A NECESSARY CONDITION OF LOCAL INTEGRABILITY FOR A NOWHERE–ZERO COMPLEX VECTOR FIELD IN $\mathbb{R}^2$.

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Abstract. Let X_2 be a nowhere-zero C^∞ complex vector field in $\mathbb{R}^2$. A necessary condition for the local integrability of X_2 which belongs to a certain class of non-solvable operators is investigated.

1. Introduction

Let X_n be a nowhere-zero C^∞ complex vector field defined near a point P in $\mathbb{R}^n$. We shall say that X_n is locally integrable at P if there exist a neighborhood Ω of P and functions $u_i (i = 1, 2, ..., n-1)$ satisfying $X_n u_i = 0$ in Ω such that $du_1 \wedge du_2 \wedge \cdots \wedge du_{n-1}(\mathrm{P}) \neq 0$.

In [2] and [3], Lewy showed the holomorphic extension of the solutions to homogeneous first-order partial differential equations $X_n u = 0$ ($n = 3$ and 4) and proposed a problem whether X_n is locally integrable. These papers assumed a new aspect to the concept of holomorphic hull.

So far, it is known that X_n is locally integrable at P if X_n is real-analytic or locally solvable at P (see Treves [12]); on the other hand Nirenberg [7] gave a non-solvable vector field in $\mathbb{R}^2$ which has no local integrability (That vector field in fact has the property that $X_2 u = 0$ admits no even non-trivial solutions near the origin (see also [6])). It is an open problem to obtain a necessary and sufficient condition for local integrability of X_n, though there are several partial results mainly when $n = 2$([1], [5], [8], [10], [11], for instance).

In this paper, we investigate the case when $n = 2$. The equation $X_2 u = 0$ near P is transformed into that of the form

$$Lu \equiv (\partial_t + ia(t,x)\partial_x)u = 0$$

near the origin in $\mathbb{R}^2$, where $a(t,x)$ is a real-valued C^∞ function. Our problem is to seek a necessary and sufficient condition for $Lu = 0$ to have a solution near the origin such that $\partial_x u \neq 0$. We know that L is locally integrable at the origin if $a(t,x)$ is real-analytic with respect to x or the function $t \to a(t,x)$ does not change sign in $\{t; (t,x) \in \mathcal{O}\}$ for every x by taking a neighborhood $\mathcal{O}$ of the origin. No one has obtained a necessary and sufficient condition yet when the function $t \to a(t,x)$ changes sign in $\{t; (t,x) \in \mathcal{O}\}$ for some x by taking any neighborhood $\mathcal{O}$ of the origin, except for the particular case of Mizohata type vector fields.

In §2 our results are stated. Our main theorem (Theorem 3) is prove d in §3. In §4, the proof of Proposition 5 which concerns the existence of non-trivial solutions is given. In §5, it is proved that the example given in §2 satisfies the required conditions.

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2. Results

The definition of *Mizohata type* is as follows:

Definition. X_2 is called a Mizohata type vector field if the following conditions hold:
(i) $X_2(0)$ and $\bar{X}_2(0)$ are C-linearly dependent.
(ii) $X_2(0)$ and $[X_2(0), \bar{X}_2(0)]$ are C-linearly independent.

Remark. L is a Mizohata type vector field if $a(0,0) = 0$ and $a_t(0,0) \neq 0$.

Treves [10] (see also Sjöstrand [8]) proved the following theorem:

Theorem 1. *Let X_2 be a Mizohata type vector field. X_2 is locally integrable at the origin if and only if there is a change of local coordinates such that X_2 becomes a (suitable non-vanishing C^∞ function) multiple of the Mizohata operator $\partial_{x_1} + ix_1\partial_{x_2}$.*

This is a beautiful result; it does not seem, however, to be really useful for deciding whether L is locally integrable or not. We present a necessary condition which is given by an estimate.

For a function $f(t,x)$, denote by $f_e(t,x)$ the even part of $f(t,x)$ with respect to t and by $f_o(t,x)$ the odd one. In [5] we remarked that the form of $\sup a_e(t,x)$ affects the local integrability: Let $ta(t,x) > 0$ for $t \neq 0$. In case of $\sup a_e(t,x) = \emptyset$, L is locally integrable at the origin. If $\sup a_e(t,x) \neq \emptyset$, there is a differential operator L which has no local integrability at the origin and the property that $\sup a_e(t,x) \cap U$ contains an open disc for every neighborhood U of the origin.

Now we shall require the following assumption:

(a.0): $a(0,x)$ vanishes identically.
(a.1): There is a neighborhood ω of the origin such that

$$ta_o(t,x) > 0 \qquad \text{in} \quad \{t \neq 0\} \cap \omega \tag{a.1.1}$$

and

$$a_e(t,x) \geq 0 \qquad \text{in} \quad \omega. \tag{a.1.2}$$

First we have the following

Lemma 2. ([5]). *Assume* (a.1.1). *Then there exist a neighborhood Ω_w of the origin and a function $w(t,x) \in C^1(\Omega_w)$ such that*

$$\min\left(\inf_{\Omega_w} Re w_x, \inf_{\Omega_w} Im\, w_x\right) > 0 \qquad \text{and} \qquad (\partial_t + ia_o(t,x)\partial_x)w = 0 \quad \text{in} \quad \Omega_w.$$

Set

$$m(w, \Omega_w) = \min\left(\inf_{\Omega_w} \mathrm{Re}\ w_x, \inf_{\Omega_w} \mathrm{Im}\ w_x\right)$$

for a function w and a neighborhood Ω_w satisfying Lemma 2. Then our main result is stated in the following form:

Theorem 3. *Assume* (a.0) *and* (a.1). *Let w and Ω_w be any one of the pairs of a function and a neighborhood satisfying Lemma 2. Assume that $Lu = 0$ has a C^1 solution near the origin such that $u_x(0) \neq 0$. Then, there exists a positive constant T_0 which is independent of w and Ω_w such that, for any simply connected domain D contained in $(0, T_0) \times (-T_0, T_0) \cap \Omega_w$ with piecewise smooth boundary ∂D, the inequality*

$$m(w,\Omega_w)\iint_D a_e\,dtdx \le \sup_{\partial D}|w|\cdot|\partial D|$$

holds, where $|\partial D|$ denotes the length of the boundary ∂D.

To give an example, set $a_0(t,x)=2t$. Then

$$w=(1-i)(t^2+ix) \quad \text{and} \quad \Omega_w=\mathbb{R}^2$$

satisfy Lemma 2. Since $|w|=\{2(t^4+x^2)\}^{\frac{1}{2}}$, we have $m(w,\Omega_w)=1$. Taking a positive integer N_0 such that $N_0^{-1}<T_0$, for every integer p satisfying $p>N_0$, we have

$$\iint_D a_e(t,x)\,dtdx \le 8p^{-2}$$

for $D=\left(0,\frac{1}{p}\right)\times\left(0,\frac{1}{p}\right)$. Therefore we obtain the following

Corollary 4. *Let $a_o(t,x)=2t$. Assume* (a.1.2) *and the following condition:*
(b) *There exist positive constants c,d, and a monotonously increasing sequence $\{p_n\}$ of positive integers such that*

$$\iint_{D(p_k)} a_e(t,x)\,dtdx > \frac{9}{(p_k+d)(p_k+c)} \tag{2.1}$$

for every sufficiently large k, where $D(p_k)=\left(0,\frac{1}{p_k}\right)\times\left(0,\frac{1}{p_k}\right)$. Then L is not locally integrable at the origin.

Let us give an example:

Example. (This is obtained by modifying an example of Nirenberg [7](p.8).)

Let n and p be arbitrary positive integers. Set $a_{n,p}=\frac{1}{(n+p-1)(n+p)}$. It is noted that $\sum_{n=1}^{\infty}a_{n,p}=p^{-1}$ and $a_{1,p}=\frac{1}{p(p+1)}$. Let $B_{n,p}$ be the non-overlapping open disc in the (t,x) plane with center $(t,x)=\left(\frac{1}{p}-(a_{1,p}+a_{2,p}+...+a_{n-1,p}+\frac{a_{n,p}}{2}),\frac{1}{p+1}+\frac{1}{2p(p+1)}\right)$ and radius $\frac{a_{n,p}}{2}$, and $C_{n,p}$ the closed disc in the (t,x) plane with radius $\frac{a_{n,p}}{4}$ and the same center as $B_{n,p}$.

Let $f_{n,p}$ be a C^∞ function having the following properties:
(i) $0\le f_{n,p}\le\frac{64\cdot 18}{\pi(n+p+1)^2}$.
(ii) $f_{n,p}$ vanishes outside of $B_{n,p}$ and equals $\frac{64\cdot 18}{\pi(n+p+1)^2}$ inside of $C_{n,p}$.

Let us define the C_o^∞ function $r(t,x)$ as follows:
(iii) $r(-t,x)=r(t,x)$.
(iv) $r(t,x)=f_{n,p}$ in $B_{n,p}$.
(v) $r(t,x)$ vanishes out side of the union of all the $B_{n,p}$.

Set $a_e(t,x)=r(t,x)$. Then the condition (b) in Corollary 4 is satisfied.

Concerning on the existence of non-constant solutions for $Lu=0$ near the origin, we incidentally obtain the following necessary condition:

Proposition 5. *Assume* (a.1) *and* (a.2). *Let w and Ω_w be any one of the pairs of a function and a neighborhood satisfying Lemma 1. Assume that the $Lu = 0$ has a non-constant C^1 solution near the origin. Then, for any small positive constant ε, there exist a real value x_0 in $(-\varepsilon, \varepsilon) \cap [\Omega_w \cap t = 0]$ and a positive value T_{x_0}, which satisfies $(-T_{x_0}, T_{x_0}) \times (-T_{x_0} + x_0, T_{x_0} + x_0) \subset \Omega_w$, such that, for any simply connected domain D contained in $(0, T_{x_0}) \times (-T_{x_0} + x_0, T_{x_0} + x_0)$ with piecewise smooth boundary ∂D, the inequality*

$$m(w, \Omega_w) \iint_D a_e \, dtdx \le \sup_{\partial D} |w| \cdot |\partial D|$$

holds.

This can be proved in the similar way as in the proof of Theorem 3.

3. Proof of Theorem 3

Multiplying u by a suitable constant $e^{i\theta}$, where θ is a real number, we can assume that $\mathrm{Re}(e^{i\theta} u_e)_x(0,0)$ and $\mathrm{Im}(e^{i\theta} u_e)_x(0,0)$ are positive. So we can assume that $\mathrm{Re}\partial_x u_e(0,0) \equiv \alpha$ and $\mathrm{Im}\, \partial_x u_e(0,0) \equiv \beta$ are positive.

Let us set $\delta = \min(\alpha, \beta)$. Since $u_o(0,x) = 0$, we see that $\partial_x u_o(0,x) = 0$. From this and $\partial_t u_o(0,x) = 0$, there is a positive constant T such that

$$Lu = 0 \ in \quad U_T = (-T,T) \times (-T,T) \quad \text{and} \quad M = \max\Big(\sup_{U_T} |\partial_t u_o|, \sup_{U_T} |\partial_x u_o|\Big) \le \frac{\delta}{4}.$$

There exists a positive constant T_1 such that

$$\mathrm{Re}\ \partial_x u_e > \frac{\delta}{2}, \quad \mathrm{Im}\, \partial_x u_e > \frac{\delta}{2} \quad \text{in} \quad U_{T_1} = (-T_1, T_1) \times (-T_1, T_1).$$

Set $T_0 = \min(T, T_1)$ and $v = \frac{2u}{\delta}$. Then we have

$$\inf_{U_{T_o}} \partial_x \mathrm{Re}\ v_e \ge \frac{\delta}{2} \cdot \frac{2}{\delta} = 1 \quad \text{and} \quad \inf_{U_{T_o}} \partial_x\ \mathrm{Im}\, v_e \ge \frac{\delta}{2} \cdot \frac{2}{\delta} = 1.$$

And we also have

$$\max\Big(\sup_{U_{T_o}} |\partial_t v_o|, \sup_{U_{T_o}} |\partial_x v_o|\Big) \le \frac{2M}{\delta} < \frac{1}{2}.$$

Since $Lv = \frac{2}{\delta} Lu = 0$, we have

$$(\partial_t + i a_o(t,x)\partial_x) v = -i a_e(t,x) \partial_x v_e. \tag{3.1}$$

Hence we have the following:

Lemma 6. *For any simply connected domain D contained in $(0, T_0) \times (-T_0, T_0) \cap \Omega_w$ with piecewise smooth boundary,*

$$i \iint_D a_e w_x \partial_x v_e \, dtdx = \int_{\partial D} w \partial_t v_o dt + w \partial_x v_o \, dx. \tag{3.2}$$

Proof. By (3.1),

$$-w_x \{(\partial_t + i a_o \partial_x) v_o\} = i a_e w_x \partial_x v_e.$$

And hence we have

$$\iint_D -w_x\{(\partial_t + ia_o\partial_x)v_o\}\,dtdx = \iint_D ia_e w_x\,\partial_x v_e\,dtdx. \tag{3.3}$$

The left-hand side of (3.3)

$$\begin{aligned} = \iint_D -\{w_x\partial_t v_o - w_t\partial_x v_o\}\,dtdx &= \iint_D d\{w(t,x)dv_o(t,x)\} \\ &= \int_{\partial D} w\partial_t v_o\,dt + w\partial_x v_o\,dx, \end{aligned}$$

which completes the proof of Lemma 6. □

By Lemma 6, we have

$$\iint_D [a_e\{\mathrm{Re}\partial_x v_e\,\mathrm{Im}\,w_x + \mathrm{Im}\,\partial_x v_e\mathrm{Re}w_x\}]\,dtdx \le \int_{\partial D} |w\partial_t v_o\,dt + w\partial_x v_o\,dx|. \tag{3.4}$$

Denote $\min\left(\inf_{U_{T_o}} \mathrm{Re}\partial_x v_e, \inf_{U_{T_o}} \mathrm{Im}\,\partial_x v_e\right)$ by m_o, then we have

$$\mathrm{Re}\partial_x v_e\,\mathrm{Im}\,w_x + \mathrm{Im}\,\partial_x v_e\mathrm{Re}w_x \ge m(w,\Omega_w)\cdot m_0 \ \ in\ D.$$

And so from (3.4), we have

$$m(w,\Omega_w)\cdot m_0 \iint_D a_e(t,x)\,dtdx \le \int_{\partial D} |w\partial_t v_o\,dt + w\partial_x v_o\,dx|.$$

As we have remarked that $m_0 \ge 1$ and $M = \max\left(\sup_{U_{T_o}} |\partial_t v_o|, \sup_{U_{T_o}} |\partial_x v_o|\right) \le \frac{1}{2}$, we obtain the following inequality:

$$m(w,\Omega_w)\iint_D a_e\,dtdx \le \sup_{\partial D}|w|\cdot|\partial D|,$$

which completes the proof of Theorem 3. □

4. Proof of Proposition 5

Suppose that $Lu = 0$ holds in a neighborhood U of the origin. We may assume that $U = U_{T_2} = (-T_2, T_2)\times(-T_2, T_2)$, where T_2 is a positive constant such that $T_2 < \varepsilon$.

Now we have the following:

Lemma 7. *There exists a real value x_0 such that $\partial_x u(0,x_0) \ne 0$ and $|x_0| < \varepsilon$.*

Proof. Assume that $\partial_x u(0,x_0)$ vanishes in $(-T_2,T_2)$. Since $a(t,x)$ is positive in $(0,T_2)\times(-T_2,T_2)$ from the assumption (a.1), L is elliptic for $t>0$. And hence $u\in C^\infty$ for $t>0$. Setting $v = \partial_x u$, we have $Lv + ia_x v = 0$ *in* $(0,T_2)\times(-T_2,T_2)$. Since $v(0,x) = 0$ in $(-T_2,T_2)$, applying the uniqueness theorem (see [4] or [9]), v vanishes in $[0,T_2)\times(-T_2,T_2)$. By the assumption (a.1), $a(t,x)$ is negative in $(-T_2,0)\times(-T_2,T_2)$. Applying the uniqueness theorem to the equation $Lv + ia_x v = 0$ in $(-T_2,0)\times(-T_2,T_2)$, $v = 0$ in $(-T_2,0)\times(-T_2,T_2)$. Therefore v vanishes in $(-T_2,T_2)\times(-T_2,T_2)$. This implies that u is constant, which is a contradiction. Hence we see Lemma 7 holds. □

Take T_2 such that $U_{T_2} \subset \Omega_w$, if necessary, we can assume that x_0 in $(-\varepsilon, \varepsilon) \cap [\Omega_w \cap t = 0]$. Multiplying u by a suitable constant and applying Lemma 7, we can assume that there is a positive constant t_{x_0} such that

$$Lu = 0 \text{ in } U_{t_{x_0}} = (-t_{x_0}, t_{x_0}) \times (-t_{x_0} + x_0, t_{x_0} + x_0),$$

$$\max\Big(\sup_{U_{t_{x_0}}} |\partial_t u_o|, \sup_{U_{t_{x_0}}} |\partial_x u_o|\Big) \le \frac{1}{2},$$

$$\min\Big(\inf_{U_{t_{x_0}}} \operatorname{Re} \partial_x u_e, \inf_{U_{t_{x_0}}} \operatorname{Im} \partial_x u_e\Big) \ge 1.$$

Take a positive constant T_{x_0} such that

$$(-T_{x_0}, T_{x_0}) \times (-T_{x_0} + x_0, T_{x_0} + x_0) \subset \Omega_w \cap U_{t_{x_0}}.$$

Then we can derive the inequality:

$$m(w, \Omega_w) \iint_D a_e \, dtdx \le \sup_{\partial D} |w| \cdot |\partial D|.$$

This ends the proof. □

5. Example

We shall show that the a_e defined in Example satisfies the condition (b).

First we set $c = 1$, $d = 2$, and $\{p_k\} = \{1, 2, \dots\}$. By putting $p_k = p$, the left-hand side of the inequality of (2.1)

$$\begin{aligned}
&\ge \sum_{n=1}^{\infty} \iint_{C_{n,p}} a_e \, dtdx + \sum_{k=p+1}^{\infty} \iint_{C_{1,k}} a_e dtdx \\
&= \sum_{n=1}^{\infty} \frac{\pi a_{n,p}^2}{16} \cdot \frac{64 \cdot 18}{\pi (n+p+1)^2} + \sum_{k=p+1}^{\infty} \frac{\pi a_{1,k}^2}{16} \cdot \frac{64 \cdot 18}{\pi (k+2)^2} \\
&= \sum_{n=1}^{\infty} 18 \cdot \left[\frac{2}{(n+p-1)(n+p)(n+p+1)}\right]^2 + \sum_{k=p+1}^{\infty} 18 \cdot \left[\frac{2}{k(k+1)(k+2)}\right]^2.
\end{aligned}$$

Since

$$\begin{aligned}
\frac{2}{(n+p-1)(n+p)(n+p+1)} &= \frac{1}{(n+p-1)(n+p)} - \frac{1}{(n+p+1)(n+p)} \\
&= \frac{1}{n+p-1} - \frac{1}{n+p} - \left\{\frac{1}{n+p} - \frac{1}{n+p+1}\right\} \\
&= \frac{1}{n+p-1} - \frac{2}{n+p} + \frac{1}{n+p+1}
\end{aligned}$$

and

$$\frac{2}{k(k+1)(k+2)} = \frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2},$$

we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} 18\cdot\left[\frac{2}{(n+p-1)(n+p)(n+p+1)}\right]^2+\sum_{k=p+1}^{\infty} 18\cdot\left[\frac{2}{k(k+1)(k+2)}\right]^2\\
=&18\sum_{n=1}^{\infty}\left[\frac{1}{(n+p-1)^2}+\frac{4}{(n+p)^2}+\frac{1}{(n+p+1)^2}-4\left\{\frac{1}{n+p-1}-\frac{1}{n+p}\right\}\right.\\
&\left.+\left\{\frac{1}{n+p-1}-\frac{1}{n+p+1}\right\}-4\left\{\frac{1}{n+p}-\frac{1}{n+p+1}\right\}\right]\\
&+18\sum_{k=p+1}^{\infty}\left[\frac{1}{k^2}+\frac{4}{(k+1)^2}+\frac{1}{(k+2)^2}-4\left\{\frac{1}{k}-\frac{1}{k+1}\right\}\right.\\
&\left.-4\left\{\frac{1}{k+1}-\frac{1}{k+2}\right\}+\left\{\frac{1}{k}-\frac{1}{k+2}\right\}\right]\\
=&18\left[\sum_{n=1}^{\infty}\left\{\frac{1}{(n+p-1)^2}+\frac{4}{(n+p)^2}+\frac{1}{(n+p+1)^2}\right\}-\frac{4}{p}+\left\{\frac{1}{p}+\frac{1}{p+1}\right\}-\frac{4}{p+1}\right]\\
&+18\left[\sum_{k=p+1}^{\infty}\left\{\frac{1}{k^2}+\frac{4}{(k+1)^2}+\frac{1}{(k+2)^2}\right\}-\frac{4}{p+1}-\frac{4}{p+2}+\frac{1}{p+1}+\frac{1}{p+2}\right]\\
&+18\left[\sum_{n=1}^{\infty}\left\{\frac{1}{(n+p-1)^2}+\frac{4}{(n+p)^2}+\frac{1}{(n+p+1)^2}\right\}-\frac{3}{p}-\frac{3}{p+1}\right]\\
&+18\left[\sum_{k=p+1}^{\infty}\left\{\frac{1}{k^2}+\frac{4}{(k+1)^2}+\frac{1}{(k+2)^2}\right\}-\frac{3}{p+1}-\frac{3}{p+2}\right]\\
=&18\left[12\sum_{k=p+2}^{\infty}\frac{1}{n^2}+\frac{1}{p^2}+\frac{6}{(p+1)^2}-\frac{1}{(p+2)^2}-\frac{3}{p}-\frac{6}{p+1}-\frac{3}{p+2}\right].
\end{aligned}
$$

Put

$$S(p)=12\sum_{n=p+2}^{\infty}\frac{1}{n^2}+\frac{1}{p^2}+\frac{6}{(p+1)^2}-\frac{1}{(p+2)^2}-\frac{3}{p}-\frac{6}{p+1}-\frac{3}{p+2}.$$

Then we have only to prove

$$S(p)\geq\frac{1}{2(p+1)(p+2)}.$$

This proof will be given in the the following Lemma 8.

Lemma 8. *For every positive integer p,*

$$S(p)>\frac{1}{2(p+1)(p+2)}.$$

Proof. We shall show this by mathematical induction.

First,

$$
\begin{aligned}
S(1) =& 12\ \ 3^{-2}+4^{-2}+5^{-2}+\dots\big)+1+\frac{3}{2}-\frac{1}{9}-3-3-1\\
=& 2\left[6\{(1^{-2}+2^{-2}+3^{-2}+4^{-2}+\dots)-1^{-2}-2^{-2}\}\right]-\frac{9}{2}-\frac{1}{9}\\
=& 2\left[6\left(\frac{\pi^2}{6}-1-\frac{1}{4}\right)\right]-\frac{9}{2}-\frac{1}{9}\\
=& 2\left(\pi^2-6-\frac{3}{2}-\frac{9}{4}\right)-\frac{1}{9}\\
=& 0.1196\dots\ ,
\end{aligned}
$$

which is grater than $\frac{1}{12}$. Hence

$$S(1) > \frac{1}{2\cdot 2\cdot 3}.$$

Next assume

$$S(p) > \frac{1}{2(p+1)(p+2)}.$$

Then

$$
\begin{aligned}
& S(p+1)-\frac{1}{2(p+2)(p+3)}\\
=& 12\sum_{n=p+3}^{\infty}\frac{1}{n^2}\\
& +\frac{1}{(p+1)^2}+\frac{6}{(p+2)^2}-\frac{1}{(p+3)^2}-\frac{3}{p+1}-\frac{6}{p+2}-\frac{3}{p+3}-\frac{1}{2(p+2)(p+3)}\\
=& 12\left\{\sum_{n=p+2}^{\infty}\frac{1}{n^2}-\frac{1}{(p+3)^2}\right\}\\
& +\frac{1}{(p+1)^2}+\frac{6}{(p+2)^2}-\frac{1}{(p+3)^2}-\frac{3}{p+1}-\frac{6}{p+2}-\frac{3}{p+3}-\frac{1}{2(p+2)(p+3)}\\
>& \frac{1}{2(p+1)(p+2)}+\frac{3}{p}+\frac{6}{p+1}+\frac{3}{p+2}+\frac{1}{(p+2)^2}-\frac{6}{(p+1)^2}-\frac{1}{p^2}\\
& -\frac{12}{(p+3)^2}+\frac{1}{(p+1)^2}+\frac{6}{(p+2)^2}-\frac{1}{(p+3)^2}-\frac{3}{p+1}\\
& -\frac{6}{p+2}-\frac{3}{p+3}-\frac{1}{2(p+2)(p+3)}\\
=& \frac{1}{(p+1)(p+2)(p+3)}+\frac{3}{p(p+1)}+\frac{6}{(p+1)(p+2)}+\frac{3}{(p+2)(p+3)}\\
& +\frac{7}{(p+2)^2}-\frac{5}{(p+1)^2}-\frac{1}{p^2}-\frac{13}{(p+3)^2}\\
=& \frac{A_1}{p(p+1)(p+2)(p+3)}+\frac{A_2}{[p(p+1)(p+2)(p+3)]^2},
\end{aligned}
$$

where

$$A_1 \equiv p+3(p+2)(p+3)+6p(p+3)+3p(p+1) = 12p^2+37p+18$$

and

$$\begin{aligned}
A_2 \equiv & 7\{p(p+1)(p+3)\}^2 - 5\{p(p+2)(p+3)\}^2 - \{(p+1)(p+2)(p+3)\}^2 \\
& - 13\{p(p+1)(p+2)\}^2 7p^2(p^2+4p+3)^2 - 5p^2(p^2+5p+6)^2 \\
& - 13p^2(p^2+3p+2)^2 - (p^3+6p^2+11p+6)^2 \\
= & 7p^2(p^4+8p^3+22p^2+24p+9) - 5p^2(p^4+10p^3+37p^2+60p+36) \\
& - 13p^2(p^4+6p^3+13p^2+12p+4) - (p^6+12p^5+58p^4+144p^3+193p^2+132p+36) \\
= & (7p^6+56p^5+154p^4+168p^3+63p^2) - (5p^6+50p^5+185p^4+300p^3+180p^2) \\
& - (13p^6+78p^5+169p^4+156p^3+52p^2) \\
& - (p^6+12p^5+58p^4+144p^3+193p^2+132p+36) \\
= & -(12p^6+84p^5+258p^4+432p^3+362p^2+132p+36).
\end{aligned}$$

Setting

$$S = \frac{A_1}{p(p+1)(p+2)(p+3)} + \frac{A_2}{[p(p+1)(p+2)(p+3)]^2},$$

we have

$$\begin{aligned}
S = & \Big\{(12p^2+37p+18)p(p+1)(p+2)(p+3) \\
& - (12p^6+84p^5+258p^4+432p^3+362p^2+132p+36)\Big\} \Big/ [p(p+1)(p+2)(p+3)]^2 .
\end{aligned}$$

And we see that the numerator of S

$$\begin{aligned}
= & (12p^2+37p+18)p(p^3+6p^2+11p+6) \\
& - (12p^6+84p^5+258p^4+432p^3+362p^2+132p+36) \\
= & p(12p^5+72p^4+132p^3+72p^2+37p^4+222p^3 \\
& + 407p^2+222p+18p^3+108p^2+198p+108) \\
& - (12p^6+84p^5+258p^4+432p^3+362p^2+132p+36) \\
= & 25p^5+114p^4+155p^3+58p^2-24p-36 \geq 292.
\end{aligned}$$

This completes the proof. □

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