer Mode (ATM) networks, multiprocessor systems, distributed computation, distributed data bases, and so forth. Levy and Sidi[21], and Takagi[29],[30] have given detailed analyses and surveys on this subject. The special case of polling systems—that consisting of two queues and a single server, has an important application for modelling communication network systems with two different types of traffic: real-time traffic (such as voice and video) and non-real-time traffic (such as data), for example, hybrid switching voice/data transmission systems, and packet-switched voice/data transmission systems. In order to be able to meet the quality of service requirements for different types of traffic, various service schedules such as the exhaustive, gated, k-limited, Bernoulli and threshold service schedules or mixture of these service schedules have been considered ([1],[2],[3],[4],[7],[8],[9],[10],[14], [15],[16],[17],[19],[20],[21]).

Recently, some polling systems with the threshold-based service schedules have been analyzed by many authors. In [16], Lee considers a two-queue model with a single-server where the high priority queue is served exhaustively; the low priority queue is served by k-limited service. In [15], Lee and Sengupta analyze a model with a mixture of 1-limited and threshold service schedules, where a customer of each queue is served alternatively if the queue length of the high priority queue does not exceed a certain threshold level; otherwise only customers from the high priority queue are served until its queue length is back to the

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threshold level. In [20], Ozawa deals with a model with mixed exhaustive and k-limited service schedules, and in [14], Katayama and Takahashi analyze a model with mixture of 1-limited and Bernoulli service schedules. In [4], Boxma and Down consider a model with a mixture of exhaustive and threshold service schedules that is different from one in [15] only when the queue length of the high priority queue does not exceed a threshold level, the server serves the two queues exhaustively. In [10], Feng et al. analyze a model with two threshold M and $N(0 \leq M < N)$, where the server returns back to the low priority queue when the queue length of the high priority queue is less than or equal to the threshold level M. They determine respectively the generating functions of joint queue-length distributions and the Laplace-Stieltjes transforms of waiting time distribution. For polling systems with Bernoulli service schedule where the server decides with a probability which queue is going to be served next, Lee [17] considers a model without switching times, and Feng et al. [9] analyze the same model with switching times. Using the approach of the Riemann-Hilbert boundary value problem they derive the generating functions of joint queue-length distributions, the Laplace-Stieltjes transforms of waiting time distributions, and the mean waiting times.

In the present paper, we consider a single-server two-queue model with a mixture of Bernoulli and threshold service schedules called *Bernoulli-Threshold service schedules*. A threshold $N(0 < N)$ is set up in one of the two queues, say, the second queue. At each epoch of service completion in the first queue where the queue is not empty, if the queue length of the second queue exceeds the threshold N, the server switches the service to the second queue; otherwise with the probability p_1, it continues to serve the customers in the first queue, and with the probability $q_1 = 1 - p_1$, it switches the service to the second queue. At each epoch of service completion in the second queue, if the the queue-length in the second queue is larger than the threshold N, the server continues to sever the customers in the second queue, otherwise with the probability p_2 it continues to serve the customers in the second queue, and with the probability $q_2 = 1-p_2$, it switches the service to the first queue.

We are motivated to consider such a Bernoulli-Threshold service discipline for the polling system by the following two-fold. The first is its application-oriented. In modern telecommunication networks employing ATM switching technology, one important problem is to be able to meet the quality of service requirements for different types of traffic. One way of accomplishing this is that the server should have more flexibility to decide which queue should be served next, as long as the constraint conditions are guaranteed. To a communication network system with real-time traffic and non-real-time traffic, for example, when the queue length (or waiting time) of the real-time traffic is below a certain threshold, the server should be able to easily assign different priorities to the two traffics. The Bernoulli-Threshold service discipline, as can be seen, is one satisfying such requirements because that (i) the control threshold N can be simply determined, and (ii) below N, the server can devote more of its processing power to a queue with high priority by simply choosing an appropriate parameter p_i for each queue. In particular, if $p_1 = p_2 = 1$, the Bernoulli discipline reduces to the exhaustive discipline, then we have a model studied in [4]. Further, if $p_1 = p_2 = 0$, the Bernoulli discipline reduces to the 1-limited discipline, then we have a model studied in [15]. The second motivation is the interesting feature that it can be taken as an approximation of the Bernoulli service discipline. When $N = \infty$, the server serves two queues with the Bernoulli service discipline completely, then we get a model studied in [17] and [9]. In those two papers the approach of the Riemann-Hilbert boundary value problem is used to derive the generating functions of joint queue-length distributions, which is complex and difficult both in theoretical analysis and numerical calculation. Therefore when N is sufficiently large, the result obtained here by a different method can be taken as an approximation of that obtained in [17] and [9]. The main aim of this paper is to derive the generating functions of the joint stationary queue-length distributions, the Laplace-Stieltjes transforms of waiting time distributions, and the mean waiting times.

The organization of the paper is as follows. In Section 2 the model is described in detail, and the ergodicity condition and the system equations of the generating functions of the joint stationary queue-length distributions are established. The solutions of the system equations are derived in Section 3. A special case is considered in Section 4. By use of these solutions, the Laplace-Stieltjes transforms of waiting time distributions and the mean waiting times are given in Section 5. In Section 6, a conclusion is included.

2. The model and the generating function equations

We consider a cyclic-service queueing system consisting of two-parallel queues, Q_1 and Q_2 with infinite buffer capacities, which are served by a single server. The arrival processes of customers at Q_1(corresponding to the real-time traffic) and Q_2(corresponding to the non-real-time traffic) are Poisson processes with rates λ_1 and λ_2, respectively. For $i = 1, 2$, the service times at Q_i are independent, identically distributed random variables with general distribution $B_i(\cdot)$. Their first moment, second moment and LST (Laplace-Stieltjes Transform) are denoted by $b_i, b_i^{(2)}$, and $\hat{B}_i(\cdot)$, and assumed to be finite. A threshold $N(0 < N)$ is set up in the queue Q_2. The server serves two queues in accordance with a called Bernoulli-threshold dynamic service schedule described as follows:

(1) At each epoch of service completion in Q_1 at which the queue is not empty, if the queue-length in Q_2 exceeds the threshold N the server switches the service to Q_2; otherwise with the probability p_1, it continues to serve the customers in Q_1, and with the probability $q_1 = 1 - p_1$, it switches the service to Q_2.

(2) At each epoch of service completion in Q_2, if the queue-length in Q_2 is larger than the threshold N the server continues to serve the customers in Q_2, otherwise with probability p_2, it continues to serve the customers in Q_2, and with the probability $q_2 = 1 - p_2$, it switches the service to Q_1.

(3) Whenever the queue being served becomes empty at an epoch of service completion, if another queue is not empty the server switches the service to that queue; otherwise, the server remains idle at the present queue until the arrival of the next coming customer between Q_1 and Q_2.

The service is first-come-first-served within each queue and nonpreemptive. The server experiences a switching time in the transition from one queue to another. For $i = 1, 2$, the successive switching times from Q_i to $Q_{(i+1)mod 2}$ form independent, identically distributed random variables with general distribution $S_i(\cdot)$. Their first moment, second moment and LST are denoted by $s_i, s_i^{(2)}$, and $\hat{S}_i(\cdot)$, and assumed to be finite. All arrival, service and switching processes are assume to be independent.

2.1. Ergodicity condition

We introduce the following notations.

$$\begin{aligned}
&\lambda \equiv \lambda_1 + \lambda_2; && r_i = \lambda_i/\lambda, \quad i = 1, 2 \\
&\rho_i \equiv \lambda_i b_i, \quad i = 1, 2; && \rho \equiv \rho_1 + \rho_2 \\
&s \equiv s_1 + s_2; && s^{(2)} \equiv s_1^{(2)} + 2 s_1 s_2 + s_2^{(2)}.
\end{aligned} \tag{2.1}$$

ρ_i is the utilization at Q_i for $i = 1, 2$, and ρ is the total utilization of the server. s and $s^{(2)}$ are respectively the first moment, second moment of the total switching time during one cycle.

For a generally periodic polling systems with a mixture of various service schedules, Fricker and Jaïbi[11] have presented the following necessary and sufficient condition for the stability.

$$\rho + \max_{1\le i\le 2}(\lambda_i/L_i^*)s < 1 \tag{2.2}$$

where for $i = 1, 2$, L_i^* is the maximum expected number of customers served in the queue Q_i during a cycle. Appealing to the conclusion (2.2) we derive stable condition for the system considered here. Since N is finite, the service schedule in Q_2, in fact, is an exhaustive-type one. Especially, when $p_2 = 1$, it becomes a pure exhaustive service schedule. Hence, we have that $L_2^* = \infty$. The calculation of L_1^* for the queue Q_1 is more complicated because it depends not only on the number of the customers left by the server when it switched the service from Q_2 to Q_1, but also on the time that number of customers in the queue Q_2 reaches to the threshold N. For $i = 1, 2$, define the random variable L_i as follows.

$$L_i = \begin{cases} 1 & \text{if } p_i = 0, q_i = 1 \\ Y_i & \text{if } 0 < p_i, q_i < 1 \\ N & \text{if } p_i = 1, q_i = 0, \end{cases} \tag{2.3}$$

where Y_i is a random variable having the geometric distribution with parameter p_i, i.e., $P(Y_i = n) = q_i p_i^n, n \ge 0$. Then L_i represents the number of the customers served at Q_i during one visit cycle of the server. Let

$$\tau_2 = \min\{n;\quad S_2 + \sum_{j=1}^{n} B_1^j > \sum_{j=1}^{L_2} A_2^j\} \tag{2.4}$$

and
$$\tau_1 = \min\{L_1, \tau_2\}. \tag{2.5}$$

where S_2 is a generic switching time from Q_2 to Q_1; A_2^j and B_1^j are, for each j, generic interarrival times and service times for Q_2 and Q_1, respectively. Then we have $L_1^* = E[\tau_1]$. Now, the stable condition corresponding to our system is

$$\rho + \frac{\lambda_1 s}{E[\tau_1]} < 1. \tag{2.6}$$

According to the service discipline, if $\rho_2 < 1$, the set $\{0, 1, \cdots, N\}$ is regeneration one in the sense that the state of the queue Q_2 entries into it infinitely often. Therefore, we can also use the similar argument as in Boxma and Down [4] to give an explanation of the condition (2.6). In particular, when $p_1 = p_2 = 1, q_1 = q_2 = 0$, the condition (2.6) is consistent with one given there. Throughout the paper we assume that the condition (2.6) holds.

2.2. The generating function equations

Let $\{t_k, k \ge 1\}$ be the successive epochs of service completion, $X_k^{(i)}$, $i = 1, 2, k \ge 1$, the number of customers at Q_i at instant immediately after t_k, and J_k, $k \ge 1$, the type of the departing customer at t_k, i.e., $J_k = i$ if the kth departing customer is from Q_i. Then $\{(X_k^{(1)}, X_k^{(2)}, J_k)\}_{k\ge 1}$ forms an imbedded vector Markov chain. Let $\{\pi_{n,m,i}; n, m \ge 0, i =$

$1, 2\}$ denote the equilibrium probabilities of $\{(X_k^{(1)}, X_k^{(2)}, J_k)\}_{k\geq 1}$, namely,

$$\pi_{n,m,i} \equiv \lim_{k\to\infty} P((X_k^{(1)}, X_k^{(2)}, J_k) = (n, m, i)). \tag{2.7}$$

For $\mid z_1 \mid\leq 1; \quad \mid z_2 \mid\leq 1$, define the generating functions

$$\Phi(z_1, z_2) \equiv \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \pi_{n,m,1} z_1^n z_2^m, \tag{2.8}$$

$$\Psi(z_1, z_2) \equiv \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \pi_{n,m,2} z_1^n z_2^m. \tag{2.9}$$

Considering the transition probabilities of the imbedded vector Markov chain during two successive service completion epochs, we derive the following equations for the generating functions $\Phi(z_1, z_2)$ and $\Psi(z_1, z_2)$:

$$\begin{aligned}\Phi(z_1, z_2) = {} & r_1\hat{B}_1(\lambda_1(1-z_1)+\lambda_2(1-z_2))[\Phi(0,0)+\hat{S}_2(\lambda_1(1-z_1)+\lambda_2(1-z_2))\Psi(0,0)] \\ & +z_1^{-1}\hat{B}_1(\lambda_1(1-z_1)+\lambda_2(1-z_2))\{\Phi(z_1,0)-\Phi(0,0) \\ & +\hat{S}_2(\lambda_1(1-z_1)+\lambda_2(1-z_2))(\Psi(z_1,0)-\Psi(0,0))+p_1\sum_{m=1}^{N}\sum_{n=1}^{\infty}\pi_{n,m,1}z_1^n z_2^m \\ & +q_2\hat{S}_2(\lambda_1(1-z_1)+\lambda_2(1-z_2))\sum_{m=1}^{N}\sum_{n=1}^{\infty}\pi_{n,m,2}z_1^n z_2^m\},\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}\Psi(z_1, z_2) = {} & r_2\hat{B}_2(\lambda_1(1-z_1)+\lambda_2(1-z_2))[\hat{S}_1(\lambda_1(1-z_1)+\lambda_2(1-z_2))\Phi(0,0)+\Psi(0,0)] \\ & +z_2^{-1}\hat{B}_2(\lambda_1(1-z_1)+\lambda_2(1-z_2))\{\hat{S}_1(\lambda_1(1-z_1)+\lambda_2(1-z_2)) \\ & \quad\times(\Phi(0,z_2)-\Phi(0,0))+(\Psi(0,z_2)-\Psi(0,0)) \\ & +\hat{S}_1(\lambda_1(1-z_1)+\lambda_2(1-z_2))[q_1\sum_{m=1}^{N}\sum_{n=1}^{\infty}\pi_{n,m,1}z_1^n z_2^n+\sum_{m=N+1}^{\infty}\sum_{n=1}^{\infty}\pi_{n,m,1}z_1^n z_2^n] \\ & +p_2\sum_{m=1}^{N}\sum_{n=1}^{\infty}\pi_{n,m,2}z_1^n z_2^n+\sum_{m=N+1}^{\infty}\sum_{n=1}^{\infty}\pi_{n,m,2}z_1^n z_2^n\}.\end{aligned} \tag{2.11}$$

For clarity, define

$$B_i^*(z_1, z_2) = \hat{B}_i(\lambda_1(1-z_1)+\lambda_2(1-z_2)), \qquad i = 1, 2,$$

$$S_i^*(z_1, z_2) = \hat{S}_i(\lambda_1(1-z_1)+\lambda_2(1-z_2)), \qquad i = 1, 2,$$

and for $\mid z_1 \mid\leq 1$, define the one-dimensional generating functions of the joint equilibrium probabilities $\{\pi_{n,m,1}; n \geq 0\}$ and $\{\pi_{n,m,2}; n \geq 0\}, \quad 0 \leq m \leq N$,

$$\varphi_m(z_1) \equiv \sum_{n=0}^{\infty} \pi_{n,m,1} z_1^n, \qquad 0 \leq m \leq N, \tag{2.12}$$

$$\psi_m(z_1) \equiv \sum_{n=0}^{\infty} \pi_{n,m,2} z_1^n, \qquad 0 \leq m \leq N. \tag{2.13}$$

We have

$$\sum_{m=N+1}^{\infty}\sum_{n=1}^{\infty}\pi_{n,m,1}z_1^n z_2^n = \Phi(z_1,z_2)-\Phi(0,z_2)-\sum_{m=0}^{N}(\varphi_m(z_1)-\varphi_m(0))z_2^m, \tag{2.14}$$

$$\sum_{m=N+1}^{\infty}\sum_{n=1}^{\infty}\pi_{n,m,2}z_1^n z_2^n = \Psi(z_1,z_2)-\Psi(0,z_2)-\sum_{m=0}^{N}(\psi_m(z_1)-\psi_m(0))z_2^m, \tag{2.15}$$

and in particular,

$$\varphi_0(0)=\Phi(0,0)=\pi_{0,0,1}, \qquad \psi_0(0)=\Psi(0,0)=\pi_{0,0,2}. \tag{2.16}$$

Using the above notations and relations, equations (2.10) and (2.11) can be rewritten as

$$\begin{aligned}\Phi(z_1,z_2) &= (r_1-z_1^{-1})B_1^*(z_1,z_2)[\varphi_0(0)+S_2^*(z_1,z_2)\psi_0(0)]+z_1^{-1}B_1^*(z_1,z_2)\\ &\times\{\ \varphi_0(z_1)+S_2^*(z_1,z_2)\psi_0(z_1)+p_1\sum_{m=1}^{N}(\varphi_m(z_1)-\varphi_m(0))z_2^m\\ &+q_2S_2^*(z_1,z_2)\sum_{m=1}^{N}(\psi_m(z_1)-\psi_m(0))\,z_2^m\ \},\end{aligned} \tag{2.17}$$

$$\begin{aligned}\Psi(z_1,z_2) &= \frac{B_2^*(z_1,z_2)}{z_2-B_2^*(z_1,z_2)}\{\ [r_2z_2+(r_1-z_1^{-1})B_1^*(z_1,z_2)]S_1^*(z_1,z_2)\varphi_0(0)\\ &+[r_2z_2+(r_1-z_1^{-1})B_1^*(z_1,z_2)S_1^*(z_1,z_2)S_2^*(z_1,z_2)]\psi_0(0)\\ &+\frac{(B_1^*(z_1,z_2)-z_1)S_1^*(z_1,z_2)}{z_1}[\varphi_0(z_1)+p_1\sum_{m=1}^{N}(\varphi_m(z_1)-\varphi_m(0))\,z_2^m]\\ &+\frac{B_1^*(z_1,z_2)S_1^*(z_1,z_2)S_2^*(z_1,z_2)-z_1}{z_1}[\psi_0(z_1)+q_2\sum_{m=1}^{N}(\psi_m(z_1)-\psi_m(0))\,z_2^m]\ \}.\end{aligned} \tag{2.18}$$

As can be seen, the generating functions $\Phi(z_1,z_2)$ and $\Psi(z_1,z_2)$ are completely determined by the one-dimensional generating functions $\varphi_m(z_1)$ and $\psi_m(z_1), 0\le m\le N$. In order to solve $\varphi_m(z_1)$ and $\psi_m(z_1), 0\le m\le N$, we need to derive more equations about these unknown functions. This can be done by considering the balance equations for $\{\pi_{n,m,1}; n\ge 0\}$ and $\{\pi_{n,m,2}; n\ge 0\}$, $0\le m\le N$. For every m, $1\le m\le N$, first, we have

$$\begin{aligned}\pi_{n,m,1} &= q_2\sum_{i=1}^{n+1}\sum_{j=0}^{m}\pi_{i,j,2}\int_0^\infty\frac{(\lambda_1t)^{n-i+1}}{(n-i+1)!}e^{-\lambda_1t}\frac{(\lambda_2t)^{m-j}}{(m-j)!}e^{-\lambda_2t}dF_{S_2+B_1}(t)\\ &+p_2\sum_{i=1}^{n+1}\pi_{i,0,2}\int_0^\infty\frac{(\lambda_1t)^{n-i+1}}{(n-i+1)!}e^{-\lambda_1t}\frac{(\lambda_2t)^{m}}{m!}e^{-\lambda_2t}dF_{S_2+B_1}(t)\\ &+r_1\pi_{0,0,2}\int_0^\infty\frac{(\lambda_1t)^{n}}{n!}e^{-\lambda_1t}\frac{(\lambda_2t)^{m}}{m!}e^{-\lambda_2t}dF_{S_2+B_1}(t)\\ &+p_1\sum_{i=1}^{n+1}\sum_{j=0}^{m}\pi_{i,j,1}\int_0^\infty\frac{(\lambda_1t)^{n-i+1}}{(n-i+1)!}e^{-\lambda_1t}\frac{(\lambda_2t)^{m-j}}{(m-j)!}e^{-\lambda_2t}dB_1(t)\\ &+q_1\sum_{i=1}^{n+1}\pi_{i,0,1}\int_0^\infty\frac{(\lambda_1t)^{n-i+1}}{(n-i+1)!}e^{-\lambda_1t}\frac{(\lambda_2t)^{m}}{m!}e^{-\lambda_2t}dB_1(t)\end{aligned} \tag{2.19}$$

$$+r_1\pi_{0,0,1}\int_0^\infty \frac{(\lambda_1 t)^n}{n!}e^{-\lambda_1 t}\frac{(\lambda_2 t)^m}{m!}e^{-\lambda_2 t}dB_1(t), \qquad 0\le n<\infty,$$

and for $m=0$, we have

$$\pi_{n,0,1}=\sum_{i=1}^{n+1}\pi_{i,0,2}\int_0^\infty \frac{(\lambda_1 t)^{n-i+1}}{(n-i+1)!}e^{-\lambda_1 t}e^{-\lambda_2 t}dF_{S_2+B_1}(t) \tag{2.20}$$

$$+r_1\pi_{0,0,2}\int_0^\infty \frac{(\lambda_1 t)^n}{n!}e^{-\lambda_1 t}e^{-\lambda_2 t}dF_{S_2+B_1}(t)$$

$$+\sum_{i=1}^{n+1}\pi_{i,0,1}\int_0^\infty \frac{(\lambda_1 t)^{n-i+1}}{(n-i+1)!}e^{-\lambda_1 t}e^{-\lambda_2 t}dB_1(t)$$

$$+r_1\pi_{0,0,1}\int_0^\infty \frac{(\lambda_1 t)^n}{n!}e^{-\lambda_1 t}e^{-\lambda_2 t}dB_1(t), \qquad 0\le n<\infty,$$

where $F_{S_2+B_1}(\cdot)$ denotes the distribution of the sum of the service time in Q_1 and the switching time from Q_2 to Q_1.

From (2.19) and (2.20), multiplying nth equation by z_1^n and summing yield

$$\varphi_m(z_1)=\frac{1}{z_1}\{\ q_2\sum_{j=0}^{m}H_{1,m-j}(z_1)(\psi_j(z_1)-\psi_j(0))+p_2H_{1,m}(z_1)(\psi_0(z_1)-\psi_0(0)) \tag{2.21}$$

$$+p_1\sum_{j=0}^{m}G_{1,m-j}(z_1)(\varphi_j(z_1)-\varphi_j(0))+q_1G_{1,m}(z_1)(\varphi_0(z_1)-\varphi_0(0))$$

$$+r_1z_1H_{1,m}(z_1)\psi_0(0)+r_1z_1G_{1,m}(z_1)\varphi_0(0)\ \}, \qquad 1\le m\le N,$$

and

$$\varphi_0(z_1)=\frac{1}{z_1}\{\ H_{1,0}(z_1)(\psi_0(z_1)-\psi_0(0))+G_{1,0}(z_1)(\varphi_0(z_1)-\varphi_0(0)) \tag{2.22}$$

$$+r_1z_1H_{1,0}(z_1)\psi_0(0)+r_1z_1G_{1,0}(z_1)\varphi_0(0)\ \},$$

where

$$H_{1,j}(z_1)\equiv\int_0^\infty \frac{(\lambda_2 t)^j}{j!}e^{-\lambda_2 t}e^{-\lambda_1(1-z_1)t}dF_{S_2+B_1}(t)$$

$$=\frac{1}{j!}\frac{\partial^j}{\partial z_2^j}\hat{F}_{S_2+B_1}(\lambda_1(1-z_1)+\lambda_2(1-z_2))\,|_{z_2=0}, \qquad 0\le j\le N,$$

$$G_{1,j}(z_1)\equiv\int_0^\infty \frac{(\lambda_2 t)^j}{j!}e^{-\lambda_2 t}e^{-\lambda_1(1-z_1)t}dB_1(t)$$

$$=\frac{1}{j!}\frac{\partial^j}{\partial z_2^j}\hat{B}_1(\lambda_1(1-z_1)+\lambda_2(1-z_2))\,|_{z_2=0}, \qquad 0\le j\le N.$$

Note that $H_{1,j}(1)$ is the probability that there are j arrivals during the period of the switching time from Q_2 to Q_1 and the service time at Q_1, and $G_{1,j}(1)$ the probability that there are j arrivals during the period of the service time at Q_1. Next, for every m, $0\le m\le N-1$, we have

$$\pi_{n,m,2}=q_1\sum_{j=1}^{m+1}\sum_{i=0}^{n}\pi_{i,j,1}\int_0^\infty \frac{(\lambda_1 t)^{n-i}}{(n-i)!}e^{-\lambda_1 t}\frac{(\lambda_2 t)^{m-j+1}}{(m-j+1)!}e^{-\lambda_2 t}dF_{S_1+B_2}(t) \tag{2.23}$$

$$+p_1 \sum_{j=1}^{m+1} \pi_{0,j,1} \int_0^\infty \frac{(\lambda_1 t)^n}{n!} e^{-\lambda_1 t} \frac{(\lambda_2 t)^{m-j+1}}{(m-j+1)!} e^{-\lambda_2 t} dF_{S_1+B_2}(t)$$

$$+r_2 \pi_{0,0,1} \int_0^\infty \frac{(\lambda_1 t)^n}{n!} e^{-\lambda_1 t} \frac{(\lambda_2 t)^m}{m!} e^{-\lambda_2 t} dF_{S_1+B_2}(t)$$

$$+p_2 \sum_{j=1}^{m+1} \sum_{i=0}^{n} \pi_{i,j,2} \int_0^\infty \frac{(\lambda_1 t)^{n-j}}{(n-j)!} e^{-\lambda_1 t} \frac{(\lambda_2 t)^{m-j+1}}{(m-j+1)!} e^{-\lambda_2 t} dB_2(t)$$

$$+q_2 \sum_{j=1}^{m+1} \pi_{0,j,2} \int_0^\infty \frac{(\lambda_1 t)^n}{n!} e^{-\lambda_1 t} \frac{(\lambda_2 t)^{m-j+1}}{(m-j+1)!} e^{-\lambda_2 t} dB_2(t)$$

$$+r_2 \pi_{0,0,2} \int_0^\infty \frac{(\lambda_1 t)^n}{n!} e^{-\lambda_1 t} \frac{(\lambda_2 t)^m}{m!} e^{-\lambda_2 t} dB_2(t), \qquad 0 \le n < \infty,$$

where $F_{S_1+B_2}(\cdot)$ denotes the distribution of the sum of the service time in Q_2 and the switching time from Q_1 to Q_2. In particular, we have

$$\pi_{0,0,2} = \pi_{0,1,1} \int_0^\infty e^{-(\lambda_1+\lambda_2)t} dF_{S_1+B_2}(t) + r_2 \pi_{0,0,1} \int_0^\infty e^{-(\lambda_1+\lambda_2)t} dF_{S_1+B_2}(t) \tag{2.24}$$

$$+\pi_{0,1,2} \int_0^\infty e^{(-\lambda_1+\lambda_2)t} dB_2(t) + r_2 \pi_{0,0,2} \int_0^\infty e^{(-\lambda_1+\lambda_2)t} dB_2(t).$$

From (2.23) and (2.24), multiplying nth equation by z_1^n and summing yield

$$\psi_m(z_1) = q_1 \sum_{j=1}^{m+1} H_{2,m-j+1}(z_1)\varphi_j(z_1) + p_1 \sum_{j=1}^{m+1} H_{2,m-j+1}(z_1)\varphi_j(0) \tag{2.25}$$

$$+p_2 \sum_{j=1}^{m+1} G_{2,m-j+1}(z_1)\psi_j(z_1) + q_2 \sum_{j=1}^{m+1} G_{2,m-j+1}(z_1)\psi_j(0)$$

$$+r_2 H_{2,m}(z_1)\varphi_0(0) + r_2 G_{2,m}(z_1)\psi_0(0), \qquad 0 \le m \le N-1,$$

where

$$H_{2,j}(z_1) \equiv \int_0^\infty \frac{(\lambda_2 t)^j}{j!} e^{-\lambda_2 t} e^{-\lambda_1(1-z_1)t} dF_{S_1+B_2}(t)$$

$$= \frac{1}{j!} \frac{\partial^j}{\partial z_2^j} \hat{F}_{S_1+B_2}(\lambda_1(1-z_1) + \lambda_2(1-z_2)) \,|_{z_2=0}, \qquad 0 \le j \le N,$$

$$G_{2,j}(z_1) \equiv \int_0^\infty \frac{(\lambda_2 t)^j}{j!} e^{-\lambda_2 t} e^{-\lambda_1(1-z_1)t} dB_2(t)$$

$$= \frac{1}{j!} \frac{\partial^j}{\partial z_2^j} \hat{B}_2(\lambda_1(1-z_1) + \lambda_2(1-z_2)) \,|_{z_2=0}, \qquad 0 \le j \le N.$$

Also note that $H_{2,j}(1)$ is the probability that there are j arrivals during the switching time from Q_1 to Q_2 and the service time at Q_2, and $G_{2,j}(1)$ the probability that there are j arrivals during the service time at Q_2. Finally, we get $\psi_m(0)$ by substituting $z = 0$ into (2.25)

$$\psi_m(0) = \sum_{j=1}^{m+1} H_{2,m-j+1}(0)\varphi_j(0) + \sum_{j=1}^{m+1} G_{2,m-j+1}(0)\psi_j(0) \tag{2.26}$$

$$+r_2 H_{2,m}(0)\varphi_0(0) + r_2 G_{2,m}(0)\psi_0(0).$$

3. Determination of the generating functions

In this section, we derive the generating functions $\Phi(z_1, z_2)$ and $\Psi(z_1, z_2)$ of the equilibrium probabilities of the queue lengths. The function equations (2.17) and (2.18) show that $\Phi(z_1, z_2)$ and $\Psi(z_1, z_2)$ can be obtained as long as the one-dimensional generating functions $\varphi_m(z_1)$ and $\psi_m(z_1)$, $0 \leq m \leq N$ are determined. Therefore, the main aim here is to deduce system equations about these one-dimensional generating functions by using (2.18), (2.21),(2.22) and (2.25), and obtain their solutions.

3.1. The solution of the equations

First, we consider the equation (2.18). According to Takás Theorem, we have that for each fixed z_1 with $| z_1 |\leq 1$, the equation

$$z_2 - B_2(\lambda_1(1-z_1) + \lambda_2(1-z_2)) = 0 \tag{3.1}$$

has exactly one root in the region $| z_2 |\leq 1$. Actually, the root satisfies

$$z_2 = \hat{V}(\lambda_1(1-z_1)) \tag{3.2}$$

where $\hat{V}(s)$ is the LST of the busy period distribution of an M/G/1 queue with arrival rate λ_2 and service time distribution $B_2(\cdot)$. Denoting this root by $z_2 = \eta(z_1)$, we have $\eta(z_1) = \hat{V}(\lambda_1(1-z_1))$. Furthermore, $\eta(1) = 1$, and

$$\frac{d}{dz_1}\eta(z_1) \mid_{z_1=1} = \frac{\lambda_1 b_2}{1-\rho_2}, \qquad \frac{d^2}{dz_1^2}\eta(z_1) \mid_{z_1=1} = \frac{\lambda_1^2 b_2^{(2)}}{(1-\rho_2)^3}. \tag{3.3}$$

Since $\Psi(z_1, z_2)$ should be regular for $| z_2 |< 1$, and continuous for $| z_2 |\leq 1$, for each fixed z_1 with $| z_1 |\leq 1$ the numerator of (2.18) must vanish at $z_2 = \eta(z_1)$. Substituting this root into (2.18) and rearranging items we have

$$\begin{aligned}
&(B_1^*(z,\eta(z)) - z)S_1^*(z,\eta(z))[\varphi_0(z) + p_1\sum_{m=1}^{N}(\varphi_m(z) - \varphi_m(0))\eta^m(z)] \\
&+(B_1^*(z,\eta(z))S_1^*(z,\eta(z))S_2^*(z,\eta(z)) - z)[\psi_0(z) + q_2\sum_{m=1}(\psi_m(z) - \psi_m(0))\eta^m(z)] \\
&+[r_2 z\eta(z) + (r_1 z - 1)B_1^*(z,\eta(z))]S_1^*(z,\eta(z))\varphi_0(0) \\
&+[r_2 z\eta(z) + (r_1 z - 1)B_1^*(z,\eta(z))S_1^*(z,\eta(z)))S_2^*(z,\eta(z))]\psi_0(0) = 0.
\end{aligned} \tag{3.4}$$

Since the equations discussed hereafter are mainly those about the argument z_1, we write z instead of z_1 for simplicity. Next, rewrite (2.21) and (2.22) as follows

$$\begin{aligned}
&-G_{1,m}(z)\varphi_0(z) - p_1\sum_{j=1}^{m-1}G_{1,m-j}(z)\varphi_j(z) + (z - p_1G_{1,0}(z))\varphi_m(z) \\
&= H_{1,m}(z)\psi_0(z) + q_2\sum_{j=1}^{m}H_{1,m-j}(z)(\psi_j(z) - \psi_j(0)) - p_1\sum_{j=1}^{m}G_{1,m-j}(z)\varphi_j(0) \\
&\qquad +(r_1 z - 1)G_{1,m}(z)\varphi_0(0) + (r_1 z - 1)H_{1,m}(z)\psi_0(0), \qquad 1 \leq m \leq N,
\end{aligned} \tag{3.5}$$

and

$$(z - G_{1,0}(z))\varphi_0(z) = H_{1,0}(z)\psi_0(z) + (r_1 z - 1)H_{1,0}(z)\psi_0(0) + (r_1 z - 1)G_{1,0}(z)\varphi_0(0). \tag{3.6}$$

Then rewrite (2.25) as follows

$$-p_2 \sum_{j=1}^{m-1} G_{2,m-j+1}(z)\psi_j(z) + (1-p_2G_{2,1}(z))\psi_m(z) - p_2G_{2,0}(z)\psi_{m+1}(z) \tag{3.7}$$

$$= q_1 \sum_{j=1}^{m+1} H_{2,m-j+1}(z)\varphi_j(z) + p_1 \sum_{j=1}^{m+1} H_{2,m-j+1}(z)\varphi_j(0) + q_2 \sum_{j=1}^{m+1} G_{2,m-j+1}(z)\psi_j(0)$$
$$+r_2H_{2,m}(z)\varphi_0(0) + r_2G_{2,m}(z)\psi_0(0), \quad 1 \le m \le N-1,$$

and

$$\psi_0(z) - p_2G_{2,0}(z)\psi_1(z) = q_1H_{2,0}(z)\varphi_1(z) + p_1H_{2,0}(z)\varphi_1(0) + q_2G_{2,0}(z)\psi_1(0) \tag{3.8}$$
$$+r_2H_{2,0}(z)\varphi_0(0) + r_2G_{2,0}(z)\psi_0(0).$$

Furthermore, we have from (2.26)

$$-r_2G_{2,m}(0)\psi_0(0) - \sum_{j=1}^{m-1} G_{2,m-j+1}(z)\psi_j(0) + (1-G_{2,1}(0))\psi_m(0) - G_{2,0}(0)\psi_{m+1}(0)$$

$$= \sum_{j=1}^{m+1} H_{2,m-j+1}\varphi_j(0) + r_2H_{2,m}(0)\varphi_0(0), \quad 1 \le m \le N-1, \tag{3.9}$$

and

$$(1-r_2G_{2,0}(0))\psi_0(0) - G_{2,0}(0)\psi_1(0) = H_{2,0}(0)\varphi_1(0) + r_2H_{2,0}(0)\varphi_0(0). \tag{3.10}$$

Define the vectors

$$\boldsymbol{\varphi}(z) = (\varphi_0(z), \varphi_1(z), \cdots, \varphi_N(z))^\tau, \quad \boldsymbol{\psi}(z) = (\psi_0(z), \psi_1(z), \cdots, \psi_N(z))^\tau. \tag{3.11}$$

Writing (3.5) and (3.6) in the matrix form we obtain a matrix equation

$$A_1(z)\boldsymbol{\varphi}(z) = H_{1a}(z)\boldsymbol{\psi}(z) + H_{1b}(z)\boldsymbol{\psi}(0)) + G_1(z)\boldsymbol{\varphi}(0) \tag{3.12}$$

where $A_1(z) = (a^1_{ij}(z))$, $H_{1a}(z) = (h^{1a}_{ij}(z))$, $H_{1b}(z) = (h^{1b}_{ij}(z))$, and $G_1(z) = (g^1_{ij}(z))$ are all the $(N+1)\times(N+1)$ matrices.

$$a^1_{ij}(z) = \begin{cases} z - G_{1,0}(z) & \text{if } j = i = 1 \\ z - p_1G_{1,0}(z) & \text{if } j = i, i = 2, \cdots, N+1 \\ -G_{1,i-1}(z) & \text{if } j = 1, i = 2, \cdots, N+1 \\ -p_1G_{1,i-j}(z) & \text{if } j < i, i = 3, \cdots, N+1 \\ 0 & \text{if } j > i, i = 1, 2, \cdots, N, \end{cases}$$

$$h^{1a}_{ij}(z) = \begin{cases} H_{1,i-1}(z) & \text{if } j = 1, i = 1, \cdots, N+1 \\ q_2H_{1,i-j}(z) & \text{if } j \le i, i = 2, \cdots, N+1 \\ 0 & \text{if } j > i, i = 1, 2, \cdots, N, \end{cases}$$

$$h^{1b}_{ij}(z) = \begin{cases} (r_1z-1)H_{1,i-1}(z) & \text{if } j = 1, i = 1, \cdots, N+1 \\ -q_2H_{1,i-j}(z) & \text{if } j \le i, i = 2, \cdots, N+1 \\ 0 & \text{if } j > i, i = 1, 2, \cdots, N, \end{cases}$$

$$g^1_{ij}(z) = \begin{cases} (r_1z-1)G_{1,i-1}(z) & \text{if } j = 1, i = 2, \cdots, N+1 \\ -p_1G_{1,i-j}(z) & \text{if } j \le i, i = 2, \cdots, N+1 \\ 0 & \text{if } j > i, i = 1, 2, \cdots, N. \end{cases}$$

Note that $z = 1$ is the unique, simple zero of the equation $z - B_1^*(z, \eta(z))S_1^*(z, \eta(z)) S_2^*(z, \eta(z)) = 0$. We get the following equation from (3.4).

$$\psi_0(z) + q_2 \sum_{m=1}^{N} \psi_m(z)\eta^m(z) = \alpha(z)\varphi_0(z) + p_1\alpha(z) \sum_{m=1}^{N} (\varphi_m(z) - \varphi_m(0))\eta^m(z) \tag{3.13}$$

$$+\beta_1(z)\varphi_0(0) + q_2 \sum_{m=1}^{N} \psi_m(0)\eta^m(z) + \beta_2(z)\psi_0(0),$$

where

$$\alpha(z) \equiv \frac{(B_1^*(z, \eta(z)) - z)S_1^*(z, \eta(z))}{z - B_1^*(z, \eta(z))S_1^*(z, \eta(z))S_2^*(z, \eta(z))}, \tag{3.12}$$

$$\beta_1(z) \equiv \frac{(r_2 z\eta(z) + (r_1 z - 1)B_1^*(z, \eta(z)))S_1^*(z, \eta(z))}{z - B_1^*(z, \eta(z))S_1^*(z, \eta(z))S_2^*(z, \eta(z))}, \tag{3.15}$$

$$\beta_2(z) \equiv \frac{r_2 z\eta(z) + (r_1 z - 1)B_1^*(z, \eta(z)))S_1^*(z, \eta(z))S_2^*(z, \eta(z))}{z - B_1^*(z, \eta(z))S_1^*(z, \eta(z))S_2^*(z, \eta(z))}, \tag{3.16}$$

and for $z = 1$,

$$\alpha(1) \equiv \lim_{z\to 1} \alpha(z) = -\frac{1 - \rho_1 - \rho_2}{1 - \rho_1 - \rho_2 - \lambda_1 s_1},$$

$$\beta_1(1) \equiv \lim_{z\to 1} \beta_1(z) = \frac{1 - \rho_2 + r_2(\rho_1 - \lambda_1 b_2)}{1 - \rho_1 - \rho_2 - \lambda_1 s_1},$$

$$\beta_2(1) \equiv \lim_{z\to 1} \beta_2(z) = \frac{1 - \rho_2 + r_2(\rho_1 + \lambda_1(s_1 + s_2) - \lambda_1 b_2)}{1 - \rho_1 - \rho_2 - \lambda_1 s_1}.$$

Rewriting (3.7), (3.8) and (3.13) in the matrix form yields another matrix equation.

$$G_{2a}(z)\boldsymbol{\psi}(z) = A_2(z)\boldsymbol{\varphi}(z) + H_2(z)\boldsymbol{\varphi}(0) + G_{2b}(z)\boldsymbol{\psi}(0) \tag{3.17}$$

where $A_2(z) = (a_{ij}^2(z))$, $G_{2a}(z) = (g_{ij}^{2a}(z))$, $G_{2b}(z) = (g_{ij}^{2b}(z))$, and $H_2(z) = (h_{ij}^2(z))$ are all the $(N+1) \times (N+1)$ matrices.

$$a_{ij}^2(z) = \begin{cases} q_1 H_{2,i-j+1}(z) & \text{if } 2 \le j \le i+1, i = 1, \cdots, N \\ p_1\alpha(z)\eta^{j-1}(z) & \text{if } j = 2, \cdots, N+1, i = N+1 \\ \alpha(z) & \text{if } j = 1, i = N+1 \\ 0 & \text{if } j = 1, i = 1, \cdots, N \\ & \text{and } j > i+1, i = 1, \cdots, N-1, \end{cases}$$

$$g_{ij}^{2a}(z) = \begin{cases} -p_2 G_{2,i-j+1}(z) & \text{if } 2 \le j \le i+1, j \ne i, i = 1, \cdots, N \\ 1 - p_2 G_{2,1}(z) & \text{if } j = i, i = 2, \cdots, N \\ q_2\eta^{j-1}(z) & \text{if } j = 2, \cdots, N+1, i = N+1 \\ 1 & \text{if } j = 1, i = 1, N+1 \\ 0 & \text{if } j = 1, i = 2, \cdots, N \\ & \text{and } j > i+1, i = 1, \cdots, N-1, \end{cases}$$

$$g_{ij}^{2b}(z) = \begin{cases} q_2 G_{2,i-j+1}(z) & \text{if } 2 \le j \le i+1, i = 1, \cdots, N \\ r_2 G_{2,i-1}(z) & \text{if } j = 1, i = 1, \cdots, N \\ q_2\eta^{j-1}(z) & \text{if } j = 2, \cdots, N+1, i = N+1 \\ \beta_2(z) & \text{if } j = 1, i = N+1 \\ 0 & \text{if } j > i+1, i = 1, \cdots, N-1, \end{cases}$$

$$h^2_{ij}(z) = \begin{cases} p_1 H_{2,i-j+1}(z) & \text{if } 2 \le j \le i+1, i = 1, \cdots, N \\ r_2 H_{2,i-1}(z) & \text{if } j = 1, i = 1, \cdots, N \\ -p_1 \alpha(z) \eta^{j-1}(z) & \text{if } j = 2, \cdots, N+1, i = N+1 \\ \beta_1(z) & \text{if } j = 1, i = N+1 \\ 0 & \text{if } j > i+1, i = 1, \cdots, N-1. \end{cases}$$

Furthermore, we can write (3.9) and (3.10) as

$$V_G \hat{\boldsymbol{\psi}}(0) = V_H \boldsymbol{\varphi}(0) + \psi_N(0) u, \tag{3.18}$$

where $\hat{\boldsymbol{\psi}}(0) = (\psi_0(0), \psi_1(0), \cdots, \psi_{N-1}(0))^\tau$, $u = (0, \cdots, 0, G_{2,0}(0))^\tau$, and $V_G = (v^G_{ij})$ is a $N \times N$ matrix, and $V_H = (v^H_{ij})$ is a $N \times (N+1)$

$$v^G_{ij} = \begin{cases} -G_{2,i-j+1}(0) & \text{if } 2 \le j \le i+1, j \ne i, i = 1, \cdots, N-1 \\ 1 - G_{2,1}(0) & \text{if } j = i, i = 2, \cdots, N \\ 1 - r_2 G_{2,1}(0) & \text{if } j = i = 1 \\ -r_2 G_{2,i-1}(0) & \text{if } j = 1, i = 2, \cdots, N \\ 0 & \text{if } j > i+1, i = 1, \cdots, N-1, \end{cases}$$

$$v^H_{ij} = \begin{cases} H_{2,i-j+1}(0) & \text{if } 2 \le j \le i+1, i = 1, \cdots, N \\ r_2 H_{2,i-1}(0) & \text{if } j = 1, i = 1, \cdots, N \\ 0 & \text{if } j > i+1, i = 1, \cdots, N-1. \end{cases}$$

Note that both the matrices $G_{2a}(z)$ and V_G are quasi-lower-triangular, i.e., $g^{2a}_{ij}(z) = 0$ for $i+1 < j, i = 1, \cdots, N-1$ and $v^G_{ij} = 0$ for $i+1 < j, i = 1, \cdots N-2$. Since $0 <\mid p_2 G_{2,0}(z) \mid \le 1$ for all $\mid z \mid \le 1$, obviously, the first two columns of the matrix $G_{2a}(z)$ are mutually independent, and that is also true for the last two rows. Similar conclusion also holds for the matrix V_G. We have that the inverses of $G_{2a}(z)$ and V_G exist. Substituting (3.17) into (3.12), and then (3.18) into the resulting formula yield

$$\mathcal{N}(z) \boldsymbol{\varphi}(z) = F(z) \boldsymbol{\varphi}(0) + \psi_N(0) E(z) \tag{3.19}$$

where

$$E(z) \equiv V_G^{-1} u + v(z),$$

$$\mathcal{N}(z) \equiv A_1(z) - H_{1a}(z) G_{2a}^{-1}(z) A_2(z),$$

$$F(z) \equiv H_{1a}(z) G_{2a}^{-1}(z) H_2(z) + G_1(z) + H_G(z) V_G^{-1} V_H,$$

and $H_G(z)$ is a $(N+1) \times N$ matrix obtained from the matrix $H_{1a}(z) G_{2a}^{-1}(z) G_{2b}(z) + H_{1b}(z)$ by deleting its $(N+1)$th column, and $v(z)$ is a $(N+1)$-dimensional vector equal to the $(N+1)$th column of the matrix $H_{1a}(z) G_{2a}^{-1}(z) G_{2b}(z) + H_{1b}(z)$.

Whenever $\mathcal{N}(z)$ is non-singular, the solutions of (3.19) are given by

$$\boldsymbol{\varphi}(z) = \mathcal{N}^{-1}(z) \{F(z) \boldsymbol{\varphi}(0) + \psi_N(0) E(z)\} \tag{3.20}$$

which may also be written as

$$\boldsymbol{\varphi}(z) = \frac{[adj \mathcal{N}(z)] \{F(z) \boldsymbol{\varphi}(0) + \psi_N(0) E(z)\}}{det \mathcal{N}(z)}. \tag{3.21}$$

Since we seek $\boldsymbol{\varphi}(z)$ which is analytic in $\mid z \mid \le 1$, the numerator of the right-hand side of (3.21) must vanish at the zeros of $det \mathcal{N}(z)$ inside the unit circle $\mid z \mid = 1$. Therefore, in solving (3.19) we have to consider the character of those zeros. Suppose there are K such zeros, $z_1, z_2, \cdots, z_K$. Let $\mathcal{N}_i(z)$ be the matrix obtained from $\mathcal{N}(z)$ by replacing the ith column by the column vector $F(z)\boldsymbol{\varphi}(0) + \psi_N(0) E(z)$. According to Cramer's rule, $\varphi_i(z) = det \mathcal{N}_i(z) / det \mathcal{N}(z)$ for $i = 0, \cdots, N$. The analyticity of $\varphi_i(z)$ in $\mid z \mid \le 1$ implies that

$$det \mathcal{N}_i(z_j) = 0 \tag{3.22}$$

for $j = 1, \cdots, K$, or

$$[adj\mathcal{N}(z_j)]\{F(z_j)\boldsymbol{\varphi}(0) + \psi_M(0)E(z_j)\} = 0 \tag{3.23}$$

for $j = 1, \cdots, K$. For each given $\psi_N(0)$, using (3.22) or (3.23), every zero z_j yields one equation relating the $N+1$ unknown $\varphi_i(0), i = 0, 1, \cdots, N$. We shall argue that under the ergodic condition (2.6), a set of $N+1$ independent equations results, determining the unique solution for the constants $\varphi_i(0), i = 0, 1, \cdots, N$. Indeed, the Kolmogorov equations for the equilibrium distribution of the Markov chain $\{(X_n^1, X_n^2, J_n), n = 1, 2, \cdots\}$, along with the normalizing condition $\Phi(1,1) + \Psi(1,1) = 1$, have a unique absolutely convergent solution, and using generating functions, we have transformed those Kolmogorov equation plus the normalizing equation into the $(N+1)$-dimensional matrix equation (3.19). If $K = N+1$, then as there exists a unique solution, the equations generated by (3.22) or (3.23) must be independent. Now suppose that $K < N+1$. Then we would obtain too few equations to determine all $N+1$ unknown constants uniquely, and we would find multiple solutions for them-which is impossible. Finally, if $K > N+1$, then we would find too many equations for the $N+1$ unknown constants. Once again, as it is known that there is a unique solution, there must be exactly $N+1$ independent equations amongst those derived by using (3.22) or (3.23). Summarizing the above, we have the following theorem.

Theorem 3.1. For each given $\psi_M(0)$, there exists a matrix $\mathcal{F}$ and a vector $\mathcal{E}$ such that: there exists a solution $\boldsymbol{\varphi}(z)$, $\boldsymbol{\varphi}(0)$ of the equation (3.19) which is analytic and bounded in $| z |\leq 1$ if and only if there exist a solution $\boldsymbol{\varphi}(0)$ of the equation

$$\mathcal{F}\boldsymbol{\varphi}(0) - \psi_M(0)\mathcal{E} = 0. \tag{3.24}$$

The solutions of (3.19) correspond one-to-one the solutions of (3.24). The matrix $\mathcal{F}$ can be taken to be $(N+1) \times (N+1)$, and the vector $\mathcal{E}$ be $(N+1)$-dimensional. Each solution $\boldsymbol{\varphi}(z)$ is actually analytic in $| z |\leq 1$.

Remark. Here we still need to emphasize the same problem as in Boxma and Down[4]. In principle, it is possible that there are more than $N+1$ zeros, but that the ensuring linear equations for the $\varphi_i(0), i = 0, \cdots, N$ are dependent. The difficulty of estimating this is to calculate the determinant $det\mathcal{N}(z)$. Only for some special values of $p_i, q_i, i = 1, 2$, the number of zeros of $det\mathcal{N}(z)$ can be directly determined by the approach of the homotopy type of argument used in Gail et al. [12] [13], and Lee and Sengupta [15]. For example, when $p_1 = p_2 = 0$, Lee and Sengupta[15] have proved that $det\mathcal{N}(z)$ has exactly $N+1$ zeros in $| z |< 1$. In the section 4, we consider the case that $p_2 = 0, 0 \leq p_1 \leq 1$ and give a similar proof by using the homotopy type of argument. As long as the number of zeros of $det\mathcal{N}(z)$ is determined, the direct proof of theorem 3.1 may follow using techniques similar to those in Gail et al. [12] [13].

3.2. The determination of $\psi_N(0)$

The remained work is to determine the unknown constant $\psi_N(0)$. It follows from Theorem 3.1 that the matrix $\mathcal{F}$ is made up of $N+1$ independent row. The equation (3.24) can be written as

$$\boldsymbol{\varphi}(0) = \psi_N(0)\boldsymbol{\xi} \tag{3.25}$$

where $\boldsymbol{\xi} = (\xi_0, \cdots, \xi_N)^\tau \equiv \mathcal{F}^{-1}\mathcal{E}$. In particular, $\varphi_0(0) = \psi_N(0)\xi_0$ and $\varphi_1(0) = \psi_N(0)\xi_1$. Let $z_1, \cdots, z_K$ be K zeros of $det\mathcal{N}(z)$ on $| z |\leq 1$, and d_j be the multiplicity of the zero z_j.

Then $\sum_{j=1}^{K} d_j = N+1$. Define $\boldsymbol{\varphi}(z)$ by substituting (3.25) into (3.20)

$$\boldsymbol{\varphi}(z) = \psi_N(0)\mathcal{N}^{-1}(z)\{F(z)\boldsymbol{\xi} + E(z)\} = \psi_N(0)\frac{[adj\mathcal{N}(z)]\{F(z)\boldsymbol{\xi} + E(z)\}}{det\mathcal{N}(z)}. \tag{3.26}$$

Then $\boldsymbol{\varphi}(z)$ is clearly analytic in $\mid z \mid < 1$ except on the zero set of $det\mathcal{N}(z)$. Since the equation (3.24) holds the numerator of (3.26) vanish with sufficient order on the zeros of $det\mathcal{N}(z)$. Furthermore, since $z_k, k = 1, \cdots, K$ are distinct, and d_k is finite for $k = 1, \cdots K$, $\boldsymbol{\varphi}(z)$ is locally bounded. Thus $\boldsymbol{\varphi}(z)$ may be extended to a function which is analytic in $\mid z \mid < 1$ by Riemann removable singularity theorem. For simplicity, we still use the notation $\boldsymbol{\varphi}(z)$ to denote its extended function. When $z = 1$, in particular, we have,

$$\boldsymbol{\varphi}(1) = \psi_N(0)\mathcal{N}^{-1}(1)\{F(1)\boldsymbol{\xi} + E(1)\} \tag{3.27}$$

Hence

$$\boldsymbol{\varphi}(1) - \boldsymbol{\varphi}(0) = \psi_N(0)\mathcal{N}^{-1}(1)\{[F(1) + \mathcal{N}(1)]\boldsymbol{\xi} + E(1)\} \equiv \psi_N(0)w_{\phi}. \tag{3.28}$$

Substituting $z = 1$ into (3.17), and then substituting (3.18), (3.25) and (3.27) into the resulting formula, we have

$$\boldsymbol{\psi}(1) - \boldsymbol{\psi}(0) = G_{2a}^{-1}(1)(A_1(1)\boldsymbol{\varphi}(1) + H_2(1)\boldsymbol{\varphi}(0)) + (G_{2a}^{-1}(1)G_{2b}(1) - I)\boldsymbol{\psi}(0) \tag{3.29}$$

$$= \psi_M(0)\{G_{2a}^{-1}(1)[A_1(1)\mathcal{N}^{-1}(1)\{F(1)\boldsymbol{\xi} + E(1)\} + H_2(1)\boldsymbol{\xi}] + G_{ab}V_G^{-1}V_H\boldsymbol{\xi} + V_G^{-1}u + g\}$$

$$\equiv \psi_M(0)w_{\psi}$$

where I is the $(N+1)\times(N+1)$ unit matrix, G_{ab} is the $(N+1)\times N$ obtained from $G_{2a}^{-1}(1)G_{2b}(1) - I$ by deleting its $(N+1)$th column and g is a $(N+1)$-dimensional vector equal to $(N+1)$th column of $G_{2a}^{-1}(1)G_{2b}(1) - I$.

Substituting $z_1 = 1$ and $z_2 = 1$ into (2.17) we have

$$\Phi(1,1) = -r_2(\varphi_0(0) + \psi_0(0)) + \varphi_0(1) + \psi_0(1) + p_1\sum_{m=1}^{N}(\varphi_m(1) - \varphi_m(0)) \tag{3.30}$$

$$+q_2\sum_{m=0}^{N}(\psi_m(1) - \psi_m(0)).$$

Substituting $z_1 = 1$ into (2.18) and subsequently letting $z_2 \to 1$ we have

$$\Psi(1,1) = \frac{r_2(1-\lambda_2 b_1)}{1-\rho_2}\varphi_0(0) + \frac{r_2(1-\lambda_2(b_1+s_1+s_2))}{1-\rho_2}\psi_0(0) \tag{3.31}$$

$$+\frac{\lambda_2 b_1}{1-\rho_2}\{\varphi_0(1) + p_1\sum_{m=1}^{N}(\varphi_m(1) - \varphi_m(0))\} + \frac{\lambda_2(b_1+s_1)}{1-\rho_2}\{\psi_0(1) + \sum_{m=1}^{N}(\psi_m(1) - \psi_m(0))\}.$$

Therefore, using the normalization condition:

$$\Phi(1,1) + \Psi(1,1) = 1, \tag{3.32}$$

the relations (3.28), (3.29) and noting that from (3.18), there exist a vector e_i and a constant δ_i such that $\psi_i(0) = e_i\boldsymbol{\varphi}(0) + \delta_i\psi_N(0) = \psi_N(0)(e_i\boldsymbol{\xi} + \delta_i)$ for $i = 0, 1$, the unknown constant $\psi_N(0)$ may be determined by

$$\psi_N(0) = (1-\rho_2)\kappa^{-1} \tag{3.33}$$

here

$$\kappa \equiv r_2(\rho_2 - \lambda_2 b_1)\xi_0 + r_2(\rho_2 - \lambda_2(b_1+s_1+s_2))(e_0\boldsymbol{\xi} + \delta_0)$$

$$+(1-\rho_2+\lambda_2 b_1)(\xi_1 + p_1 < w_{\varphi}, 1 > +(1-\rho_2+\lambda_2(b_1+s_1+s_2))(e_1\boldsymbol{\xi} + \delta_1 + q_2 < w_{\psi}, 1 >$$

where $< \cdot, \cdot >$ represents the usual inner product between vectors, and $1 = (0, 1, \cdots, 1)$ the $(N+1)$-dimensional vector.

4. The special case

In the present section, we consider the special case that $p_2 = 0, q_2 = 1$, i.e., when the queue length in the queue Q_2 is less than or equal to the threshold N, the service discipline in Q_2 is 1-limited one, and in Q_1 is Bernoulli one with the probabilities $0 \le p_1 \le 1, q_1 = 1 - p_1$. Hereafter we write p, q instead of p_1, q_1 for convenience. We prove that $det\mathcal{N}(z)$ has exactly $N+1$ zeros in $\mid z \mid < 1$ by using the homotopy type of argument. If $p_2 = 0, q_2 = 1$, the matrix $G_{2a}(z) = (g_{ij}^{2a}(z))$ becomes

$$g_{ij}^{2a}(z) = \begin{cases} 1 & \text{if } j = i, i = 1, \cdots, N \\ \eta^{j-1}(z) & \text{if } j = 1, \cdots, N+1, i = N+1 \\ 0 & \text{otherwise} \end{cases}$$

So we can easily get the inverse $G_{2a}^{-1}(z) = (g_{ij}^{-2a}(z))$ of the matrix $G_{2a}(z)$ as follows

$$g_{ij}^{-2a}(z) = \begin{cases} 1 & \text{if } j = i, i = 1, \cdots, N \\ -\eta^{j-N-1}(z) & \text{if } j = 1, \cdots, N, i = N+1 \\ \eta^{-N}(z) & \text{if } j = i = N+1 \\ 0 & \text{otherwise} \end{cases}$$

Then we can directly calculate the matrix $\mathcal{N}(z) = A_1(z) - H_{1q}(z)G_{2a}^{-1}(z)A_2(z) \equiv (n_{ij}(z))$.

$$n_{ij}(z) = \begin{cases} -qH_{1,0}(z)H_{2,0}(z) & \text{if } j = i+1, i = 1, \cdots, N \\ z - pG_{1,0}(z) - q\sum_{l=0}^{1} H_{1,1-l}(z)H_{2,l}(z) & \text{if } j = i, i = 2, \cdots, N \\ -pG_{1,i-j+1}(z) - q\sum_{l=0}^{i-j+1} H_{1,i-j+1-l}(z)H_{2,l}(z) & \text{if } 2 \le j < i, i = 2, \cdots, N \\ z - G_{1,0}(z) & \text{if } j = i = 1 \\ -G_{1,i-1}(z) & \text{if } j = 1, i = 2, \cdots, N \\ -G_{1,N}(z) - \alpha(z)\eta^{-N}(z)H_{1,0}(z) & \text{if } j = 1, i = N+1 \\ -p(G_{1,N-j+1}(z) + \alpha(z)\eta^{j-N-1}(z)H_{1,0}(z)) & \\ \quad -q\sum_{l=0}^{N-j+1}(H_{1,N-l-j+2}(z) - H_{1,0}(z) & \text{if } j = 2, \cdots, N+1, \\ \qquad \times\eta^{l+j-N-2}(z))H_{2,l}(z) & \quad i = N+1 \\ 0 & \text{if } j > i+1, i = 1, \cdots, N-1. \end{cases}$$

Now we identify the singularities of the matrix $\mathcal{N}(z)$, i.e., determine the number of zeros of $\det\mathcal{N}(z)$. To do this, we study the determinant of another matrix which is deduced from $\mathcal{N}(z)$ by the following steps: (i) multiplying Nth row by $\eta^{-1}(z)$, and then adding it to $N+1$th row; (ii) to the resulting matrix, multiplying Nth row by $-p\alpha(z)/qH_{2,0}(z)$, and then adding it to $N+1$th row again. Denoting the final resulting matrix by $\mathcal{M}(z)$, we have $det\mathcal{N}(z) = det\mathcal{M}(z)$ by construction. Hence, it is sufficient to discuss the number of the zeros of $det\mathcal{M}(z)$. Note that all entries of $\mathcal{M}(z)$ are analytic $\mid z \mid < 1$. This implies that $det\mathcal{M}(z)$ is analytic on $\mid z \mid < 1$.

Theorem 4.1. $det\mathcal{M}(z)$ has exactly $N+1$ zeros on $\mid z \mid \le 1$.

Proof. We prove the conclusion by the approach of the homotopy type of argument used in the proof of Theorem 1 in Lee and Sengupta[18](also see Gail, et.al. [12],[13] for details). First, we write $\mathcal{M}(z) = \mathcal{D}(z) + \mathcal{O}(z)$, where $\mathcal{D}(z)$ is the diagonal matrix and $\mathcal{O}(z)$ is the off-diagonal matrix corresponding to $\mathcal{M}(z)$, namely, the diagonal entries of $\mathcal{O}(z)$ are all zero. We have that the diagonal entries of $\mathcal{D}(z)$ are $d_1(z) = z - G_{1,0}(z); d_i(z) = z - pG_{1,0}(z) -$

$q(H_{1,0}(z)H_{2,1}(z)+H_{1,1}(z)H_{2,0}(z)), i=2,\cdots,N; d_{N+1}(z)=z-pG_{1,0}(z)-qH_{1,1}(z)H_{2,0}(z)$. Define

$$\mathcal{M}(z,t)=\mathcal{D}(z)+t\mathcal{O}(z), \qquad 0\le t\le 1. \tag{4.1}$$

Note that $\mathcal{M}(z,1)=\mathcal{M}(z)$. We first show that $det\mathcal{D}(z)$ has exactly $N+1$ zeros on $\mid z\mid\le 1$. Since $det\mathcal{D}(z)=(z-G_{1,0}(z))(z-pG_{1,0}(z)-q(H_{1,0}(z)H_{2,1}(z)+H_{1,1}(z)H_{2,0}(z))^{N-1}(z-pG_{1,0}(z)-qH_{1,1}(z)H_{2,0}(z))$, we only need to prove that each equation of (i) $z-G_{1,0}(z)=0$, (ii) $z-pG_{1,0}(z)-q(H_{1,0}(z)H_{2,1}(z)+H_{1,1}(z)H_{2,0}(z))$, and (iii) $z-pG_{1,0}(z)-qH_{1,1}(z)H_{2,0}(z)=0$ has exactly one root on $\mid z\mid\le 1$. First we consider the equation (i), let $f(z)=z, g(z)=G_{1,0}(z)$. On $\mid z\mid=1$, we have

$$\mid g(z)\mid\le\mid\int_0^\infty e^{-(\lambda_2+\lambda_1(1-z))t}dB_1(t)\mid<\sum_{i=0}^{\infty}\mid\int_0^\infty\frac{\lambda_2 te^{-\lambda_2 t}}{i!}e^{-\lambda_1(1-z))t}dB_1(t)\mid\le 1=\mid f(z)\mid .$$

An simple application of Rouché's Theorem shows that $z-G_{1,0}(z)=0$ has exactly one root on $\mid z\mid\le 1$. Next consider the equation (ii). Let $f(z)=z, g(z)=pG_{1,0}(z)+q(H_{1,0}(z)H_{2,1}(z)+H_{1,1}(z)H_{2,0}(z))$. Note that for $j\ge 0$, $H_{1,j}(1)$ is the probability that there are j arrivals during the switching time from Q_2 to Q_1 and the service time at Q_1, and $G_{1,j}(1)$ the probability that there are j arrivals during the service time at Q_1. Then by the definition of $G_{1,j}(z)$, $H_{1,j}(z)$ and $H_{2,j}(z)$, we have that

$$\begin{aligned}\mid g(z)\mid &\le pG_{1,0}(1)+q(H_{1,0}(1)H_{2,1}(1)+H_{1,1}(1)H_{2,0}(1))\le pG_{1,0}(1)+q(H_{1,0}(1)+H_{1,1}(1))\\ &< pG_{1,0}(1)+q\sum_{j=0}^{\infty}H_{1,j}(1)\le p+q=1=\mid f(z)\mid .\end{aligned}$$

Again by applying Rouché's Theorem, we get that the equation $z-pG_{1,0}(z)-q(H_{1,0}(z)H_{2,1}(z)+H_{1,1}(z)H_{2,0}(z))=0$ has exactly one root on $\mid z\mid\le 1$. The proof of (iii) is similar to (ii).

Next, we consider $det\mathcal{M}(z,t)$. Note that there exists an $\epsilon>0$ such that $det\mathcal{M}(z,t)$ is analytic on $\mid z\mid\le 1+\epsilon$ for $t\in[0,1]$. Since zeros of an analytic function which is not identically zero must be isolated(see, for example, Churchill and Brown[5]), we can take any closed contour $\hat{C}$ in the region $1<\mid z\mid\le 1+\epsilon$ such that $det\mathcal{M}(z,t)\ne 0$ for $z\in\hat{C}$ and $t\in[0,1]$. Let $m(t)$ be the number of zeros of $det\mathcal{M}(z,t)$ on the region enclosed by $\hat{C}$. By the argument principle,

$$m_{\hat{C}}(t)=\frac{1}{2\pi i}\oint_{\hat{C}}\frac{\frac{\partial}{\partial z}det\mathcal{M}(z,t)}{det\mathcal{M}(z,t)}dz. \tag{4.2}$$

Observing that $\mathcal{M}(z,t)$ is a continuous function of t we have that $m(t)$ is also a continuous integer-valued function of t. Therefore, $m_{\hat{C}}(t)=m_{\hat{C}}(0)$. Since the contour $\hat{C}$ is arbitrary on the region $1<\mid z\mid\le 1+\epsilon$, it follows that $m_C(1)=m_C(0)=N+1$. These complete the proof. $\square$

5. Waiting times

In this section we consider the LST of the waiting time distributions and the mean waiting times at $Q_i, i=1,2$. Let W_i represent the waiting time at Q_i, and $\hat{W}_i(s)$ its LST for $i=1,2$. Since the customers present in Q_i just after the instant of service completion of type i customer are just the customers who had arrived during the waiting time and service time of that type i customer, we have the following relations:

$$r_1\hat{W}_1(\lambda_1(1-z_1))\hat{B}_1(\lambda_1(1-z_1))=\Phi(z_1,1), \quad \mid z_1\mid\le 1, \tag{5.1}$$

$$r_2\hat{W}_2(\lambda_2(1-z_2))\hat{B}_2(\lambda_2(1-z_2))=\Psi(1,z_2), \quad \mid z_2\mid\le 1. \tag{5.2}$$

Therefore

$$E[W_1] = \frac{1}{r_1\lambda_1}\frac{d}{dz_1}\Phi(z_1,1)|_{z_1=1} - b_1, \tag{5.3}$$

$$E[W_2] = \frac{1}{r_2\lambda_2}\frac{d}{dz_2}\Psi(1,z_2)|_{z_2=1} - b_2. \tag{5.4}$$

Substituting $z_2 = 1$ into (2.12), and then differentiating in z_1, we get

$$\frac{d}{dz_1}\Phi(z_1,1)|_{z_1=1} = r_1(\varphi_0(0)+\psi_0(0)) - \lambda_1 s_2[r_2\psi_0(0)+\psi_0(1)+q_2\sum_{m=1}^{N}(\psi_m(1)-\psi_m(0)))]$$

$$-(1-\rho_1)\Phi(1,1)+\varphi_0^{'}(1)+\psi_0^{'}(1)+p_1\sum_{m=1}^{N}\varphi_m^{'}(1)+q_1\sum_{m=1}^{N}\psi_m^{'}(1). \tag{5.5}$$

Next, substituting $z_1 = 1$ into (2.13), we get $\Psi(1,z_2) = \zeta(z_2)/\nu(z_2)$ where

$$\nu(z_2) \equiv z_2 - \hat{B}_2(\lambda_2(1-z_2))$$

$$\zeta(z_2) \equiv \hat{B}_2(\lambda_2(1-z_2))\{\ (r_2(z_2-\hat{B}_1(\lambda_2(1-z_2))\hat{S}_1(\lambda_2(1-z_2))\varphi_0(0)$$

$$+r_2(z_2-\hat{B}_1(\lambda_2(1-z_2))\hat{S}_1(\lambda_2(1-z_2))\hat{S}_1(\lambda_2(1-z_2)))\psi_0(0)$$

$$+(\hat{B}_1(\lambda_2(1-z_2))-1)\hat{S}_1(\lambda_2(1-z_2))[\varphi_0(1)+p_1\sum_{m=1}^{N}(\varphi_m(1)-\varphi_m(0))z_2^m]$$

$$+(\hat{B}_1(\lambda_2(1-z_2))\hat{S}_1(\lambda_2(1-z_2))\hat{S}_2(\lambda_2(1-z_2))-1)[\psi_0(1)+q_2\sum_{m=1}^{N}(\psi_m(1)-\psi_m(0))z_2^m].$$

Since $\nu(1) = \zeta(1) = 0$, using L'Hospital's rule, we get

$$\frac{d}{dz_2}\Psi(1,z_2)|_{z_2=1} = \frac{\zeta^{''}(1)\nu^{'}(1)-\zeta^{'}(1)\nu^{''}(1)}{2(\nu^{'}(1))^2} \tag{5.6}$$

where $\nu^{'}(1) = 1-\rho_2$, $\nu^{''}(1) = -\lambda_2 b_2^{(2)}$ and

$$\zeta^{'}(1) = r_2(1-\lambda_2 b_1)\varphi_0(0)+r_2(1-\lambda_2(b_1+s_1+s_2))\psi_0(0)+\lambda_2 b_1[\varphi_0(1)+p_1\sum_{m=1}^{N}(\varphi_m(1)-\varphi_m(0))]$$

$$+\lambda_2(b_1+s_1+s_2)[\psi_0(1)+q_2\sum_{m=1}^{N}(\psi_m(1)-\psi_m(0))],$$

and

$$\zeta^{''}(1) = 2\rho_2\zeta^{'} + r_2[2(1-\lambda_2 b_1)\lambda_2 s_1 - \lambda_2^2 b_1^{(2)}]\varphi_0(0) - r_2\lambda_2^2(b_1+s_1+s_2)^{(2)}\psi_0(0)$$

$$+\lambda_2^2(b_1^{(2)}+2b_1 s_1)[\varphi_0(1)+p_1\sum_{m=1}^{N}(\varphi_m(1)-\varphi_m(0))]+2\lambda_2 b_1 p_1\sum_{m=1}^{N}m(\varphi_m(1)-\varphi_m(0))$$

$$+\lambda_2^2(b_1+s_1+s_2)^{(2)}[\psi_0(1)+q_2\sum_{m=1}^{N}(\psi_m(1)-\psi_m(0))]+2\lambda_2(b_1+s_1+s_2)q_2\sum_{m=1}^{N}m(\psi_m(1)-\psi_m(0)).$$

where $(b_1+s_1+s_2)^{(2)} = b_1^{(2)}+s_1^{(2)}+s_2^{(2)}+2b_1s_1+2b_1s_2+2s_1s_2$. As shown in (5.5), the differential values $\varphi_m^{'}(1)$, $\psi_m^{'}(1), 0 \le m \le N$ are necessary to obtain $d\Phi^{(1)}(z_1,1)/dz_1|_{z_1=1}$.

One can easily calculate these values by differentiating (3.20) and (3.17) in z and then letting $z \to 1$.

$$\frac{d}{dz}\boldsymbol{\varphi}(z)|_{z=1} = \mathcal{N}^{-1}(z)\left[\psi_N(0)((\frac{d}{dz}F(z))\boldsymbol{\xi} + \frac{d}{dz}E(z)) - (\frac{d}{dz}\mathcal{N}(z))\boldsymbol{\varphi}(z)\right]_{z=1}$$

and

$$\frac{d}{dz}\boldsymbol{\psi}(z)|_{z=1} = G_{2a}^{-1}(z)\left[(\frac{d}{dz}A_2(z))\boldsymbol{\varphi}(z) + A_2(z)(\frac{d}{dz}\boldsymbol{\varphi}(z)) + (\frac{d}{dz}H_2(z))\boldsymbol{\varphi}(0)\right.$$
$$\left. +(\frac{d}{dz}G_{2b}(z))\boldsymbol{\psi}(0) - (\frac{d}{dz}G_{2a}(z))\boldsymbol{\psi}(z)\right]_{z=1}.$$

6. Conclusions

In this paper, we have presented a service schedule of the Bernoulli-Threshold service schedule for a polling system consisting of two-parallel queues and single server. This service schedule is more flexible because only by simply choosing the threshold values N and the probabilities p_1, p_2 for the queues Q_1, Q_2, one can easily assign a higher priority to a queue, for example, a higher priority to the real-time traffic over the non-real-time traffic. Hence it can be used for meeting the quality of various service requirements by different type of traffic. For this model, we have carried out the analysis of the system performance and derived the generating functions of the joint stationary distribution of the queue lengths at the service completion instants. Furthermore, we also have determined the Laplace-Stieltjes transforms of waiting times for both queues, and obtained their mean waiting times. Utilizing these results, we can determine other performances of the system, for example, the optimal threshold values. As have been seen, our model includes those studied by Lee and Sengupta[15], and Boxma and Down[4] as the special cases. This generalizes the known results about polling systems with two queues and signal server.

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