ON THE PERIOD OF ELEMENT OF BCI-ALGEBRAS

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Received June 15, 1995

ABSTRACT. In this paper, we study the period of element of BCI-algebras. First, some properties of P-semisimple BCI-algebras are generalized to general BCI-algebra. Finally, we give some characterizations of some BCI-algebras by them.

Let X be a BCI-algebra. For any $x, a \in X$, we denote

$$a * x^n = (\dots((a * x) * x) * \dots) * x.$$

Definition. Let X be a BCI-algebra. For any $x \in X$, if there exists a leas natural number n such that $0 * x^n = 0$, then n is called to be the period of x, and denote $\rho(x) = n$. If, for any natural number n, $0 * x^n \neq 0$ then the period of x is called to be infinite and denote $\rho(x) = \infty$.

Obviously, $\rho(0) = 1$.

Proposition 1. *Let X be a P-semisimple BCI-algebra and $x \in X$. Then the period of x is equal to the period of x in the Abel group which is equivalent to X.*[1]

Corollary 1. *Let X be a P-semisimple BCI-algebra. Then, for any $x, y \in X$,*

(1) $\rho(x) = \rho(0 * x)$;
(2) $\rho(x * y) = \rho(y * x)$;
(3) $\rho(x * y) \mid [\rho(x), \rho(y)]$, *where* $[\rho(x), \rho(y)]$ *is the least common multiple of* $\rho(x)$ *and* $\rho(y)$.
(4) $A = \{x \in X \mid \rho(x) < \infty\}$ *is a close ideal of X (i.e. A is both an ideal and a subalgebra).*

Proposition 2. *Let X be a BCI-algebra. Then $f_0 : x \mapsto 0 * x$, for any $x \in X$, is an endomorphism of X and* $\operatorname{Im} f_0^2 = \operatorname{Im} f_0 = \{x \in X | x = 0 * (0 * x)\}$.[4]

It is easy to obtain that $\operatorname{Im} f_0$ is the greastest P-semisimple BCI-subalgebra and we have

Proposition 3. *Let X be a BCI-algebra. Then the following are equivalent:*

(1) *X is a P-semisimple BCI-algebra;*
(2) $X = \operatorname{Im} f_0$;
(3) *f_0 is an automorphism;*
(4) $f_0^2 = 1_X$ *(1_X is a unite map of X).*

Definition. Let X be a BCI-algebra and $x \in X$. If $y \leq x$ implies $y = x$ then x is called to be a minimal element of X. Obviously, 0 is a minimal element of X.

Proposition 4. *Let X be a BCI-algebra. Then X is a P-semisimple iff every element of X is a minimal element.*

Proof. First, we prove the necessity.

Key words and phrases. BCI-algebra, the period of element.

For any $x \in X$, if $y \leq x$, then $y * x = 0$ and $0 = 0 * (y * x) = (0 * y) * (0 * x)$, i.e., $0 * y \leq 0 * x$. On the other hand, $0 * y \geq 0 * x$ since $y \leq x$. Hence $0 * y = 0 * x$. By Proposition 3, $x = y$. So x is a minimal element of X.

Now we prove the sufficiency. If every element of X is a minimal element of X then, for any $x \in X$, $0 * (0 * x) = x$ since $0 * (0 * x) \leq x$. So X is P-semisimple.

Corollary 2. *Let X is a BCI-algebra. Then X is a BCK-algebra iff X has and only has a minimal element* 0.

Corollary 3. *Let X be a BCI-algebra. If $y \leq x$ then $0 * y = 0 * x$.*

Now we generalize Corollary of Proposition 1 to the general BCI-algebra.

Proposition 5. *Let X be a BCI-algebra. Then, for any $x, y \in X$,*

(1) $\rho(x) = \rho(0 * x)$;
(2) $\rho(x * y) = \rho(y * x)$;
(3) $\rho(x * y) \mid [\rho(x), \rho(y)]$.

Proof. (1) By the induction, it is easy to prove that

$$0 * (0 * x)^n = 0 * (0 * x^n), \quad \text{for any } n \in N.$$

So $\rho(x) = \rho(0 * x)$.

(2) By (1), we have $\rho(x * y) = \rho(0 * (x * y))$ and $\rho(y * x) = \rho(0 * (y * x))$. On the other hand, $0 * x, 0 * y \in \operatorname{Im} f_0$. By Corollary 1,

$$\rho(0 * (x * y)) = \rho((0 * x) * (0 * y)) = \rho((0 * y) * (0 * x)) = \rho(0 * (y * x)).$$

Thus $\rho(x * y) = \rho(y * x)$.

(3) By (1) and Corollary 1, $\rho(x * y) = \rho(0 * (x * y)) = \rho((0 * x) * (0 * y))$ can integral divide $[\rho(0 * x), \rho(0 * y)] = [\rho(x), \rho(y)]$.

Proposition 6. *Let X be a BCI-algebra. If $y \leq x$ then $\rho(y) = \rho(x)$.*

Proof. By Corollary 3 and Proposition 5, if $y \leq x$ then $0 * y = 0 * x$ and $\rho(y) = \rho(0 * y) = \rho(0 * x) = \rho(x)$.

Theorem. *Let X be a BCI-algebra and denote $A = \{x \in X | \rho(x) < \infty\}$. Then A is a close ideal of X and X/A is a P-semisimple BCI-algebra.*

Proof. (i) Obviously, $0 \in A$. Suppose $x, y \in A$ and denote $n = \rho(x)$ and $m = \rho(y)$. By Proposition 5, $\rho(x * y) \mid [n, m]$. So $x * y \in A$ and A is a subalgebra of X.

(ii) If $x, y * x \in A$, by Proposition 5, then $\rho(x * y) = \rho(y * x) < \infty$ and $x * y \in A$. Thus, by (i), $0 * y = (x * y) * x \in A$ and $\rho(y) = \rho(0 * y) < \infty$, by Proposition 5. Hence $y \in A$ and A is an ideal of X.

(iii) By (i) and (ii), we have $A = A_0$. If $A_0 * A_x = A_0$ then $A_{0 * x} = A_0$ and $0 * x \in A$. By Proposition 5, $x \in A = A_0$ and $A_x = A_0$. Thus X/A is P-semisimple.

Finally we use the period of element to give some characterizations of some BCI-algebras.

Proposition 7. *If X is a BCI-algebra then X is a BCK-algebra iff $\rho(x) = 1$ for any $x \in X$.*

Proof. $\rho(x) = 1 \Leftrightarrow 0 * x = 0$.

By Proposition 7, we have

Proposition 8. *If X is a BCI-algebra. Then X is a proper BCI-algebra iff there exists $x \in X$ such that $\rho(x) > 1$.*

Proposition 9. *If X is a BCI-algebra. Then X is P-semisimple iff $\rho(x) > 1$ for any $0 \neq x \in X$.*

Proof. X is P-semisimple $\Leftrightarrow$ The BCK-part of X is equal to $\{0\}$.

Proposition 10. *If X is a BCI-algebra then X is associate iff $\rho(x) = 2$ for any $0 \neq x \in X$.*

Proof. X is associate $\Leftrightarrow$ $0 * x = x$ for any $x \in X$ $\Leftrightarrow$ The BCK-part of X is equal to $\{0\}$ and $\rho(x) = 2$, for any $0 \neq x \in X$.

Definition. [5]BCK-algebra X is called to be quasi-associate, if $(x * y) * z \leq x * (y * z)$, for any $x, y, z \in X$.

Proposition 11. *Let X be a BCI-algebra. Then X is quasi-associate iff $\rho(x) \leq 2$, for any $x \in X$.*

Proof. (Necessity) If $x \in X$ then

$$x = x * (x * x) \geq (x * x) * x = 0 * x,$$

i.e., $(0 * x) * x = 0$ and $\rho(x) \leq 2$.

(Sufficiency) For any $x, y, z \in X$, we have $((x*y)*z)*(x*(y*z)) = ((x*y)*(x*(y*z))*z \leq ((y*z)*y)*z = (0*z)*z = 0$ since $\rho(z) \leq 2$. So $(x*y)*z \leq x*(y*z)$ and X is quasi-associate.

BCI-algebra X is said to be well in case every ideal of X is close.

Lemma. [6] *If A is an ideal of BCI-algebra X then A is close iff $0 * x \in A$ for any $x \in A$.*

Proposition 12. *Let X be a BCI-algebra. Then X is a well BCI-algebra iff $\rho(x) < \infty$ for any $x \in X$.*

Proof. If X is a well BCI-algebra, then, for any $x \in X$, $< x >$ is a close ideal of X and $0 * x \in < x >$. So there exists $n \in N$ such that $(0 * x) * x^n = 0$ and $\rho(x) < \infty$. On the contrary, for any $A \lhd X$ and $x \in A$, denote $\rho(x) = n$, i.e., $0 * x^n = 0$, we have $0 * x \in A$. Thus A is close and X is a well BCI-algebra.

BCI-algebra $X \neq \{0\}$ is said to be simple if X has no non-trivial ideal.

Obviously, simple BCI-algebras are well BCI-algebras. So $\rho(x) < \infty$ for any $x \in X$.

Proposition 13. *If X is a simple BCI-algebra then $\rho(x) = \rho(y)$, for any $x, y \in X \backslash \{0\}$.*

Proof. If $x, y \notin \{0\}$ then $X = < x > = < y >$ since X is a simple. On the other hand, the P-radical P of X is an ideal of X. So $P = \{0\}$ or $P = X$ since X is simple, i.e., X is P-semisimple or X is a BCK-algebra.

(i) If X is a BCK-algebra then $\rho(x) = \rho(y) = 1$.

(ii) If X is P-semisimple then, for any $0 \neq x \in X$ and $y \in X = < x >$ there exists $m \in N$ such that $y * x^m = 0$. Thus $0 * y = 0 * x^m$ and $0 * y^n = 0 * x^{mn} = 0$, where $n = \rho(x)$. Hence $\rho(y) \leq n = \rho(x)$. Similarly, $\rho(x) \leq \rho(y)$ and $\rho(x) = \rho(y)$.

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