GROUP TOPOLOGIES ON THE COMPLEX NUMBERS WITH SPECIAL CONVERGENCE II

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Abstract. Let $(\mathbb{C}, +)$ be the additive group of complex numbers and $z \in \mathbb{C} \setminus \mathbb{R}$ with $|z| > 1$. For each $k \in N$, let $I'_k(z)$ be the set of all complex numbers of a form $\alpha_1 z^{k_1} + \alpha_2 z^{k_2} + \cdots + \alpha_n z^{k_n}$, where $\alpha_i \in \mathbb{Z}$, $k_i \in N$ $(i = 1, 2, \cdots, n)$, $k \le k_1 < k_2 < \cdots < k_n$ and $n \in N$. We prove that $\inf\{|w| : w \in I'_k(z)\} \to \infty$ $(k \to \infty)$ if and only if z is an algebraic integer with degree 2. In this case, we can easily define a metrizable group topology τ on $(\mathbb{C}, +)$ such that the sequence $\{z^n : n \in N\}$ converges to 0 in the topo logical group $(\mathbb{C}, +, \tau)$.

We use the same notation as in [2]. Let $(\mathbb{C}, +)$ be the additive group of complex numbers. For each $z \in \mathbb{C}$ and each $k \in N$, let $I'_k(z)$ be the set of all complex numbers w which can be written as a form

$$w = \alpha_1 z^{k_1} + \alpha_2 z^{k_2} + \cdots + \alpha_n z^{k_n},$$

where $\alpha_i \in \mathbb{Z}$, $k_i \in N$ $(i = 1, 2, \cdots, n)$, $k \le k_1 < k_2 < \cdots < k_n$ and $n \in N$. In the previous paper [3], we proved that for every $z \in \mathbb{C}$ with $|z| > 1$, there exists a metrizable group topology τ on $(\mathbb{C}, +)$ such that τ is coarser than the Euclidean topology and the sequence $\{z^n : n \in N\}$ converges to 0 in the topological group $(\mathbb{C}, +, \tau)$. In particular, if z satisfies that

$$\inf\{|w| : w \in I'_k(z) \setminus \{0\}\} \to \infty \quad (k \to \infty), \tag{1}$$

then such a topology can easily be obtained by simply taking the family $\mathcal{B}(z) = \{u + U_k : u \in \mathbb{C}, k \in N\}$ as a base, where $U_k = \bigcup_{w \in I'_k(z)} \{u \in \mathbb{C} : |u - w| < 1/2^k\}$ for each $k \in N$. The main purpose of this paper is to determine a complex number z satisfying (1) by proving the following thorem:

Theorem 1. *Let $z \in \mathbb{C} \setminus \mathbb{R}$ with $|z| > 1$. Then, z satisfies* (1) *if and only if z is an algebraic integer with degree* 2*, i.e., $z^2 + \alpha z + \beta = 0$ for some $\alpha, \beta \in \mathbb{Z}$.*

The authors proved in [3] that a real number z satisfies (1) if and onl y if $z \in \mathbb{Z}$, but they had been unable to determine such a complex number z. For $z \in \mathbb{C} \setminus \mathbb{R}$ with $|z| > 1$ and satisfying (1), the topology on $(\mathbb{C}, +)$ generated by the base $\mathcal{B}(z)$ is called the *simple topology* induced by z and is denoted by $\tau'(z)$.

To prove Theorem 1, we need some notation and a lemma. As usual, let $\mathbb{Z}[x]$ denote the set of all polynomials with integral coefficients and $r\mathbb{Z} = \{rn : n \in \mathbb{Z}\}$ for each $r \in \mathbb{R}$. Further, let $\mathbb{Z}_0[x]$ be the subset of $\mathbb{Z}[x]$ consisting of all polynomials such that the coefficient of the term with the maximum degree is 1. For a set A, $\sharp A$ denotes the cardinality of A.

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Lemma 2. *Let $z \in \mathbb{C} \setminus \mathbb{R}$ with $|z| > 1$. Assume that z is not an algebraic integer with degree 2. Then, there exists $f(x) \in \mathbb{Z}[x]$ such that $0 < |f(z)| < 1$.*

Proof. First, we consider the case that z is not an algebraic number with degree 2. In this case, z is either an transcendental number or an algebraic number with degree ≥ 3. Fix $n \in N$ with $|z| \leq n$ and let $\delta = 2^5$. Let L be the set of all complex numbers w of the form

$$w = z^3 + a_1 z^2 + a_2 z + a_3,$$

where $a_i \in \mathbb{Z}$ $(i = 1, 2, 3)$, $|a_1| \leq \delta n$, $|a_2| \leq \delta n^2$ and $|a_3| \leq \delta n^3$. Then, L contains no same elements, because z is neither an algebraic number with degree 2 nor a real number. Hence, we have that

$$\sharp L = (2\delta n + 1)(2\delta n^2 + 1)(2\delta n^3 + 1) > 8\delta^3 n^6 = 2^{18} n^6. \tag{2}$$

On the other hand, $|w| \leq n^3 + \delta n^3 + \delta n^3 + \delta n^3 = (1 + 3\delta)n^3$ for each $w \in L$, and hence, L is included in a square S with an edge of length $2(1 + 3\delta)n^3$. Now, we decompose S into $16(1 + 3\delta)^2 n^6$ many small squares with an edge o f length $1/2$. Note that $16(1 + 3\delta)^2 n^6 < 2^4 (4\delta)^2 n^6 = 2^{18} n^6$. This combined with (2) implies that at least one of the small squares contains two distinct elements $w, w' \in L$. Since $0 < |w - w'| < 1$, this means that there is $f(x) \in \mathbb{Z}[x]$ such that $0 < |f(z)| < 1$.

Next, we consider the case that z is an algebraic number with degree 2. Since z is not an algebraic integer, there exist $a_0, a_1, a_2 \in \mathbb{Z}$ such tha t

$$a_0 z^2 + a_1 z + a_2 = 0, \tag{3}$$

where $a_0 \geq 2$ and the greatest common divisor $(a_0, a_1, a_2) = 1$. If $d = (a_0, a_1) \geq 2$, then $a_0/d, a_1/d \in \mathbb{Z}$, but $a_2/d \notin \mathbb{Z}$ because $(a_0, a_1, a_2) = 1$. Define $f(x) = (a_0/d)x^2 + (a_1/d)x + c$, where c is the smallest integer greater than a_2/d. Then, $0 < |f(z)| = |(a_2/d) - c| < 1$, because $(a_0/d)z^2 + (a_1/d)z + (a_2/d) = 0$ by (3) .. Thus, we need only consider the case that $(a_0, a_1) = 1$. In this case, suppose on the contrary that there is no $f(x) \in \mathbb{Z}[x]$ such that $0 < |f(z)| < 1$. Then, we have the following claim:

Claim. *If $g(x) \in \mathbb{Z}_0[x]$ and $0 < |\operatorname{Re}(g(z))| < 1/2$ and $\operatorname{Im}(g(z)) \neq 0$, then there is $h(x) \in \mathbb{Z}_0[x]$ such that $0 < |\operatorname{Re}(h(z))| < 1/2$ and $0 < |\operatorname{Im}(h(z))| < |\operatorname{Im}(g(z))|$.*

Proof. Since $\operatorname{Re}(g(z)) \neq 0$ and $\operatorname{Im}(g(x)) \neq 0$, $g(z)^2 \notin \mathbb{R}$, and hence, we can write $g(z)^2 = \beta + \gamma i$ for some $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R} \setminus \{0\}$. Since $(g(z)^2 - \beta)^2 = -\gamma^2$,

$$g(z)^4 - 2\beta g(z)^2 + \beta^2 + \gamma^2 = 0. \tag{4}$$

If $\beta \in (1/2)\mathbb{Z}$, then $\beta^2 + \gamma^2 \notin \mathbb{Z}$, because z is not an algebraic integer. Let $f(x) = g(x)^4 - 2\beta g(x)^2 + c$, where c is the smallest integer greater tha n $\beta^2 + \gamma^2$. Then, $0 < |f(z)| = |c - (\beta^2 + \gamma^2)| < 1$ by (4), which contradicts our assumption. Hence, we have $\beta \notin (1/2)\mathbb{Z}$, which implies that $0 < |\beta - \alpha| < 1/2$ for some $\alpha \in \mathbb{Z}$. Let $h(x) = g(x)^2 - \alpha$. Since $\operatorname{Re}(h(z)) = \beta - \alpha$, we have $0 < |\operatorname{Re}(h(z))| < 1/2$. Since $\operatorname{Im}(g(z)^2) \neq 0$ and $|\operatorname{Re}(g(z))| < 1/2$, we have

$$0 < |\operatorname{Im}(h(z))| = |\operatorname{Im}(g(z)^2)| = 2|\operatorname{Re}(g(z))| \cdot |\operatorname{Im}(g(z))| < |\operatorname{Im}(g(z))|.$$

This completes the proof. $\square$

Let us return to the proof of Lemma 2. Note that

$$z = \frac{-a_1 \pm \sqrt{D}}{2a_0}, \quad \text{where } D < 0 \tag{5}$$

by (3). Since $(a_0, a_1) = 1$, there is $k \in \mathbb{Z}$ such that $0 < |(-a_1/2a_0) - k| < 1/2$. Define $g_1(x) = x - k \in \mathbb{Z}_0[x]$. Then, $\mathrm{Re}(g_1(z)) = (-a_1/2a_0) - k$ by (5). Hence, $0 < |\,\mathrm{Re}(g_1(z))| < 1/2$ and $\mathrm{Im}(g_1(z)) = \mathrm{Im}(z) \neq 0$. By Claim, we can inductively define a sequence $\{g_i(x) : i \in N\} \subseteq \mathbb{Z}_0[x]$ such that $0 < |\,\mathrm{Re}(g_i(z))| < 1/2$ and $0 < |\,\mathrm{Im}(g_{i+1}(z))| < |\,\mathrm{Im}(g_i(z))|$ for each $i \in N$. Thus, we can find distinct $i, j \in N$ such that $0 < |g_i(z) - g_j(z)| < 1$. Since $f(x) = g_i(x) - g_j(x) \in \mathbb{Z}[x]$, this contradicts our assumption. $\square$

Let $z \in \mathbb{C}\backslash\mathbb{R}$ be an algebraic integer with degree 2. Then, z is contained in the imaginary quadratic field $K = \mathbb{Q}(\sqrt{m})$, wher e m is a negative square free integer. As is well known, the ring $_K$ of algebraic integers in K is a lat tice, i.e., a free $\mathbb{Z}$-module of rank 2 whose basis are 1 and u, where $u = (1+\sqrt{m})/2$ if $m \equiv 1 \pmod 4$ and $u = \sqrt{m}$ if $m \equiv 2$ or $3 \pmod 4$.

Proof of Theorem 1. Let $z \in \mathbb{C} \setminus \mathbb{R}$ with $|z| > 1$. If z is an algebraic integer with degree 2, then $f(z) \in {}_K$ for each $f(x) \in \mathbb{Z}[x]$, where $_K$ is defined as above. Since $_K$ is a lattice, we have $\alpha = \min\{|f(z)| : f(z) \neq 0, f(x) \in \mathbb{Z}[x]\} > 0$. For each $w \in I'_k(z) \setminus \{0\}$, w can be written as $w = z^k f(z)$ for some $f(x) \in \mathbb{Z}[x]$, and thus,

$$|w| = |z|^k |f(z)| \geq |z|^k \alpha. \tag{6}$$

Hence, $\inf\{|w| : w \in I'_k(z) \setminus \{0\}\} = |z|^k \alpha$, which implies that z satisfies (1). Conversely, assume that z is not an algebraic integer with degree 2. By Lemma 2, there is $f(x) \in \mathbb{Z}[x]$ such that $0 < |f(z)| < 1$. Let $k \in N$ be fixed. Then, $z^k f(z)^n \in I'_k(z) \setminus \{0\}$ for each $n \in N$. Since $|z^k f(z)^n| = |z|^k |f(z)|^n \to 0$ $(n \to \infty)$, we have

$$\inf\{|w| : w \in I'_k(z) \setminus \{0\}\} = 0.$$

Hence, z fails to satisfy (1), which completes the proof. $\square$

Corollary 3. *Assume that either $z \in \mathbb{Z}$ or z is an imaginary algebraic integer with degree 2, and that $|z| > 1$. Then, there exists a metrizable group topology τ on $(\mathbb{C}, +)$ such that τ is coarser than the Euclidean topology and the sequence $\{\alpha^n z^n : n \in N\}$ coverges to 0 in the topological group $(\mathbb{C}, +, \tau)$ for each $\alpha \in \mathbb{Z}$.*

Proof. By Theorem 1, z satisfies (1). Hence, the simple topology $\tau'(z)$ induced by z is a required topology; i nfact, $\{\alpha^n z^n : n \in N\}$ converges to 0 in $(\mathbb{C}, +, \tau'(z))$ for each $\alpha \in Z$, because $\alpha^n z^n \in I'_k(z)$ whenever $n \geq k$, for every $k, n \in N$. $\square$

Remark 1. It is open whether, for every two $z_1, z_2 \in \mathbb{C}$ with $z_1 \neq z_2$ and $|z_i| > 1$ $(i = 1, 2)$, there is a metrizable group topology τ on $(\mathbb{C}, +)$ such that τ is coarser than the Euclidean topology and both $\{z_1^n : n \in N\}$ and $\{z_2^n : n \in N\}$ converge to 0 in $(\mathbb{C}, +, \tau)$. In particular, the following question asked by Hattori [1] still remains open: Does there exist a metrizable group topology τ on $(\mathbb{R}, +)$ such that τ is coarser than the Euclidean topology and both $\{2^n : n \in N\}$ and $\{3^n : n \in N\}$ converge to 0 in the topological group $(\mathbb{R}, +, \tau)$?

Remark 2. Theorem 1 enables us to construct the simple topology $\tau'(z)$ by a geometrical method. To show this, let $z \in \mathbb{C}\backslash\mathbb{R}$ be a complex number, with $|z| > 1$, satisfying (1). Then, z is an algebraic integer with degree 2 by Theorem 1. Let $_K$ be the same as the one defined before the proof of Theorem 1. Let $k \in N$ be fixed for a while. Since $I'_k(z)$ is a subgroup of $_K$, $I'_k(z)$ is also a lattice, a nd hence, the quotient topological group $T_k = \mathbb{C}/I'_k(z)$ is homeomorphic to the torus. Let $h_k : \mathbb{C} \to T_k$ be the natural homomorphism. If we define $h_k : \mathbb{C} \to T_k$ for each $k \in N$, then we have a continuous homo morphism

$$h : \mathbb{C} \to T = \prod_{k \in N} T_k$$

such that $h_k = \pi_k \circ h$ for each $k \in N$, where $\pi_k : T \to T_k$ is the projection. Let $\rho(z)$ be the relative topology on $h[\mathbb{C}]$ induced by the product topology on T. Since $z^n \in I'_k(z)$ for each $k \leq n$, the sequence $\{h(z^n) : n \in N\}$ co nverges to $h(0)$ with respect to the topology $\rho(z)$. Now, observe that condition (1) implies that h is a monomorphism. Moreover, it is not difficult to see that the map $h : (\mathbb{C}, \tau'(z)) \to (h[\mathbb{C}], \rho(z))$ is a homeomorphism. Hence, we can consider that $\rho(z) = \tau'(z)$.

For an integer $r \in \mathbb{Z}$, $I'_k(r)$ coincides with the set of all integral multiples of r^k, i.e., $I'_k(r) = r^k\mathbb{Z}$ for each $k \in N$. If $|r| > 1$, then the topology $\tau'_{\mathbb{R}}(r)$ on $\mathbb{R}$ generated by a base $\{s + V_k : s \in \mathbb{R}, k \in N\}$, where $V_k = \bigcup_{n \in \mathbb{Z}} \{x \in \mathbb{R} : |x - r^k n| < 1/2^k\}$, is also a metrizable group topology on $\mathbb{R}$ such that $\tau'_{\mathbb{R}}(r)$ is coarser than the Euclidean topology and the sequence $\{r^n : n \in N\}$ converges to 0 in the topological group $(\mathbb{R}, +, \tau'_{\mathbb{R}}(r))$. The topology $\tau'_{\mathbb{R}}(r)$ was first studied by Hattori [2] for $r = 2$. Similarly to the above, $\tau'_{\mathbb{R}}(r)$ is obtained as a relative topology induced by the product topology on the product of countably many circles $\{\mathbb{R}/r^k\mathbb{Z} : k \in N\}$.

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