TRANSLATABLE RADII OF AN OPERATOR IN THE DIRECTION OF ANOTHER OPERATOR

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Abstract. We introduce translatable radii of an operator corresponding to the norm in the direction of another operator and generalise an inequality developed by Fujii and Prasanna[2], namely, $sup_{\|x\|=1}\|Tx-(Tx,x)x\| \geq w_T$.

Introduction.

Let T and A be bounded linear operators on a complex Hilbert space H with the inner product (,). Consider the generalized eigenvalue problem $Tx = \lambda Ax, x \in H$, where λ is called the eigenvalue of the above equation and x the corresponding eigenvector. Mikhlin [4] has studied this problem in a way similar to the problem $Tx = \lambda x$.

The nonnegative functional $S(x) = \|Tx-(Tx,x)x\|$ gives the deviation of a unit vector x from being an eigenvector . Bjorck and Thomee have shown [1] that

$$sup_{\|x\|=1}\{\|Tx\|^2 - |\ (Tx,x)\ |^2\}^{1/2} = R_T$$

for a normal operator T where R_T is the radius of the smallest circle containing the spectrum of T.

Garske [3] improved the result to obtain the inequality

$$sup_{\|x\|=1}\{\|Tx\|^2 - |\ (Tx,x)\ |^2\}^{1/2} \geq R_T$$

which was later improved by Fujii and Prasanna [2].

Translatable radii of an operator in the direction of another operator.

If 0 does not belong to approximate point spectrum of A let

$$\begin{aligned} M_A(T) &= sup_{\|x\|=1}\{\|Tx\|^2 - \frac{|\ (Tx,Ax)\ |^2}{(Ax,Ax)}\}^{1/2} \\ \text{i.e.}\quad M_A(T) &= sup_{\|x\|=1}\{\|Tx - \frac{(Tx,Ax)}{(Ax,Ax)}Ax\|\} \end{aligned}$$

Also if $0 \notin \overline{W(A)}$ then let

$$\hat{M}_A(T) = sup_{\|x\|=1}\{\|Tx - \frac{(Tx,x)}{(Ax,x)}Ax\|\}$$

Clearly $M_A(T) = M_A(T+\lambda A)$ and $\hat{M}_A(T) = \hat{M}_A(T+\lambda A)$ so that both are translation invariant in the sense of A. We define $M_A(T)$ and $\hat{M}_A(T)$ as the translatable radii of T in the direction of A.Geometrically $Tx - \frac{(Tx,Ax)}{(Ax,Ax)}Ax$ is the vector perpendicular from Tx to Ax and $Tx - \frac{(Tx,x)}{(Ax,x)}Ax$ is a vector perpendicular to x.

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Let $W_A(T) = \{\frac{(Tx,Ax)}{(Ax,Ax)} : \|x\| = 1\}$ and $\hat{W}_A(T) = \{\frac{(Tx,x)}{(Ax,x)} : \|x\| = 1\}$. Clearly $W_A(T)$ is convex. Examples can be given to show that $\hat{W}_A(T)$ need not be convex. Let $m_A(T)$ $(resp.\ \hat{m}_A(T)\ )$ denote the radius of the smallest circle containing the set $W_A(T)$ $(resp.\ \hat{W}_A(T)\ )$. Also let $\mid W_A(T) \mid = sup\{\mid z \mid : z \in W_A(T)\}$ and $\mid \hat{W}_A(T) \mid = sup\{\mid z \mid : z \in \hat{W}_A(T)\}$.

Main Result. Fujii and Prasanna [2] proved that for any bounded linear operator T

$$sup_{\|x\|=1}\|Tx - (Tx,x)x\| \geq w_T$$

In this paper we generalize the result to prove that

$$if\ 0 \notin \overline{W(A)}\ then\ \hat{M}_A(T) \geq M_A(T) \geq m_A(T)/\|A^{-1}\|.$$

To prove this we need the following lemmas.

Lemma 1.

$$m_A(T) = min_z \mid W_A(T - zA) \mid = min_z \mid W_A(T) - z \mid$$
$$\hat{m}_A(T) = min_z \mid \hat{W}_A(T - zA) \mid = min_z \mid \hat{W}_A(T) - z \mid$$

Proof. The proof is clear from the definitions.

Lemma2. $\|T\| \leq \|T - zA\|\ \forall z \in C$ iff there exists a sequence $\{x_n\}$ of unit vectors such that $(A^*Tx_n, x_n) \to 0$ and $\|Tx_n\| \to \|T\|$.

Proof. We prove the necessary part only, sufficient part follows easily.

Let us consider the set $W_0(A) = \{\lambda \in C/\exists\{x_n\} \subset H, \|x_n\| = 1 \ni (Tx_n, Ax_n) \to \lambda$ and $\|Tx_n\| \to \|T\|\ \}$, which is non-empty, closed and convex.

Let us first assume that $\|A\| \leq 1$. If possible let $0 \notin W_0(A)$. Then as $W_0(A)$ is closed and convex by rotating T suitably we can assume that Re $W_0(A) > \eta > 0$.

Let M $= \{x \in H/\|x\| = 1$ and Re $(Tx, Ax) \leq \eta/2\}$ and $\beta = sup_{x\in M}\|Tx\|$. Clearly $\beta < \|T\|$. Let $z_0 =$ min $\{\ \eta, (\|T\| - \beta)/\|A\|\}$. Now if $x \in$ M, then $\|(T - z_0A)x\| \leq \|Tx\| + \mid z_0 \mid \|Ax\| < \beta + \{(\|T\| - \beta)/\|A\|\}\|A\| = \|T\|$ and if $x \notin M$, then let $Tx = (a + ib)Ax + y$, where $(Ax, y) = 0$.

$$\begin{aligned} So, \|(T - z_0A)x\|^2 &= \{(a - z_0)^2 + b^2\}\|Ax\|^2 + \|y\|^2 \\ &= \|Tx\|^2 + ({z_0}^2 - 2az_0\|Ax\|^2) + {z_0}^2(\|Ax\|^2 - 1) \\ &< \|Tx\|^2, for\ Re(Tx, Ax) = a\|Ax\|^2 > \eta/2 > z_0/2\ and\ \|Ax\| \leq 1 \end{aligned}$$

Thus in all cases, $\|(T - z_0A)x\|^2 < \|T\|^2$ so that $\|T\| \geq \|T - z_0A\|$ - This is a contradiction. Hence there exists a sequence $\{x_n\}$ of unit vectors such that $(Tx_n, Ax_n) \to 0$ and $\|Tx_n\| \to \|T\|$.

Next let $\|A\| > 1$. Then let $B = A/\|A\|$. Proceeding as above we can find a sequence $\{y_n\}$ of unit vectors such that $(Ty_n, By_n) \to 0$ and $\|Ty_n\| \to \|T\|$. So $\|T\| \leq \|T - z(A/\|A\|)\|$ forall z $\in$ C iff there exists a sequence $\{y_n\}$ of unit vectors such that $((A^*T/\|A\|)y_n, y_n) \to 0$ and $\|Ty_n\| \to \|T\|$.

Hence for any bounded linear operator A, $\|T\| \leq \|T - zA\|$ forall z $\in$ C iff there exists a sequence $\{x_n\}$ of unit vectors such that $(A^*Tx_n, x_n) \to 0$ and $\|Tx_n\| \to \|T\|$. This completes the proof of Lemma 2.

Lemma 3, stated below, is a generalization of a result by S.Prasanna [6] , namely , for any bounded linear operator T , $min_z\|T - zI\| = sup_{\|x\|=1}\{\|Tx\|^2 - \mid (Tx, x) \mid^2\}^{1/2}$. This paper is based on the result stated in Lemma 3. Detailed proofs of both Lemma 2 and Lemma 3 are given in [5].

Lemma 3. $M_A(T) = min_z\|T - zA\|$.

Proof. We may assume the existence of $M_A(T) = \min_z \|T - zA\|$ by $\lim_{|z|\to\infty} \|T - zA\| =$

$+\infty$. Since $M_A(T)$ is translation invariant in the sense of A i.e., $M_A(T) = M_A(T - zA)$ for all z in C, it suffices to prove that if $\|T\| \leq \|T - zA\|$ $\forall z \in$ C then $M_A(T) = \|T\|$.
We have $\|Tx\|^2 \geq \|Tx\|^2 - \frac{|(Tx,Ax)|^2}{\|Ax\|^2}$ for all unit vectors x $\in$ H.
so that $\|T\| \geq M_A(T)$.
Again by Lemma 2 there exists a sequence $\{x_n\}$ of unit vectors such that $(Tx_n, Ax_n) \to 0$ and $\|Tx_n\| \to \|T\|$.
Now

$$\begin{aligned} \|T\|^2 &= \lim_{n\to\infty} \|Tx_n\|^2 \\ &= \lim_{n\to\infty} \{\|Tx_n\|^2 - \frac{|\ (Tx_n, Ax_n)\ |^2}{\|Ax_n\|^2}\} \\ &\leq M_A(T)^2 \end{aligned}$$

Thus $\|T\| = M_A(T)$. This completes the proof of Lemma 3.
We now prove our main result in Theorem 1.
Theorem1. If $0 \notin \overline{W(A)}$ then $\hat{M}_A(T) \geq M_A(T) \geq m_A(T)/\|A^{-1}\|$.

Proof. Since $\frac{|(Tx,Ax)|}{(Ax,Ax)} \leq \|T\|\|A^{-1}\|$ for a unit vector $x \in H$, we have $|\ W_A(T)\ | \leq \|T\|\|A^{-1}\|$ for all operators T, so that

$$|\ W_A(T - zA)\ | \leq \|T - zA\|\|A^{-1}\|$$

for all $z \in C$. Hence it follows from Lemma 1 and Lemma 3 that

$$m_A(T) \leq M_A(T)\|A^{-1}\|.$$

Let $Tx = \frac{(Tx,Ax)}{(Ax,Ax)}Ax + h$ and $Tx = \frac{(Tx,x)}{(Ax,x)}Ax + \hat{h}$, where $(h, Ax) = 0$ and $(\hat{h}, x) = 0$. Then

$$\hat{h} - h = \{\frac{(Tx, Ax)}{(Ax, Ax)} - \frac{(Tx, x)}{(Ax, x)}\}Ax$$

As $(h, Ax) = 0$ we get

$$\|\hat{h}\|^2 = \|h\|^2 + \{|\ \frac{(Tx, Ax)}{(Ax, Ax)} - \frac{(Tx, x)}{(Ax, x)}\ |\}^2\|Ax\|^2$$

Thus $\|\hat{h}\| \geq \|h\|$ so that

$$sup_{\|x\|=1}\{\|Tx - \frac{(Tx, x)}{(Ax, x)}Ax\|\} \geq sup_{\|x\|=1}\{\|Tx - \frac{(Tx, Ax)}{(Ax, Ax)}Ax\|\}$$

i,e, $\hat{M}_A(T) \geq M_A(T)$. So

$$\hat{M}_A(T) \geq M_A(T) \geq m_A(T)/\|A^{-1}\|.$$

This completes the proof.
Theorem2. If $0 < c \leq |\ (Ax, x)\ |$, for all unit vectors $x \in H$, then $\hat{M}_A(T) \geq M_A(T) \geq c\ \ \hat{m}_A(T)$.

Proof. We have $\frac{|(Tx,x)|}{|(Ax,x)|} \leq \|T\|/c$ for all unit vectors x $\in$ H, so that $\|T\| \geq c\ |\ \hat{W}_A(T)\ |$ for all operators T. Hence

$$c.\ |\ \hat{W}_A(T - zA)\ | \leq \|T - zA\|$$

for all $z \in C$. Hence it follows from Lemma 1 and Lemma 3 that

$$c\ \hat{m}_A(T) \leq M_A(T).$$

Since $\hat{M}_A(T) \geq M_A(T)$, $\hat{M}_A(T) \geq M_A(T) \geq c\ \ \hat{m}_A(T)$. This completes the proof.

Corollary. For A = I we get the inequality due to Fujii and Prasanna[2].

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