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ON PSEUDO-COMMUTATIVE PO-SEMIGROUPS

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Abstract. In this paper, the class of pseudo-commutative po-semigroups is studied. It is noted that pseudo-commutative po-semigroups are special weakly commutative po-semigroups. We will show that a pseudo-commutative po-semigroup can be decomposed into a semilattice of Archimedean po-semigroups and such decomposition may not be unique.

By a po-semigroup, we mean a semigroup S endowed with a partial order "$\leq$ " such that the multiplication of S is compatible with " $\leq$ ", that is, $a \leq b$ implies that $xa \leq xb$ and $ax \leq bx$ for all $x \in S$. Po-semigroups with a greatest elements e are called poe-semigroups. Poe-semigroups were firstly studied by Kehayopulu in [5], [12] and [8]. We call a po-semigroup S weakly commutative if for all $x, y \in S$, there exists a positive integer $n \in N$ such that $(xy)^n \leq yax$ for some element $a \in S$. It was announced by Kehayopulu in [11] that a po-semigroup is weakly commutative if and only if for every $x \in S$, $N(x) = \{y \in S | x^n \in (ySy] \text{ for some } n \in N\}$. Although her result is very close to the form $N(x) = \{y \in S \mid x^n \in ySy \text{ for some } n \in N\}$, for every $x \in S$, obtained by Petrich in [17], the proof is not the same since the partial order "$\leq$ " implemented on S is mathematically different from " $\in$ ". She also proved in [12] that a po$e$-semigroup S is weakly commutative if and only if $N(x) = \{y \in S | x^k \leq yey, \text{ for some } k \in N\}$, for all $x \in S$. Her results were later on re-obtained and reproved by Jing and Chen in [2].

In this paper, we investigate a special subclass of the class of weakly commutative po-semigroups, namely, the class of pseudo-commutative po-semigroups as this class of po-semigroups has some interesting properties of its own. By a right pseudo-commutative po-semigroup, we mean an ordered semigroup S such that $(xy)^n \leq xy^\lambda$ for all $x, y \in S$ and some positive integers n and λ. Left pseudo-commutative po-semigroup can be dually defined. It is clear that the left pseudo-commutative po-semigroup is the dual of the right pseudo-commutative po-semigroup. The right (left) pseudo-commutative po-semigroup will be called the pseudo-commutative po-semigroup if no possible ambiguity arises. We will show that a pseudo-commutative po-semigroup can be expressed as a semilattice of Archimedean po-semigroups and such semilattice decompositions may not be unique.

For terminologies and definitions not given in this paper, the reader is referred to Petrich [17] and Kehayopulu [12], [15]. Throughout this paper, unless otherwise stated, S is always a po-semigroup $(S, \cdot, \leq)$.

To start with, we first notice that pseudo-commutative po-semigroups are special weakly commutative po-semigroups. We now cite an example in [3] to show that there exist weakly commutative po-semigroups which are not pseudo-commutative.

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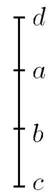
Example 1 ([3]) Let $S = \{a, b, c, d\}$ be a set with the following Cayley table and Hasse diagram

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	b	a
c	a	c	c	a
d	a	a	a	d

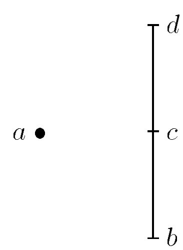


Then $(S, \cdot, \leq)$ is a po-semigroup. (For the method of checking, the reader is referred to Kehayopulu in [7], [13] and [14]). Since $(bc)^n = b$ in the above table, for all $n \in N$, we have $b = (bc)^n \not\leq c^\lambda b = c$ and $(bc)^n = b \not\leq cb^\lambda$ for all positive integers n and λ. Hence S is not pseudo-commutative. On the other hand, S is weakly commutative since $(xy)^n \leq ytx$ for some $t \in S$, for instance, $(bc)^n = b \leq cab = a$, where a is not in the subsemigroup generated by $\{b, c\}$. This example thus illustrates that the class of pseudo-commutative po-semigroups is indeed a proper subclass of the class of weakly commutative po-semigroups.

There are also some other proper subclasses of weakly commutative po-semigroups such as the classes of cyclic commutative po-semigroups, strictly cyclic-commutative po-semigroups and weakly cyclic commutative po-semigroups etc (cf. [3]). The relationships among these subclasses of weakly commutative po-semigroups, including the pseudo-commutative po-semigroups have recently been described by the authors in [3]. We now study the semilattice decomposition of pseudo-commutative po-semigroups so that the structure of this kind of po-semigroups can be further investigated and described. We point out here again that a pseudo-commutative po-semigroup is even not necessarily a weakly cyclic commutative po-semigroup. The following is an example of pseudo-commutative po-semigroup which is not weakly cyclic.

Example 2 (cf. [3]) Let $S = \{a, b, c, d\}$ be a set with Cayley table and Hasse diagram shown below:

$\cdot$	a	b	c	d
a	b	b	c	c
b	b	b	c	c
c	c	c	c	c
d	c	c	c	c



Then, by using the method of Kehayopulu ([13], [14]), we can verify that S is a po-semigroup. (The checking is omitted). Clearly S is pseudo-commutative but not weakly 3-cyclic commutative because $(adb)^n = c \not\leq b = ba$. Thus the class of weakly cyclic commutative po-semigroups and the class of pseudo-commutative po-semigroups are different sub-classes of the class of weakly commutative po-semigroups.

To study the semilattice decomposition of pseudo-commutative po-semigroups, we recall the following definitions and notations.

Definition 3 (cf. [12]**).** *A subsemigroup F of a po-semigroup S is called a filter of S if the following conditions are satisfied:*

(i) $a, b \in S$ *and* $ab \in F \Longrightarrow a \in F$ *and* $b \in F$*;*
(ii) $a \in F$ *and* $c \in S, c \geq a \Longrightarrow c \in F$.

Remark:
We note that the above definition of filter is only applied for po-semigroups. However, we would like to point out that this definition is different from the previous one given by Petrich in [17], as for algebraic semigroups, the condition (ii) is not required. Thus, the word "filter" that we are dealing with in this paper is only the "order filter", not the "algebraic filter".

Notation 4 We denote the smallest filter containing an element x of a po-semigroup S by $N(x)$ and call it the principal filter generated by x.

Definition 5. *A subset T of a po-semigroup S is called Archimedean if for each $a, b \in T$ there exists a positive integer n such that $a^n \leq \mu b\nu$ for some $\mu, \nu \in T$ (cf.* [11], [15]*).*

Definition 6. *A congruence σ on a po-semigroup S is called a semilattice congruence if and only if for all $x, y \in S$, $xy\sigma yx$ and $x^2\sigma x$. (cf.* [6]*).*

Notation 7 Let S be a po-semigroup. Define $\mathcal{N} = \{(x, y) \in S \times S | N(x) = N(y)\}$. Then it is well known that the relation $\mathcal{N}$ is a semilattice congruence on the po-semigroup S (cf. [6]).

Notation 8 Let $\mathcal{SC}(S)$ be the collection of all semilattice congruences defined above on a po-semigroup S. For any $\sigma \in \mathcal{SC}(S)$, denote the congruence class of $x \in S$ by $(x)_\sigma$. Define "$\preceq$" by $(x)_\sigma \preceq (y)_\sigma \iff (x)_\sigma = (xy)_\sigma$ on the quotient semigroup $S/\sigma = \{(x)_\sigma | x \in S\}$. Then, it is well known that $(x)_\sigma$ is a subsemigroup of S and $[(S/\sigma, \cdot, \preceq)]$ is again a po-semigroup [9].

By using the above definitions and notations, Kehayopulu gave the following characterization for the semilattice congruence classes of a po-semigroup S.

Lemma 9 (cf. [15]**).** *Let σ be a semilattice congruence on a po-semigroup S. Then a σ-congruence class $(a)_\sigma$ is Archimedean for all $a \in S$ if and only if for all $a \in S$ and all $y \in (a)_\sigma \Longrightarrow$ there exists some $n \in N$ such that $y^n \leq \mu a\nu$ for some $\mu, \nu \in (a)_\sigma$.*

The following lemma describes the principal filters in a pseudo-commutative po-semigroup S. The idea of proof follows from [16].

Lemma 10. *Let S be a pseudo-commutative po-semigroup. Then for each $x \in S, N(x) = \{a \in S \mid \exists k \in N : \ x^k \leq \mu a\nu$ for some $\mu, \nu \in S\}$.*

Proof: Let $x \in S$ and $T := \{a \in S \mid \exists k \in N : \ x^k \leq \mu a\nu$ for some $\mu, \nu \in S\}$.

We first show that T is a filter of S containing x. Clearly, $\phi \neq T \subseteq S$ since $x^3 \leq xxx$, so $x \in T$. Now, we verify the following:

(i) T is subsemigroup of S. In fact, let $a, b \in T$, then, by the definition of T, we have $x^n \leq \mu_1, a\nu_1$, for some $\mu_1, \nu_1 \in S$; and $x^m \leq \mu_2 b\nu_2$ for some $\mu_2, \nu_2 \in S$, where $n, m \in N$. Since S is pseudo-commutative, $\big((\mu_1 a)\nu_1\big)^k \leq \nu_1(\mu_1 a)^k$ for some $k \in N$. Similarly, we have $\big(\mu_2(b\nu_2)\big)^\ell \leq (b\nu_2)\mu_2^\ell$. Then we have

$$x^{nk} = (x^n)^k \leq (\mu_1 a\nu_1)^k \leq \nu_1(\mu_1 a)^k = \nu_1(\mu_1 a)^{k-1}\mu_1 a.$$

This implies that $x^{nk} \leq \nu_1\mu_1' a$, where $\mu_1' = (\mu_1 a)^{k-1}\mu_1 \in S$. (for $k = 1, \mu_1' = (\mu_1 a)^\circ \mu_1 = \mu_1 \in S$). Similarly, we have

$$x^{m\ell} \leq (\mu_2 b\nu_2)^\ell \leq b\nu_2\mu_2^\ell = b\nu_2';\ \nu_2 = \nu_2\mu^\ell \in S.$$

Thus, $x^{nk+m\ell} = x^{mk}x^{m\ell} \leq \nu_1\mu_1'(ab)\nu_2'$; with $\nu_1\mu_1', \nu_2' \in S$ and $nk+m\ell \in N$. This shows that $ab \in T$. Hence, T is a subsemigroup of S.

(ii) Let $a, b \in S$ such that $ab \in T$. We want to show that $a, b \in T$. Since $x^k \leq \mu(ab)\nu = \mu a(bv); \mu, b\nu \in S$, we have $a \in T$. Also, since $x^k \leq (\mu a)b\nu$ with $\mu a, \nu \in S$, we have $b \in T$ by the definition of T.

(iii) Let $a \in T$ such that $a \leq b$ for some $b \in S$. We need to show $b \in T$. In fact, since $a \in T$, there exists $k \in N$ such that $x^k \leq \mu a \nu$ for some $\mu, \nu \in S$. Since $a \leq b$, $x^k \leq \mu a\nu \leq \mu b\nu$; for some $\mu, \nu \in S$. Thus $b \in T$.

We now claim that T is the smallest filter containing x. If our claim is established, then $T = N(x)$, by definition.

Let F be the filter of S such that $x \in F$. If $a \in T$, then there exist some $k \in N$ such that $x^k \leq \mu a \nu$ for some $\mu, \nu \in S$. Since F is a filter containing x, $x^k \in F$. Observe that $\mu a \nu \in S$ and $\mu a \nu \geq x^k \in F$, so we have $\mu a \nu \in F$.Consequently, $a \in F$ since F is a filter. Thus, our claim is established and hence $T = N(x)$. Our proof is completed. $\square$

Remark 1. The set $T = N(x)$ in the proof of the above lemma can be re-written in the following form: $T = \{a \in S | \exists k \in N \text{ and } \exists \mu, \nu \in N(x) : x^k \leq \mu a \nu\}$. For, if $a \in T$, then there exists a $k \in N$ such that $x^k \leq \mu a \nu$ for some $\mu, \nu \in S$. This implies that $\mu a \nu \in N(x)$ by the definition of $N(x)$. Since $N(x)$ is a filter, we have $\mu, \nu \in N(x)$. Now, we can easily deduce that $T = \{a \in S | \exists k \in N \text{ and } \exists \mu, \nu \in N(x) : x^k \leq \mu a \nu\}$.

Remark 2. It was announced by Kehayopulu that a po-semigroup S is weakly commutative if and only if for each $x \in S, N(x) = \{a \in S | x^n \in (aSa] \text{ for some } n \in N\}$ (cf.[11]). As pseudo-commutative po-semigroups are special weakly commutative semigroups, their $N(x)$ must be of the same form. Indeed, by Lemma 10, if the po-semigroup S is pseudo-commutative then $N(x) = \{a \in S | x^k \in (SaS] \text{ for some } k \in N\}$ for every $x \in S$. Thus, $x^{k_1} \leq tay$ for some $t, y \in S$. By the pseudo-commutativity of S, we have $x^{k_1m_1} \leq \big((ta)y\big)^{m_1} \leq y^\lambda(ta) = (y^\lambda t)a$ or $y(ta)^\lambda = (y(ta)^{\lambda-1}t)a$ for some $m_1, \lambda \in N$. Hence, $x^{km_1} \in (Sa]$. Similarly, by $x^{k_2} \leq t(ay)$, we can prove that $x^{k_2m_2} \in (aS]$. Let $m = k_1m_1k_2m_2 \in N$. Then, we have $x^m \in (aSa]$. In other words, $N(x) = \{a \in S | \exists m \in N : x^m \in (aSa]\}$, for every $x \in S$. On the other hand, if $a \in S$ with $x^n \in (aSa]$, then $x^n \in (Sa]$ and $x^n \in (aS]$. This leads to $x^n \in (SaS]$.

Remark 3. The forms of $N(x)$ for other subclasses of the class of weakly commutative po-semigroups have been also obtained in [3].

The following lemma concerning the semilattice congruence $\mathcal{N}$ is useful in proving our theorem for pseudo-commutative po-semigroups.

Lemma 11 (See [9]**).** *For the semilattice congruence $\mathcal{N} = \{(x, y) \in S \times S | N(x) = N(y)\}$ on a po-semigroup S, $a \leq b$ implies $(a, ab) \in \mathcal{N}$.*

By using lemma 9, lemma 10 and lemma 11, we obtain the following theorem.

Theorem 12. *The pseudo-commutative po-semigroups can be expressed as semilattices of some Archimedean po-semigroups.*

Proof: It is known that the relation $\mathcal{N}$ is a semilattice congruence on S and $(x)_\mathcal{N}$ is a subsemigroup of S for every $x \in S$. (See [9, the proof of the theorem]). We only need to prove that $(y)_\mathcal{N}$ is Archimedean for every $y \in S$. For this purpose, we let $b, x \in (y)_\mathcal{N}$. By lemma 9, we need to show that there exist $\lambda \in N$ and $z, t \in (y)_\mathcal{N}$ such that $b^\lambda \leq zxt$.

Since $b, x \in (y)_\mathcal{N}, (b, x) \in \mathcal{N}$. This implies that $N(b) = N(x)$. Since S is pseudo-commutative and $b \in N(x)$, we have, by lemma 10,

$$x^m \leq \mu b \mu' \tag{1}$$

for some $m \in N$ and $\mu, \mu' \in S$. Since $x \in N(b)$ and $N(b)$ is itself a subsemigroup, $x^{m+3} \in N(b)$. Thus, by lemma 10 again, there exists $n \in N, \nu, \nu' \in S$ such that

$$b^n \leq \nu x^{m+3} \nu'. \tag{2}$$

By (2) and lemma 11, we can easily deduce that $(b^n, b^n \nu x^{m+3} \nu') \in \mathcal{N} \Longrightarrow (b, b\nu x\nu') \in \mathcal{N} \Longrightarrow (x, b\nu x\nu') \in \mathcal{N}$ (since $(b, x) \in \mathcal{N}$ and $\mathcal{N} \in \mathcal{SC}(S)$). Consequently, we have:

$$\nu' b\nu x \in (x)_{\mathcal{N}}. \tag{3}$$

Now, applying (1) and lemma 11 again, we immediately obtain $(x^m, x^m \mu b\mu') \in \mathcal{N}$ and so

$$(x, x\mu b\mu') \in \mathcal{N} \Longrightarrow x\mu b\mu' \in (x)_{\mathcal{N}}. \tag{4}$$

Since $b\nu x^{m+3}, \nu' \in S$ and S is pseudo-commutative, there exists $k \in N$ such that

$$(b\nu x^{m+3}\nu') \leq \nu'(b\nu x^{m+3})^k. \tag{5}$$

Thus, by (2), we have $b^{n+1} \leq b\nu x^{m+3}\nu' \Longrightarrow (b^{n+1})^k \leq (b\nu x^{m+3}\nu')^k$ and by (5), we obtain

$$(b^{n+1})^k \leq \nu'(bvx^{m+3})^k. \tag{6}$$

Applying (1) again, we deduce further that

$$\begin{aligned} x^{m+3} &\leq x^3\mu b\mu' \\ \Longrightarrow \quad & b\nu x^{m+3} \leq b\nu x^3 \mu b\mu' \\ \Longrightarrow \quad & (b\nu x^{m+3})^k \leq (b\nu x^3\mu b\mu')^k \\ \Longrightarrow \quad & \nu'(b\nu x^{m+3})^k \leq \nu'(b\nu x^3\mu b\mu')^k. \end{aligned}$$

Now, using (6) and (7), we get $(b^{n+1})^k \leq \nu'(b\nu x^3\mu b\mu')^k \Longrightarrow b^{(n+1)k} \leq \nu'(b\nu x^3\mu b\mu')^{k-1}$ $b\nu x^3\mu b\mu'$. By putting $z = \nu'(b\nu x^3\mu b\mu')^{k-1}b\nu x$ and $t = x\mu b\mu'$, we have $z, t \in S$ and $b^{(n+1)k} \leq zxt$, where $(n+1)k \in N$. This shows that $z, t \in (y)_{\mathcal{N}}$. Consequently, by (4), we have $t \in (x)_{\mathcal{N}} = (y)_{\mathcal{N}} \Longrightarrow t \in (y)_{\mathcal{N}}$. It still remains to show that $z \in (y)_{\mathcal{N}}$. For this purpose, we consider the following cases:

(α) If $k = 1$, then $z = \nu' b\nu x$. By using (3), we have $z \in (x)_{\mathcal{N}} = (y)_{\mathcal{N}} \Longrightarrow z \in (y)_{\mathcal{N}}$.

(β) If $k \neq 1$, then since $z = \nu'(b\nu x^3\mu b\mu')^{k-1}b\nu x$ and $\mathcal{N}$ is a semilattice congruence on S, we have

$$(z, \nu' b\nu x x\mu b\mu') \in \mathcal{N}. \tag{7}$$

By using (3), (4) and noting that $\mathcal{N} \in \mathcal{SC}(S)$, we have

$$(\nu' b' \nu x x \mu b\mu', x) \in \mathcal{N}. \tag{8}$$

Applying (8) and (9), it follows that $(z, x) \in \mathcal{N}$ and hence $z \in (x)_{\mathcal{N}} = (y)_{\mathcal{N}}$. The proof is completed. □

Proposition 13. *The semilattice congruence $\mathcal{N}$ on a po-semigroup S is the greatest semilattice congruence on S such that $(x)_{\mathcal{N}}$ is Archimedean for every $x \in S$.*

Proof: Let σ be a semilattice congruence on S. Then it can be easily seen that $(x)_\sigma$ is an Archimedean subsemigroup of S, for all $x \in S$. (Note : $(x)_\sigma$ is not necessarily a subsemigroup of S unless σ is a semilattice congruence). Let $(a, b) \in \sigma$. Then, since $a, b \in (b)_\sigma$, by the Archimedean property of $(b)_\sigma$, there exist $n \in N, \mu, \nu \in (b)_\sigma$ such that $a^n \leq \mu b\nu$. Since $a^n \in N(a)$ and $N(a)$ is a filter of S, we have $\mu b\nu \in N(a)$. This leads to $b \in N(a)$ so that $N(b) \subseteq N(a)$. Similarly, we have $N(b) \subseteq N(a)$. Consequently $N(a) = N(b)$

and so $(a,b) \in \mathcal{N}$. This implies that $\mathcal{N}$ is the greatest semilattice congruence on S such that $(x)_{\mathcal{N}}$ is Archimedean for every $x \in S$. $\square$

In fact, in example 2, one can easily check that there are two semilattice congruences on S and they are of the following forms (cf. [4]):

$$\begin{aligned} \mathcal{N} &= \{(x,y) \in S \times S | N(x) = N(y)\} = S \times S \\ \eta &= \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,a),(c,d),(d,c)\}. \end{aligned}$$

Clearly, $\eta \nsubseteq \mathcal{N}$ as η does not satisfy the condition given in lemma 11. Considering the η-classes, we can see that $(a)_\eta = (b)_\eta = \{a,b\}, (c)_\eta = (d)_\eta = \{c,d\}$. Since $a^2 = b$ and $b^2 = b$, $(a)_\eta$ is Archimedean. Similarly, since $d^2 = c, c^2 = c$, $(c)_\eta$ is also Archimedean. Thus, apart from $\mathcal{N}$, the congruence η gives a semilattice decomposition of S into Archimedean semigroups. Hence, this example illustrates that the decomposition of a pseudo-commutative po-semigroup into a semilattice of Archimedean po-semigroups is not necessarily unique.

Remark (cf. [4]) In the above example, it is clear that the semilattice congruence $\mathcal{N} = \{(x,y) \in S \times S | N(x) = N(y)\}$ on S is not the least semilattice congruence on the po-semigroup S.

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