

## LOCALLY WEAK-STAR UNIFORM ROTUNDITY OF ORLICZ SEQUENCE SPACES

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**ABSTRACT.** The criteria for locally weak-star uniform rotundity of  $l_M$  and  $l_M^0$  are given. For  $l_M$ , locally weak-star uniform rotundity is equivalent to locally uniform rotundity, and for  $l_M^0$ , locally weak-star uniform rotundity is weaker essentially than locally weak uniform rotundity.

In the last years, there were a lot of discussion on the criteria for various rotundity of Orlicz sequence spaces, and many satisfactory results were obtained, such as rotundity ( $R$ )<sup>[1][22]</sup>, uniform rotundity ( $UR$ )<sup>[2][23–25]</sup>, K-uniform rotundity ( $KUR$ )<sup>[3,4]</sup>, locally uniform rotundity ( $LUR$ )<sup>[5,6]</sup>, locally weak uniform rotundity ( $LWUR$ )<sup>[5,6]</sup>, weakly uniform rotundity( $WUR$ )<sup>[7,8]</sup>, uniform rotundity in every direction ( $URED$ )<sup>[9,10]</sup>, mid-point locally uniform rotundity ( $MLUR$ )<sup>[11]</sup>, H-rotundity ( $HR$ )<sup>[12]</sup>, Fully-K rotundity<sup>[13]</sup>,  $P$ –convexity<sup>[14]</sup>,  $B$ –convexity<sup>[15–17][26]</sup>, and so on. In this paper, we will discuss the unsolved problem namely, the criteria for locally weak-star uniform rotundity. The results and the method of the proof seem to be interesting. This concerns especially  $l_M^0$ , where  $LW * UR$  is weaker essentially than  $LUR$  and  $LWUR$ .

A Banach space  $(X, \|\cdot\|)$  is said to be  $LUR(LWUR, LW^*UR)$  provided for any  $x, \{x_n\}$  with  $\|x\| = \|x_n\| = 1$  ( $n = 1, 2, \dots$ ), if  $\|x+x_n\| \rightarrow 2$  ( $n \rightarrow \infty$ ), then  $\|x_n-x\| \rightarrow 0$  ( $x_n-x \xrightarrow{W} 0, x_n-x \xrightarrow{W^*} 0$ )<sup>[18]</sup>. It is obvious that  $LUR \Rightarrow LWUR \Rightarrow LW^*UR$ .

Let  $M(u)$  be a N-function and  $N(v)$  its complementary function,  $p_-(u)$  and  $p(u)$  be the left derivative and the right derivative of  $M(u)$ ,  $q_-(v)$  and  $q(v)$  be the left derivative and right derivative of  $N(v)$ , respectively. The condition  $M \in \Delta_2$  means that there exist  $u_0 > 0, k > 2$ , such that  $M(2u) \leq kM(u)$  for  $0 \leq u \leq u_0$ .  $M \in \nabla_2$  indicates that  $N \in \Delta_2$ ,  $M \in SC[a, b]$  means that  $M(u)$  is strictly convex in  $[a, b]$ , i.e.,  $u, v \in [a, b]$  and  $u \neq v$  imply  $M(\frac{u+v}{2}) < \frac{M(u)+M(v)}{2}$ . Let  $x = (x(j))_{j=1}^\infty$  be a real sequence. Then  $\rho_M(x) = \sum_{j=1}^\infty M(x(j))$  is said to be the modular of  $x$  with respect to  $M(u)$ . The linear space  $\{x : \exists \lambda > 0, \rho_M(\lambda x) < \infty\}$  endowed with the Luxemburg norm

$$\|x\| = \inf\{c > 0 : \rho_M\left(\frac{x}{c}\right) \leq 1\}$$

or the Orlicz norm

$$\|x\|^0 = \inf\left\{\frac{1}{k}(1 + \rho_M(kx)) : k > 0\right\}$$

are both Banach spaces, denoted by  $l_M$  and  $l_M^0$  respectively, and called an Orlicz sequence space .In this paper, the following known results will be quoted.

**Lemma 1**<sup>[10]</sup> For any  $0 < \lambda, \delta < 1$  and  $[a, b] \subset (0, 1)$ , there exists  $0 < \delta' \leq \delta$ , such that if  $u, v \geq 0$  and  $M(\lambda u + (1 - \lambda)v) \leq (1 - \delta)(\lambda M(u) + (1 - \lambda)M(v))$ , then  $M(\lambda' u + (1 - \lambda')v) \leq (1 - \delta')(\lambda' M(u) + (1 - \lambda')M(v))$  for any  $\lambda' \in [a, b]$ .

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**Lemma 2<sup>[20]</sup>** For any  $x, 0 \neq x \in l_M^0$ , the equality  $\|x\|^0 = \frac{1}{k}(1 + \rho_M(kx))$  holds if and only if  $k \in K(x) = [k_x^*, k_x^{**}]$ , where

$$k_x^* = \inf\{k > 0 : \rho_N(p(kx)) \geq 1\} \text{ and } k_x^{**} = \sup\{k > 0 : \rho_N(p(kx)) \leq 1\}.$$

**Lemma 3<sup>[21]</sup>** If  $M \in SC[0, q_-(N^{-1}(1))]$ ,  $1 = \|x\|^0 = \frac{1}{k}(1 + \rho_M(kx))$ ,  $1 = \|x_n\|^0 = \frac{1}{k_n}(1 + \rho_M(k_n x_n))$ , ( $n = 1, 2, \dots$ ),  $\|x_n + x\|^0 \rightarrow 2$  ( $n \rightarrow \infty$ ) and  $\{k_n\}$  is bounded, then  $k_n x_n(j) \rightarrow kx(j)$  as  $n \rightarrow \infty$  for every  $j$ .

See [20],[21] for other knowledge about Orlicz sequence space.

#### Main Results:

**Theorem 1:** The space  $l_M$  is  $LW^*UR$  if and only if

- i)  $M \in \Delta_2, M \in SC[0, M^{-1}(\frac{1}{2})];$
- ii)  $M \in \nabla_2 \text{ or } M \in SC[M^{-1}(\frac{1}{2}), M^{-1}(1)].$

#### Proof:

Sufficiency. From[8] it follows trivially that  $l_M$  is  $LUR$ , then it is trivial that  $l_M$  is  $LW^*UR$ .

Necessity. Since  $LW^*UR \Rightarrow R$ , by Th.2.7 of [20], the necessity of i) is obtained immediately. Suppose (ii) is not true. Then there exists a  $M(u)$ 's affine interval  $[a, b] \subset [M^{-1}(\frac{1}{2}), M^{-1}(1)]$  and  $u_n < \frac{1}{n}$ , such that  $M(\frac{u_n}{2}) > (1 - \frac{1}{n})\frac{M(u_n)}{2}$  ( $n = 1, 2, \dots$ ). Take  $m_n \in N$  satisfying

$$M(b) - M(a) - \frac{1}{n} < m_n M(u_n) \leq M(b) - M(a) \quad (n = 1, 2, \dots)$$

and  $c \geq 0$  satisfying  $M(b) + M(c) = 1$ . Put

$$\begin{aligned} x &= (c, b, 0, 0, \dots), \\ x_n &= (c, a, \underbrace{u_n, u_n, \dots, u_n}_{m_n}, 0, 0, \dots) \quad (n = 1, 2, \dots). \end{aligned}$$

Then  $\rho_M(x) = 1, \rho_M(x_n) = M(c) + M(a) + m_n M(u_n) \leq M(c) + M(b) = 1$  ( $n = 1, 2, \dots$ ). So  $\|x\| = 1, \|x_n\| \leq 1$  ( $n = 1, 2, \dots$ ). Moreover,

$$\begin{aligned} \rho_M\left(\frac{x_n + x}{2}\right) &= M(c) + M\left(\frac{a+b}{2}\right) + m_n M\left(\frac{u_n}{2}\right) \\ &> M(c) + \frac{M(a) + M(b)}{2} + (1 - \frac{1}{n})\frac{M(b) - M(a) - \frac{1}{n}}{2} \\ &\rightarrow M(c) + M(b) = 1 \text{ as } n \rightarrow \infty, \end{aligned}$$

whence  $\|x_n + x\| \rightarrow 2$  ( $n \rightarrow \infty$ ). But  $x(2) - x_n(2) = b - a$ , so  $x_n \xrightarrow{W^*} x$  does not hold, a contradiction.

**Corollary 1:** For Orlicz sequence space  $l_M$  there holds the following:

$$LUR \iff LWUR \iff LW^*UR.$$

**Theorem 2:** The space  $l_M^0$  is  $LW^*UR$  if and only if

- i)  $M \in SC[0, q_-(N^{-1}(1))];$
- ii)  $M \in \nabla_2;$
- iii) For any  $\varepsilon \in (0, 1)$ , there exist  $a > 1$ , and  $\delta \in (0, 1)$ , such that the conditions  $0 \leq \varepsilon u \leq v < u \leq 1/\varepsilon$ ,  $M(u) \geq \varepsilon p_-(u)$  and  $M\left(\frac{u+v}{2}\right) > (1 - \delta)\frac{M(u) + M(v)}{2}$ , imply  $p_-((1 - \varepsilon)u) \leq a p_-(v)$ .

**Proof:**

Sufficiency. Let

$$1 = \|x\|^0 = \frac{1}{k}(1 + \rho_M(kx)) = \|x_n\|^0 = \frac{1}{k_n}(1 + \rho_M(k_n x_n))(n = 1, 2, \dots) \text{ and}$$

$\|x + x_n\|^0 \rightarrow 2(n \rightarrow \infty)$ . From ii), we get  $\bar{k} = \sup_n k_n < \infty$ . By lemma 3, it follows that

$$\lim_{n \rightarrow \infty} k_n x_n(j) = kx(j) \quad (j = 1, 2, \dots) \quad (1)$$

If  $\lim_{n \rightarrow \infty} k_n = k$  can be proved, then  $\lim_{n \rightarrow \infty} x_n(j) = x(j)$  ( $j = 1, 2, \dots$ ) can be obtained immediately. So  $x_n \xrightarrow{h_N} x$  can be easily get, i.e.  $x_n \xrightarrow{W^*} x$ . Therefore, we only need to prove  $k_n \rightarrow k$  ( $n \rightarrow \infty$ ).

Since  $M \in \nabla_2$ , there exists  $0 < \eta < 1$ , such that if  $1 \leq u < \bar{k}M^{-1}(1)$  and  $\lambda \in [\frac{1}{1+k}, \frac{1+2\bar{k}}{2+2\bar{k}}] \subset (0, 1)$ , we have

$$M(\lambda u) \leq (1 - \eta)\lambda M(u). \quad (*)$$

Let  $\varepsilon$  be an arbitrary number satisfying  $0 < \varepsilon < \min\{\frac{\eta}{k}, \frac{1}{2}\}$ . Let  $a > 1$  and  $\delta \in (0, 1)$  be chosen according to  $\varepsilon$  in condition iii). By lemma 1, for this  $\delta, \lambda = \frac{1}{2}, [\frac{1}{1+k}, \frac{\bar{k}}{1+k}] \subset (0, 1)$ , there exists  $\delta' > 0$  such that if  $M\left(\frac{u+v}{2}\right) \leq (1 - \delta)\frac{M(u) + M(v)}{2}$ , then

$$M(\lambda u + (1 - \lambda)v) \leq (1 - \delta')(\lambda M(u) + (1 - \lambda)M(v)) \quad (2)$$

for any  $\lambda \in [\frac{1}{1+k}, \frac{\bar{k}}{1+k}]$ .

Since  $\sum_{j=1}^{\infty} M(kx(j)) \leq \sum_{j=1}^{\infty} k|x(j)|p_-(k|x(j)|) \leq k\|x\|^0 = k$ , so we can choose  $j_0$  large enough such that

$$\sum_{j>j_0} M(kx(j)) \leq \sum_{j>j_0} k|x(j)|p_-(kx(j)) < \frac{\varepsilon^2}{a} < \varepsilon. \quad (3)$$

If we fix  $j_0$ , then (1) implies that  $\left| \sum_{j=1}^{j_0} M(k_n(x_n(j))) - M(kx(j)) \right| < \varepsilon$  for  $n$  large enough. So, if  $n$  is large enough,

$$\begin{aligned} |k_n - k| &= |\rho_M(k_n x_n) - \rho_M(kx)| \\ &\leq \left| \sum_{j=1}^{j_0} (M(k_n x_n(j)) - M(kx(j))) \right| + \sum_{j>j_0} M(kx(j)) + \sum_{j>j_0} M(k_n x_n(j)) \\ &< \sum_{j>j_0} M(k_n x_n(j)) + 2\varepsilon. \end{aligned}$$

In the following, we only need to prove that

$$\limsup_{n \rightarrow \infty} \sum_{j>j_0} M(k_n x_n(j)) = 0(\varepsilon). \quad (4)$$

For each  $n$ , denote

$$\begin{aligned} A_n &= \{j : j > j_0; k_n|x_n(j)| \leq k|x(j)| \text{ or } M(k_n(x_n(j))) < \varepsilon k_n|x_n(j)|p_-(k_n|x_n(j)|)\}, \\ B_n &= \{j : j > j_0; x_n(j)x(j) < 0 \text{ or } k_n|x_n(j)| < \varepsilon k_n|x_n(j)|\} \text{ or} \\ M\left(\frac{kk_n}{k+k_n}(x_n(j) + x(j))\right) &\leq (1 - \delta')\left(\frac{k}{k+k_n}M(k_n x_n(j)) + \frac{k_n}{k+k_n}M(kx(j))\right)\}, \\ C_n &= \{j : j > j_0\} \setminus A_n \setminus B_n. \end{aligned}$$

When  $j \in A_n$ , by the definition of  $A_n$ , it follows that

$$M(k_n x_n(j)) \leq M(kx(j)) + \varepsilon \bar{k} |x_n(j)| p_-(k_n |x_n(j)|).$$

By (3), we have

$$\sum_{j \in A_n} M(k_n x_n(j)) < \varepsilon + \varepsilon \bar{k} \|x_n\|^0 = \varepsilon(1 + \bar{k}) \quad (5)$$

If  $j \in C_n$ , we get  $j > j_0$ ,  $0 \leq \varepsilon k_n |x_n(j)| \leq kx(j) < k_n |x_n(j)|$ ,  $x(j)x_n(j) \geq 0$ ,  $M(k_n x_n(j)) \geq \varepsilon k_n |x_n(j)| p_-(k_n |x_n(j)|)$  and  $M\left(\frac{k k_n}{k + k_n}(x_n(j) + x(j))\right) > (1 - \delta')\left(\frac{k}{k + k_n} M(k_n x_n(j)) + \frac{k_n}{k + k_n} M(kx(j))\right)$ . By (2), we have

$$M\left(\frac{k_n x_n(j) + kx(j)}{2}\right) > (1 - \delta) \frac{M(k_n x_n(j)) + M(kx(j))}{2}.$$

By condition(iii),  $p_-((1 - \varepsilon)|k_n x_n(j)|) \leq ap_-(k|x(j)|)$ . Hence,

$$\begin{aligned} M(k_n x_n(j)) &= M((1 - \varepsilon)k_n x_n(j) + \int_{(1-\varepsilon)k_n |x_n(j)|}^{k_n |x_n(j)|} p(s) ds) \\ &\leq (1 - \varepsilon)k_n |x_n(j)| p_-((1 - \varepsilon)k_n |x_n(j)|) + \varepsilon k_n |x_n(j)| p_-(k_n |x_n(j)|) \\ &\leq \frac{1}{\varepsilon} k |x(j)| a p_-(k |x(j)|) + \varepsilon \bar{k} |x_n(j)| p_-(k_n |x_n(j)|). \end{aligned}$$

By (3), we have

$$\sum_{j \in C_n} M(k_n x_n(j)) < \varepsilon + \varepsilon \bar{k} \|x_n\|^0 = \varepsilon(1 + \bar{k}). \quad (6)$$

If  $j \in B_n$ , and  $x_n(j)x(j) < 0$ , by (\*)

$$\begin{aligned} M\left(\frac{k_n k}{k_n + k}(x_n(j) + x(j))\right) &\leq M\left(\frac{k_n k}{k_n + k} \max(|x_n(j)|, |x(j)|)\right) \\ &\leq (1 - \eta)\left(\frac{k}{k_n + k} M(k_n x_n(j)) + \frac{k_n}{k_n + k} M(kx(j))\right). \end{aligned}$$

When  $j \in B_n$  and  $k|x(j)| < \varepsilon k_n |x_n(j)|$  is satisfied, noticing that  $\frac{\varepsilon k_n + k}{k_n + k} \leq \frac{1 + 2\bar{k}}{2 + 2\bar{k}}$ , by (\*) we have

$$\begin{aligned} M\left(\frac{k_n k}{k_n + k}(x_n(j) + x(j))\right) &\leq M\left(\frac{\varepsilon k_n + k}{k_n + k} k_n x_n(j)\right) \\ &\leq (1 - \eta) \frac{\varepsilon k_n + k}{k_n + k} M(k_n x_n(j)) = (1 - \eta) \frac{\varepsilon k_n + k}{k} \cdot \frac{k}{k_n + k} M(k_n x_n(j)) \\ &\leq (1 - \eta)(1 + \varepsilon \bar{k}) \frac{k}{k_n + k} M(k_n x_n(j)) \\ &\leq (1 - \eta^2)\left(\frac{k}{k_n + k} M(k_n x_n(j)) + \frac{k_n}{k_n + k} M(kx(j))\right). \end{aligned}$$

Denote  $\delta'' = \min\{\eta^2, \delta'\}$ . Then for  $j \in B_n$ , we have

$$M\left(\frac{k_n k}{k_n + k}(x_n(j) + x(j))\right) \leq (1 - \delta'')\left(\frac{k}{k_n + k} M(k_n x_n(j)) + \frac{k_n}{k_n + k} M(kx(j))\right).$$

From

$$\begin{aligned} 0 &\leftarrow \|x\|^0 + \|x_n\|^0 - \|x + x_n\|^0 \\ &\geq \frac{1}{k_n}(1 + \rho_M(k_n x_n)) + \frac{1}{k}(1 + \rho_M(kx)) - \frac{k + k_n}{kk_n} \rho_M\left(\frac{kk_n}{k + k_n}(x + x_n)\right) \\ &= \frac{k + k_n}{kk_n} \sum_{j=1}^{\infty} \left[ \frac{k}{k + k_n} M(k_n x_n(j)) + \frac{k_n}{k + k_n} M(kx(j)) - M\left(\frac{kk_n}{k + k_n}(x(j) + x_n(j))\right) \right] \\ &\geq \frac{k + k_n}{kk_n} \sum_{j \in B_n} \left[ \frac{k}{k + k_n} M(k_n x_n(j)) + \frac{k_n}{k + k_n} M(kx(j)) - M\left(\frac{kk_n}{k + k_n}(x(j) + x_n(j))\right) \right] \\ &\geq \frac{k + k_n}{kk_n} \sum_{j \in B_n} \delta'' \left( \frac{k}{k + k_n} M(k_n x_n(j)) + \frac{k_n}{k + k_n} M(kx(j)) \right) \geq \frac{\delta''}{k} \sum_{j \in B_n} M(k_n x_n(j)), \end{aligned}$$

we get

$$\sum_{j \in B_n} M(k_n x_n(j)) < \varepsilon \quad (7)$$

for  $n$  large enough. From (5), (6) and (7), we obtain (4), finishing the proof of the sufficiency.

Necessity:

Since  $LW^*UR \Rightarrow R$ , by Th2.9 of [20], the necessity of (i) follows.

Assume that (ii) is not true. Then there exist  $u_n \rightarrow 0$ ,  $N(p(u_n))/u_n p(u_n) < \frac{1}{2^n}$  ( $n = 1, 2, \dots$ ). Without loss of generality, let  $u_n p(u_n) < \frac{1}{n}$ . We can choose positive integers  $m_n$  such that  $1 - \frac{1}{n} < m_n u_n p(u_n) \leq 1$  for ( $n = 1, 2, \dots$ ).

Take  $c > 0$  satisfying  $N(p_-(c)) \leq 1 \leq N(p(c))$  and  $c_n > 0$  satisfying  $N(p_-(c_n)) \leq 1 - m_n N(p(u_n)) \leq N(p(c_n))$  for each  $n$ . Then  $c_n \rightarrow c$  ( $n \rightarrow \infty$ ) follows from  $m_n N(p(u_n)) < \frac{1}{2^n} \rightarrow 0$  ( $n \rightarrow \infty$ ). Take  $\eta, \eta_n \geq 0$  satisfying  $N(p_-(c) + \eta) = 1$ ,  $N(p_-(c_n) + \eta_n) + m_n N(p(u_n)) = 1$ . Clearly,  $p_-(c) + \eta \leq p(c)$ ,  $p_-(c_n) + \eta_n \leq p(c_n)$  ( $n = 1, 2, \dots$ ), because of the left continuity of  $p_-(\cdot)$ ,  $\eta_n \rightarrow \eta$  ( $n \rightarrow \infty$ ) can be easily get. Denoting

$$k = c(p_-(c) + \eta), \quad k_n = c_n(p_-(c_n) + \eta_n) + m_n u_n p(u_n) \quad (n = 1, 2, \dots),$$

we get  $k_n \rightarrow k + 1$  ( $n \rightarrow \infty$ ). Denote

$$\begin{aligned} x &= \frac{1}{k}(c, 0, 0, \dots), \\ x_n &= \frac{1}{k_n}(c_n, \overbrace{u_n, u_n, \dots, u_n}^{m_n}, 0, 0, \dots) \quad (n = 1, 2, \dots) \end{aligned}$$

It is known that  $k \in K(x)$ , and  $k_n \in K(x_n)$  for  $n = 1, 2, \dots$ . Therefore

$$\|x\|^0 = \frac{1}{k}(1 + \rho_M(kx)) = \frac{1}{k}(N(p_-(c) + \eta) + M(c)) = \frac{c(p_-(c) + \eta)}{k} = 1.$$

In the same way, we can get  $\|x_n\|^0 = 1$  ( $n = 1, 2, \dots$ ). Setting  $y_n = (p_-(c_n) + \eta_n, \overbrace{p(u_n), p(u_n), \dots, p(u_n)}^{m_n}, 0, 0, \dots)$ , we have  $\rho_N(y_n) = 1$ . Hence

$$\begin{aligned} \|x + x_n\|^0 &\geq \langle x + x_n, y_n \rangle = \left(\frac{c}{k} + \frac{c_n}{k_n}\right)(p_-(c_n) + \eta_n) + \frac{m_n u_n p(u_n)}{k_n} \\ &= 1 + \frac{c}{k}(p_-(c_n) + \eta_n) \rightarrow 2 \quad (n \rightarrow \infty). \end{aligned}$$

But  $x(1) - x_n(1) = \frac{c}{k} - \frac{c_n}{k_n} \rightarrow \frac{c}{k(k+1)}$ . It contradicts the condition  $x_n - x \xrightarrow{W^*} 0$ .

Finally, let us prove the necessity of (iii).

Suppose that (iii) does not hold. Then there exist  $\varepsilon > 0$ ,  $u_n, v_n \rightarrow 0$  as  $n \rightarrow \infty$ , such that

$$0 \leq \varepsilon u_n \leq v_n < u_n, M(u_n) \geq \varepsilon u_n p_-(u_n), M\left(\frac{u_n + v_n}{2}\right) > (1 - \frac{1}{n}) \frac{M(u_n) + M(v_n)}{2}.$$

and  $p_-(((1 - \varepsilon)u_n) > np_-(v_n)$  ( $n \rightarrow \infty$ ). Denote

$$\xi_0 = \sup \left\{ \xi > 0 : \limsup_{n \rightarrow \infty} \frac{p_-(((1 - \varepsilon)u_n)}{p_-(\xi v_n)} = \infty \right\}.$$

Then  $1 \leq \xi_0 \leq \frac{1 - \varepsilon}{\varepsilon}$ . Now, we will discuss the following two cases.

$$(I) \quad \limsup_{n \rightarrow \infty} \frac{p_-(((1 - \varepsilon)u_n)}{p_-(\xi_0 v_n)} = \infty.$$

Clearly,  $\xi_0 v_n < (1 - \varepsilon)u_n$ . For any  $\lambda > 1$ , since

$$\infty = \limsup_{n \rightarrow \infty} \frac{p_-((1 - \varepsilon)u_n)}{p_-(\xi_0 v_n)} = \limsup_{n \rightarrow \infty} \frac{p_-((1 - \varepsilon)u_n)}{p_-(\lambda \xi_0 v_n)} \cdot \frac{p_-(\lambda \xi_0 v_n)}{p_-(\xi_0 v_n)}$$

and  $\limsup_{n \rightarrow \infty} \frac{p_-((1 - \varepsilon)u_n)}{p_-(\lambda \xi_0 v_n)} < \infty$ , so  $\limsup_{n \rightarrow \infty} \frac{p_-(\lambda \xi_0 v_n)}{p_-(\xi_0 v_n)} = \infty$ . Hence, without loss of generality (passing to a subsequence if necessary), we can assume that  $p_-((1 + \frac{1}{n})\xi_0 v_n) > 2^{n+1}p_-(\xi_0 v_n)$ . Denoting  $\xi_0 v_n = w_n$ , we have

$$v_n \leq w_n < (1 - \varepsilon)u_n, \quad p_-((1 + \frac{1}{n})w_n) > 2^{n+1}p_-(w_n) \quad (8)$$

$$(II) \quad \limsup_{n \rightarrow \infty} \frac{p_-((1 - \varepsilon)u_n)}{p_-(\xi_0 v_n)} < \infty.$$

Obviously  $\xi_0 > 1$ . For any  $\lambda > 1$ ,

$$\infty = \limsup_{n \rightarrow \infty} \frac{p_-((1 - \varepsilon)u_n)}{p_-\left(\frac{\xi_0 v_n}{\lambda}\right)} = \limsup_{n \rightarrow \infty} \frac{p_-((1 - \varepsilon)u_n)}{p_-(\xi_0 v_n)} \cdot \frac{p_-(\xi_0 v_n)}{p_-\left(\frac{\xi_0 v_n}{\lambda}\right)}.$$

So  $\limsup_{n \rightarrow \infty} \frac{p_-(\xi_0 v_n)}{p_-\left(\frac{\xi_0 v_n}{\lambda}\right)} = \infty$ . Without loss of generality (passing to a subsequence if necessary), we can assume that  $p_-(\xi_0 v_n) > 2^{n+1}p_-\left(\frac{\xi_0 v_n}{1 + \frac{1}{n}}\right)$ . Denoting  $\frac{n \xi_0 v_n}{n + 1} = w_n$ , inequality (8) can be obtained.

Choose positive integer  $m_i$  such that

$$\frac{1}{2^{i+2}} < m_i N(p(w_i)) \leq \frac{1}{2^{i+1}} \quad (i = 1, 2, \dots).$$

By (8), for  $n$  large enough

$$N(p_-(u_n)) \geq N\left(p_-\left((1 + \frac{1}{n})(1 - \varepsilon)u_n\right)\right) \geq N\left(p_-\left((1 + \frac{1}{n})w_n\right)\right) > 2^{n+1}N(p_-(w_n)),$$

so  $m_n N(p_-(u_n)) > \frac{1}{2}$ . Without loss of generality, we may assume that  $N(p_-(u_n)) < \frac{1}{n}$ .

Take  $\bar{m}_n < m_n$  satisfying

$$\frac{1}{2} - \frac{1}{n} < \bar{m}_n N(p(u_n)) \leq \frac{1}{2} \quad (i = 1, 2, \dots)$$

Since  $M \in \nabla_2$ , there exists  $d > 0$  such that  $M(u) \leq u p_-(u) \leq d N(p_-(u))$  for small  $u > 0$ , Therefore, we have  $\sum_{i=1}^{\infty} m_i M(w_i) \leq d \sum_{i=1}^{\infty} m_i N(p(w_i)) \leq \frac{d}{2}$ . Denote

$$k = 1 + \sum_{i=1}^{\infty} m_i M(w_i),$$

$$k_n = 1 + \sum_{i \neq n} m_i M(w_i) + (m_n - \bar{m}_n)M(w_n) + \bar{m}_n M(u_n) \quad (n = 1, 2, \dots)$$

Then  $k < \infty$ ,  $\sup_n k_n < \infty$ . Since

$$\bar{m}_n M(u_n) \geq \bar{m}_n \varepsilon u_n p_-(u_n) \geq \varepsilon \bar{m}_n N(p_-(u_n)) \geq \varepsilon \left(\frac{1}{2} - \frac{1}{n}\right),$$

so  $\liminf_{n \rightarrow \infty} (k_n - k) \geq \frac{\varepsilon}{2}$ . Set

$$x = \frac{1}{k}(\overbrace{w_1 \dots w_1}^{m_1} \overbrace{w_2 \dots w_2}^{m_2} \dots),$$

$$x_n = \frac{1}{k_n}(\overbrace{w_1 \dots w_1}^{m_1} \dots \overbrace{w_{n-1} \dots w_{n-1}}^{m_{n-1}} \overbrace{w_n \dots w_n}^{m_n - \bar{m}_n} \overbrace{w_n \dots w_n}^{\bar{m}_n} \dots \overbrace{w_{n+1} \dots w_{n+1}}^{m_{n+1}} \dots) \quad (n = 1, 2, \dots).$$

Since  $\rho_N(p_-(kx)) \leq \frac{1}{2} < 1$ ,  $\rho_N(p_-(1+s)kx) = \sum_{i=1}^{\infty} m_i N(p_-(1+s)w_i) > \sum_{i>\frac{1}{s}} m_i N(p_-(1+s)w_i) > \sum_{i>\frac{1}{s}} 2^{i+1} m_i N(p_-(w_i)) = \infty$ , for any  $s > 0$ . So  $k \in K(x)$ . In the same way, we can prove that  $k_n \in K(x_n)$  ( $n = 1, 2, \dots$ ). Therefore,

$$\|x\|^0 = \frac{1}{k}(1 + \rho_M(kx)) = \frac{1}{k}(1 + \sum_{i=1}^{\infty} m_i M(w_i)) = 1,$$

and  $\|x_n\|^0 = 1$  ( $n = 1, 2, \dots$ ). Notice that

$$\frac{k}{k_n+k} k_n x_n(j) + \frac{k_n}{k_n+k} kx(j) = \begin{cases} \frac{k}{k_n+k} u_n + \frac{k_n}{k_n+k} w_n, & j = \sum_{i=1}^{n-1} m_i + (m_n - \bar{m}_n) + 1, \\ & \dots, \sum_{i=1}^n m_i \\ kx(j), & \text{otherwise} \end{cases}$$

In the same way, the condition  $\frac{k_n k}{k_n+k} \in K(x+x_n)$  can be proved, so

$$\begin{aligned} \|x+x_n\|^0 &= \frac{k_n+k}{k_n k} \left( 1 + \rho_M \left( \frac{k_n k}{k_n+k} (x+x_n) \right) \right) \\ &= \frac{k_n+k}{k_n k} \left( 1 + \sum_{i \neq n} m_i M(w_i) + (m_n - \bar{m}_n) M(w_n) + \bar{m}_n M \left( \frac{k}{k_n+k} u_n + \frac{k_n}{k_n+k} w_n \right) \right) \end{aligned}$$

Since  $v_n \leq w_n < u_n$  and  $M\left(\frac{u_n+v_n}{2}\right) > \left(1 - \frac{1}{n}\right) \frac{M(u_n) + M(v_n)}{2}$ , it is easy to see that  $M\left(\frac{u_n+w_n}{2}\right) > \left(1 - \frac{1}{n}\right) \frac{M(u_n) + M(w_n)}{2}$ . By lemma 1, there exist  $\delta_n \rightarrow 0$  such that

$$M\left(\frac{k_n}{k_n+k} w_n + \frac{k}{k_n+k} u_n\right) > (1 - \delta_n) \left( \frac{k_n}{k_n+k} M(w_n) + \frac{k}{k_n+k} M(u_n) \right).$$

Therefore,

$$\begin{aligned} \|x+x_n\|^0 &> (1 - \delta_n) \frac{k_n+k}{k_n k} \left( 1 + \sum_{i \neq n} m_i M(w_i) + (m_n - \bar{m}_n) M(w_n) \right. \\ &\quad \left. + \bar{m}_n \left( \frac{k}{k_n+k} M(u_n) + \frac{k_n}{k_n+k} M(w_n) \right) \right) \\ &= (1 - \delta_n) \left[ \frac{1}{k_n} \left( 1 + \sum_{i \neq n} m_i M(w_i) + (m_n - \bar{m}_n) M(w_n) \right. \right. \\ &\quad \left. \left. + \bar{m}_n M(u_n) + \frac{1}{k} \left( 1 + \sum_{i=1}^{\infty} m_i M(w_i) \right) \right) \right] \\ &= 2(1 - \delta_n) \rightarrow 2 \quad (n \rightarrow \infty). \end{aligned}$$

But  $x(1) - x_n(1) = \left(\frac{1}{k} - \frac{1}{k_n}\right) w_1 = \frac{k_n - k}{k_n k} w_1 > \frac{k_n - k}{kk} w_1$ . Therefore,

$$\liminf_{n \rightarrow \infty} (x(1) - x_n(1)) \geq \frac{w_1 \varepsilon}{2kk},$$

which contradicts the condition  $x_n - x \xrightarrow{w^*} 0$ . This completes the proof of the necessity.

Since condition (iii) in **Theorem 2** is weaker than  $M \in \Delta_2$ , comparing it with **Th2.28** (3<sup>0</sup>) of [20], we get

**Corollary 2.** For Orlicz sequence space  $l_M^0$ ,  $LW^*UR$  is weaker essentially than  $LWUR$ .

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