THE DUNFORD-PETTIS PROPERTY AND STRICT TOPOLOGIES

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ABSTRACT. Let X be a completely regular space. We denote by $C_b(X)$ the Banach space of all real-valued bounded continuous functions on X endowed with the supremun-norm. In this paper we prove some characterisations of weakly compact operators from $C_b(X)$ into a Banach space E which are continuous with respect to $\beta_t, \beta_\sigma, \beta_\tau, \beta_s$ and β_g, strict topologies. We also prove that $(C_b(X), \beta_i); i = t, \tau, p, s, g$ has the Dunford-Pettis property.

1. Preliminary result and notation. Throughout X denotes a completely regular Hausdorff space, $C_b(X)$ the space of all bounded real-valued continuous functions defined on X. $M(X)$ the Banach space dual of $C_b(X)$ with the supremum norm $\| \cdot \|$; t_p will denote the topology of pointwise convergence on X. In $C_b(X)$ has also been defined the so called strict topologies denoted by $\beta_\sigma, \beta_\tau, \beta_t, \beta_p, \beta_g, \beta_s$; which yield as dual very important subspaces of $M(X)$ commounly encountered in topological measure theory: the spaces $M_\sigma, M_\tau, M_t, M_p, M_g$ and M_s of σ-additive, τ-additive, tight, perfect, Grothendieck and separable Baire measures [12].

E always will denote a Banach space. $Ba(X)$ and $Ba^\star(X)$ will stand for the σ-algebra and algebra of Baire in X. Let $\mathcal{A}$ be an algebra of subsets of X and let $m : \mathcal{A} \to E$ a finitely additive vector measure. We say that m is strongly additive if the serie $\sum m(A_n)$ converges for each mutually disjoint sequence $\{A_n\}$ of elements of A. The set function from $\mathcal{A}$ to $\mathbf{R}$ defined by

$$\| m \| (A) = \sup\{| x' \circ m | (A) : x' \in B_{E'}\}$$

is called the semivariation of m. If $\| m \| (X)$ is finite m is said to be of bounded semivariation. Let $ba(\mathcal{A}, E)$ denote the space of all vector measures $m : \mathcal{A} \to E$ of bounded semivariation. It is well known that $ba(\mathcal{A}, E)$ is a Banach space with the norm $m \to \| m \| (X)$ [5].

The following theorem can be proved very similar to the case when X is a compact Hausdorff space ([5], [1], [4]).

Theorem 1.1. *Let $T : C_b(X) \to E$ be a bounded operator. Then there exists a unique finetely, additive vector measure $m : Ba^\star \to E''$ of bounded semivariation such that:*

1. *For every $x' \in E'$, $x' \circ m \in M(X)$*
2. *The mapping from E' into $M(X)$ defined by $x' \to x' \circ m$ is $\sigma(E', E) - \sigma(M(X), C_b(X))$-continuous.*
3. *$T(f) = \int f \, dm$ for every $f \in C_b(X)$*

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4. $\| T \| = \| m \| (X)$

Conversely, if $m : Ba^\star \to E''$ is a finetely additive vector measure of bounded semivariation which satisfies 1 y 2, then 3 definies a bounded linear operator $T : C_b(X) \to E$ that satisfies 4.

We will use freely the following results:

Theorem 1.2. Carathéodory-Hahn-Kluvanek Extension Theorem
Let $\mathcal{F}$ be algebra of subsets of X, Σ the σ-algebra generated by $\mathcal{F}$. Then any of the following statements about a weakly countably additive vector measure $m : \mathcal{F} \to E$ implies the others:

1. *m has a unique countably additive extension $\overline{m} : \Sigma \to E$*
2. *There exist a finite, no negative, countably additive scalar measure $\mu : \mathcal{F} \to \mathbf{R}$ such that $m \ll \mu$*
3. *m is strongly additive.*
4. *$m(\mathcal{F})$ is a relatively weakly compact subset of E.*

In fact the control measure μ can be taken of the form $x' \circ m$ for some $x' \in E'$ ([5, IX.2.Th.2 (Rybakov)]).

Lemma 1.3. [7] *Let K a compact Haussdorff space, if $H \subset C(K)$ is t_p-compact, then H is t_p-sequentially compact.*

2. Operators and Strict Topologies. If F and E are Banach spaces, an operator $T : F \to E$ is said to be *weakly compact* if T send bounded sets in F into relatively weakly compact subset of E. In particular we will consider operators $T : C_b(X) \to E$.

Similarly to [5] we have the following result.

Theorem 2.1. *Let $T : C_b(X) \to E$ be a β_σ continuous linear operator with representing measure m. Then T is weakly compact if and only if m takes its values in E and is strongly additive.*

Proof: It is well known that if T is weakly compact, T'' is also weakly compact. By the construction of m, ([4]), it represents the operator

$$\overline{T} = T'' \mid \mathcal{B}$$

where $\mathcal{B} = \mathcal{B}(X, Ba^\star)$ the space of uniform limits of $Ba^\star$-simple functions. Then, in fact, m takes its values in E.

Moreover:

$$m(Ba^\star) = \{\overline{T}(\chi_A) : A \in Ba^\star\}$$

and since $\chi_A \in B_{\mathcal{B}}$, the unit ball $\mathcal{B}$, it follows that the range of m is weakly compact.

Also, since $x' \circ m \in M_\sigma$ for every $x' \in E'$, it follows from Theorem 1.2 that m is strongly additive.

Conversely, if m is strongly additive, and since m represents the operator $\overline{T} : \mathcal{B} \to E''$ (see [4] or [5]), it follows that $\overline{T}$ is weakly compact and then its restriction T is also weakly compact (considered as an operator from $C_b(X)$ to E'').

Now since $T(B_{C_b(X)})$ is $\sigma(E'', E''')$-compact it easily follows that it is $\sigma(E, E')$-compact and then T is a weakly compact operator. □

Lemma 2.2. *Let E be normed space and E' its dual. If F locally convex topological vector space and $T : F \to E$ is lineal, then T is continuous if and only if $\{x' \circ T : x' \in B_{E'}\}$ is equicontinuous.*

Proof: If T is continuous, given $\epsilon > 0$ there exists a 0-neighborhood U in F such that if $f \in U$ then $\| T(f) \| \leq \epsilon$.

On the other hand if $x' \in B_{E'}$ it follows that

$$| x' \circ T(f) | \leq \| T(f) \| \leq \epsilon$$

uniformly for $x' \in B_{E'}$.

Conversely, if $\{x' \circ T : x' \in B_{E'}\}$ is equicontinuous; and (f_i) is a net en F converging to zero then; $x' \circ T(f_i) \to 0$ uniformly in $B_{E'}$. Thus

$$\| T(f_i) \| = \sup\{| x' \circ T(f_i) | : x' \in B_{E'}\} \to 0$$

which implies that T is continuous. □

The following results which appears in [1], [2] and [3] follow easily from Lemma 2.2 and the corresponding characterization of β_z-equicontinuos subsets of M_z for $z = t, \tau, p, s$ and σ ([12]).

We write:

$$\| m \|_\star (A) = \sup\{| x' \circ m |_\star (A) : x' \in B_{E'}\}$$

for $A \subset X$.

Corollary 2.3. *Let $T : C_b(X) \to E$ be a bounded linear operator and m its representing measure. The following are equivalent:*

1. *T is β_t-continuous.*
2. $(\forall \epsilon > 0)(\exists K \subset X,\ K \ \ compact\ \ )(\| m \|_\star (X \setminus K) \leq \epsilon)$

Corollary 2.4. *Let $T : C_b(X) \to E$ be a bounded linear operator and m its representing measure. The following are equivalent:*

1. *T is β_σ-continuous.*
2. $f_n \downarrow 0 \Rightarrow T(f_n) \to 0$

Corollary 2.5. *Let $T : C_b(X) \to E$ be a bounded linear operator and m its representing measure. The following are equivalent:*

1. *T is β_τ-continuous.*
2. $f_\alpha \downarrow 0 \Rightarrow T(f_\alpha) \to 0$

Corollary 2.6. *Let $T : C_b(X) \to E$ be a bounded linear operator and m its representing measure. The following are equivalent:*

1. *T is β_s-continuous.*
2. *For each partition of unity $(f_\lambda)_{\lambda \in I}$ in $C_b(X)$ and every $\epsilon > 0$ there exists a finite subset $F \subset I$ such that $| x' \circ m | (1 - \sum_{\lambda \in F} f_\lambda) < \epsilon$ for each $x' \in B_{E'}$*

Corollary 2.7. *Let $T : C_b(X) \to E$ be a bounded linear operator and m its representing measure. The following are equivalent:*

1. *T is β_p-continuous.*
2. *For each continuous map f from X onto a separable metric space Y and every $\epsilon > 0$ there exists a compact $K \subset Y$ such that $| x' \circ m | (X \setminus f^{-1}(K)) < \epsilon$ for every $x' \in B_{E'}$.*

In addition it is worth mentioning that the following results holds

Theorem 2.8. *Let $T : C_b(X) \to E$ be a weakly compact operator and m its representing measure. For $z = \sigma$ or s, T is β_z-continuos if and only if $x' \circ m \in M_z$ for every $x' \in E'$.*

Proof: Clearly if T is β_z-continuos, then $x' \circ m \in M_z(X)$ for every $x' \in E'$.

Conversely, suppose $x' \circ m \in M_z$ for every $x' \in E'$. Now since T is weakly compact T' is also weakly compact. Then $\{x' \circ m : x' \in B_{E'}\}$ is $\sigma(M(X), C_b(X)'')$-compact. Now, since M_z is a subspace of $M(X)$, then $\{x' \circ m : x' \in B_{E'}\}$ will also be $\sigma(M_z, C_b(X))$-compact. But, β_σ and β_s are Mackey; then $\{x' \circ m : x' \in B_{E'}\}$ is β_z-equicontinuos and by 2.2 T it follows that T is β_z continuos. □

3. The Dunford Pettis Property in $(C_b(X), \beta_z)$. If F is locally convex Hausdorff topological vector space, then F is said to have the Dunford Pettis property (DP) if for every Banach space E and every continuous operator $T : F \to E$ which send bounded sets into relatively weakly compact subset of E, it sends absolutely convex and weakly compact subsets of F into relatively compact subsets of E.

In what follows we will discuss the Dunford Pettis property for $(C_b(X), \beta_z)$ for $z = t, \tau, p, s, g$. In [8] Khurana proved the result for $z = t$ and in [10] Khurana and Vielma proved it for $z = \tau$ and p, in both cases using measure theoretic aproaches. Our main contribution is to prove it for $z = s$ and g. In any case we present a different proof for $z = t, \tau$ and p, for the sake of completness.

Theorem 3.1. *$(C_b(X), \beta_t)$ has the Dunford-Pettis property.*

Proof: Let $T : C_b(X) \to E$ be a β_t-continuous operator which is weakly compact and let $H \subset C_b(X)$ an absolutely convex and $\sigma(C_b(X), M_t)$-compact.

Let $\{f_n\}$ be a sequence in H. Since $\beta_t \leq \beta_\sigma$, the representing measure m of T has a positive control measure $\mu \in M_t$. Now, since every tight Baire measure can be extended to a compact-regular Borel measure, we always assume that this extension has been made.

Since $\{f_n\}$ is norm-bounded there exists an $L > 0$ such that $\| f_n \| \leq L/2$ for every $n \in \mathbf{N}$. Also for $\epsilon > 0$, there exists a $\delta > 0$ such that $\mu(F) < \delta$ implies that $\| m \| (F) < \epsilon/3L$.

Now by Corollary 2.3, there exists a compact subset $K \subset X$ such that $\| m \| (X \setminus K) < \epsilon/3L$.

Since H is t_p-compact by Lemma 1.3 $\{f_n\}$ has a subsequence, denoted by $\{f_n\}$, such that $f_n(x) \to f(x)$ for every $x \in K$.

Now, by Egoroff's Theorem, there exists a $F_\delta \in Ba(X)$ contained in K such that $\{f_n\}$ is uniformly Cauchy in $K \setminus F_\delta$ and $\mu(F_\delta) < \delta$.

Let $n_0 \in \mathbf{N}$ be such that, for $n, m \geq n_0$, we have that

$$\sup\{\| f_n(x) - f_m(x) \| : x \in K \setminus F_\delta\} < \epsilon/3M$$

with $M = \| m \| (X)$.

Then it follows that for $n, m \geq n_0$

$$\begin{aligned} &| x' \circ T(f_n) - x' \circ T(f_m) | \\ &\leq | \textstyle\int_{X \setminus K} (f_n - f_m) d(x' \circ m) | + | \int_{K \setminus F_\delta} (f_n - f_m) d(x' \circ m) | + | \int_{F_\delta} (f_n - f_m) d(x' \circ m) | \\ &\leq L \| m \| (X \setminus K) + \sup\{\| f_n - f_m \| : x \in K \setminus F_\delta\} M + L \| m \| (F_\delta) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \end{aligned}$$

for every $x' \in B_{E'}$.

If we take $\epsilon = 1$ we obtain a subsequence of $\{f_n\}$, said $\{f_n^1\}$ and $n_1 \in \mathbf{N}$ such that for $n, m \geq n_1$

$$\|x' \circ T(f_n^1) - x' \circ T(f_m^1)\| < 1$$

for every $x' \in B_{E'}$.

Inductively, we obtain $\{f_n^k\}$, a subsequence of $\{f_n^{k-1}\}$, for $k = 2, 3, \ldots$ and $n_k \in \mathbf{N}$ such that for $n, m \geq n_k$

$$\|x' \circ T(f_n^k) - x' \circ T(f_m^k)\| < 1/k$$

for every $x' \in B_{E'}$.

The sequence $\{g_n\}$ defined for $g_n = f_n^n$ is a subsequence of $\{f_n\}$ norm-convergent in E and the result follows. □

We need a definition and a lemma before we state the next theorem.

If A is a set of real-valued measurable functions on a finite measure space (Ω, Σ, μ), A is said to have *the separation property* with respect to μ if

$$f, g \in A,\ f = g \ \ ae(\mu) \ \text{ then } \ f(x) = g(x), \ \text{ on } \ \Omega$$

Lemma 3.2. *If A is t_p-compact, convex and has the separation property with respect to μ; then $t_p = t_\mu$ in A where t_μ is the topology of μ-convergence and t_p is metrizable (*[12, 6.13.]*)*

Theorem 3.3. *$(C_b(X), \beta_\tau)$ has the Dunford-Pettis property.*

Proof: Let $T : C_b(X) \to E$ be a β_τ-continuous operator which is weakly compact and let $H \subset C_b(X)$ an absolutely convex and $\sigma(C_b(X), M_\tau)$-compact.

Let $\{f_n\}$ be a sequence in H. Since $\beta_\tau \leq \beta_\sigma$, the representing measure m of T has a positive control measure $\mu \in M_\tau$. Now, since every τ-additive measure can be uniquely extend to a Borel measure, we always assume that this extension has been made.

Since $\{f_n\}$ is norm-bounded there exists an $L > 0$ such that $\| f_n \| \leq L/2$ for every $n \in \mathbf{N}$. Also, given $\epsilon > 0$, there exists a $\delta > 0$ such that if $\mu(F) < \delta$ it follows that $\| m \| (F) < \epsilon/3L$. Now, since every τ-additive measure has a non empty support we call $F = sop(\mu)$. It is clear that F is closed, and $\mu(F) = \mu(X)$.

Now, since H is t_p-compact it follows that the set $H_F = \{f \mid F : f \in H\}$ is t_p-compact and convex in $C_b(F)$.

We claim that H_F has the separation property with respect to μ_F ($\mu_F(B) = \mu(B \cap F)$ where B is Borel in X).

In fact, if f, g belong to H and $f \mid F = g \mid F \ \ ae(\mu_F)$ we let $\widehat{F} = \{x \in X : f(x) = g(x)\}$; $\widehat{F}$ is closed in X and $\mu(\widehat{F}) = \mu(X)$. Then $F \subset \widehat{F}$ and that implies that $f = g$ on F. Therefore by Lemma 3.2 $(C_b(F), t_p)$ is metrizable.

Now since H is t_p-compact, then any sequence $\{f_n\}$ in H has a t_p-accumulation point f. Then $f \mid F$ is a t_p-accumulation point of $\{f_n \mid F\}$. Therefore there exists a subsequence, that we call $\{f_n \mid F\}$ again, which convergs to $f \mid F$ pointwise.

By Egoroff's Theorem, there exists a $F_\delta \in Bo(X)$ contained in F such that $\{f_n\}$ is uniformly Cauchy in $F \setminus F_\delta$ and $\mu(F_\delta) < \delta$.

Let $n_0 \in \mathbf{N}$ be such that, for every $n, m \geq n_0$

$$\sup\{\| f_n(x) - f_m(x) \| : x \in F \setminus F_\delta\} < \epsilon/2M$$

with $M = \| m \| (X)$.

To show that $\{T(f_n)\}$ is norm-convergent we follow a similar procedure as Theorem 3.3.

Then $\{T(f_n)\}$ converge in E and the theorem follows. □

Theorem 3.4. *$(C_b(X), \beta_p)$ has the Dunford-Pettis property.*

Proof: Let $T : C_b(X) \to E$ be a β_p-continuous operator which is weakly compact operator and $H \subset C_b(X)$ absolutely convex and $\sigma(C_b(X), M_p)$-compact.

Again the representing measure of T has a positive control measure $\mu \in M_p$.

Again since $\{f_n\} \in H$ is norm bounded sequence with $\| f_n \| \leq L/2$ for some $L > 0$, then $\{f_n\}$ has a t_p-accumulation point $f \in H$ since H is t_p-compact. Also given $\epsilon > 0$, there exists a $\delta > 0$ so that $\mu(F) < \delta$ implies that $\| m \| (F) < \epsilon/2L$.

Now by a well known result of Fremlim, [6] $\{f_n\}$ has a subsequence, denoted again by $\{f_n\}$, which converges pointwise to f μ-almost everywhere.

Let $Y \subset X$ with $\mu(Y) = \mu(X)$ and $f_n(x) \to f(x)$ for every $x \in Y$. By Egoroff's Theorem, there is an $F_\delta \in Ba(X)$ contained in Y such that $\{f_n\}$ is uniformly Cauchy in $Y \setminus F_\delta$ and $\mu(F_\delta) < \delta$.

To show that $\{T(f_n)\}$ is norm convergent we follow a similar procedure as Theorem 3.3. □

Theorem 3.5. $(C_b(X), \beta_s)$ *has the Dunford-Pettis property.*

Proof: Let $T : C_b(X) \to E$ be β_s-continuous operator which is weakly compact and $H \subset C_b(X)$ absolutely convex and $\sigma(C_b(X), M_s)$-compact.

Again the representing measure m of the operator T has a positive control measure $\mu \in M_s$. Since $\{f_n\}$ in H is a norm bounded sequence with , $\| f_n \| \leq L/2$ for some $L > 0$ then $\{f_n\}$ has a t_p-accumulation point $f \in H$ since H is t_p-compact.

Also given $\epsilon > 0$, there is a $\delta > 0$ so that $\mu(F) < \delta$ implies $\| m \| (F) < \epsilon/2L$.

Let us define a continuous pseudometric on X as follows

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \mid f_n(x) - f_n(y) \mid$$

And let $Y \subset X$, a zero-set d-closed and d-separable subset of X such that $\mu(X) = \mu(Y)$ ([12]) with $\{x_n : n \in \mathbf{N}\}$ d-dense en Y.

By an standard diagonalization procedure we can found a subsequence of $\{f_n\}$, called again $\{f_n\}$, such that $f_n(x_i) \to f(x_i)$ for every $x_i \in \{x_1, x_2, \dots\}$.

We claim that in fact $f_n(x) \to f(x)$ for every $x \in Y$. Let $\epsilon > 0$, and take $i \in \mathbf{N}$ such that $d(x_i, x) < \epsilon/3$ and $\mid f(x_i) - f(x) \mid < \epsilon/3$ then it follows that $\mid f_n(x_i) - f_n(x) \mid < \epsilon$ for every $n \in \mathbf{N}$.

Let $n(i) \in \mathbf{N}$ such that $\mid f_n(x_i) - f(x_i) \mid < \epsilon/3$ if $n \geq n(i)$. Then if $n \geq n(i)$ we get

$$\mid f_n(x) - f(x) \mid \leq \mid f_n(x) - f_n(x_i) \mid + \mid f_n(x_i) - f(x_i) \mid + \mid f(x_i) - f(x) \mid < \epsilon$$

As in previous theorem we invoke Egoroff's Theorem and coclude that $\{T(f_n)\}$ converges in E. □

In the next theorem we answer positevely a question posted by Wheeler in [12, 15.11.].

Theorem 3.6. $(C_b(X), \beta_g)$ *has the Dunford-Pettis property.*

Proof: Let $T : C_b(X) \to E$ be a β_g-continuous weakly compact operator and let $H \subset C_b(X)$ be absolutely convex and $\sigma(C_b(X), M_g)$-compact.

Now since H is t_p-compact and β_g is the finnest locally convex topology on $C_b(X)$ agreeing with t_p on absolutely convex and t_p-compact sets, we get that H is β_g-compact. Now since T is β_g-continuos we obtain that $T(H)$ is norm-compact in E. □

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