ON FIRST-PASSAGE-TIME DENSITIES FOR CERTAIN SYMMETRIC MARKOV CHAINS

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Abstract. The spatial symmetry property of truncated birth-death processes studied in Di Crescenzo [6] is extended to a wider family of continuous-time Markov chains. We show that it yields simple expressions for first-passage-time densities and avoiding transition probabilities, and apply it to a bilateral birth-death process with jumps. It is finally proved that this symmetry property is preserved within the family of strongly similar Markov chains.

1 Introduction

A spatial symmetry for the transition probabilities of truncated birth-death processes has been studied in Di Crescenzo [6]. Such a property leads to simple expressions for certain first-passage-time densities and avoiding transition probabilities. In this paper we aim to extend those results to a wider class of continuous-time Markov chains.

Given a set \( \{x_n\} \) of positive real numbers and the transition probabilities \( p_{k,n}(t) \) of a continuous-time Markov chain whose state-space is \( \{0,1,\ldots,N\} \) or \( \mathbb{Z} \), in Section 2 we introduce the following spatial symmetry property:

\[
p_{N-k,N-n}(t) = \frac{x_n}{x_k} p_{k,n}(t).
\]

In section 3 we point out some properties of first-passage-time densities and avoiding transition probabilities for Markov chains that are symmetric in the sense of (1). These properties allow one to obtain simple expressions for first-passage-time densities in terms of probability current functions, and for avoiding transition probabilities in terms of the ‘free’ transition probabilities. In Section 4 we then apply these results to a special bilateral birth-death process with jumps. Finally, in Section 5 we refer to the notion of strong similarity between the transition probabilities of Markov chains, expressed by \( \tilde{p}_{k,n}(t) = \left(\beta_n/\beta_k\right) p_{k,n}(t) \) (see Pollett [16], and references therein) and show the following preservation result: if \( p_{k,n}(t) \) possesses the symmetry property (1), then also \( \tilde{p}_{k,n}(t) \) does it.

2 Symmetric Markov chains

Let \( \{X(t), \ t \geq 0\} \) be a homogeneous continuous-time Markov chain on a state-space \( \mathcal{S} \). We shall assume that \( \mathcal{S} = \{0,1,\ldots,N\} \), where \( N \) is a fixed positive integer, or \( \mathcal{S} = \mathbb{Z} \equiv \{\ldots,-1,0,1,\ldots\} \). Let

\[
p_{k,n}(t) = \Pr\{X(\tau + t) = n \mid X(\tau) = k\}, \quad k,n \in \mathcal{S}; \quad t, \tau \geq 0
\]

be the stationary transition probabilities of \( X(t) \), satisfying the initial conditions

\[
p_{k,n}(0) = \delta_{k,n} = \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases}
\]

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Let $Q$ be the infinitesimal generator of the transition function (2), i.e. the matrix whose $(k,n)$-th finite entries are:

\begin{equation}
q_{k,n} = \left. \frac{d}{dt} p_{k,n}(t) \right|_{t=0},
\end{equation}

satisfying the following relations: (a) $q_{k,n} \geq 0$ for all $k,n \in S$ such that $k \neq n$, (b) $q_{n,n} \leq 0$ for all $n \in S$, and (c) $\sum_{n \in S} q_{k,n} = 0$ for all $k \in S$.

The spatial symmetry of Markov processes allows one to approach effectively the first-passage-time problem. Indeed, it has been often exploited by various authors to obtain closed-form results for first-passage-time distributions; see Giorno et al. [13] and Di Crescenzo et al. [8] for one-dimensional diffusion processes, Di Crescenzo et al. [7] for two-dimensional diffusion processes, and Di Crescenzo [5] for a class of two-dimensional random walks. Moreover, in Di Crescenzo [6] a symmetry for truncated birth-death processes was expressed as in (1), with $x_i$ suitably depending on the birth and death rates. Such symmetry notion can be extended to the wider class of continuous-time Markov chains considered above. Indeed, for a set of positive real numbers $\{x_n; n \in S\}$ there holds:

\begin{equation}
p_{N-k,N-n}(t) = \frac{x_n}{x_k} p_{k,n}(t) \quad \text{for all } k,n \in S \text{ and } t \geq 0
\end{equation}

if and only if

\begin{equation}
q_{N-k,N-n} = \frac{x_n}{x_k} q_{k,n} \quad \text{for all } k,n \in S.
\end{equation}

The proof is similar to that of Theorem 2.1 in Di Crescenzo [6], and thus is omitted.

Eq. (4) focuses on a symmetry with respect to $N/2$, which identifies with the mid point of $S$ when $S = \{0,1,\ldots,N\}$. For each sample-path of $X(t)$ from $k$ to $n$ there is a symmetric path from $N-k$ to $N-n$, and the ratio of their probabilities is time-independent. Hence, in the following we shall say that $X(t)$ possesses a central symmetry if relation (4) is satisfied.

**Remark 2.1** If $X(t)$ possesses a central symmetry, then

$$
\frac{x_n}{x_k} = \frac{x_{N-k}}{x_{N-n}} \quad \text{for all } k,n \in S.
$$

An example of a Markov chain with finite state-space and a central symmetry is given hereafter.

**Example 2.1** Let $X(t)$ be a continuous-time Markov chain with state-space $S = \{0,1,2,3\}$, with 0 and 3 absorbing states, and infinitesimal generator

$$
Q = \begin{bmatrix}
0 & 0 & 0 & 0 \\
\alpha \varrho + \beta & -\alpha(\varrho + \varrho^2) - \beta \varrho & \beta \varrho & \alpha \varrho^2 \\
\alpha & \beta & -\alpha(\varrho + \varrho^2) - \beta \varrho & (\alpha \varrho + \beta) \varrho \\
0 & 0 & 0 & 0
\end{bmatrix},
$$

with $\alpha, \beta, \varrho > 0$ and $\varrho = 1 + \varrho$. Then, $X(t)$ has a central symmetry, with $p_{N-k,N-n}(t) = \varrho^{k-n} p_{k,n}(t)$ for all $k,n \in S$ and $t \geq 0$, and $q_{N-k,N-n}(t) = \varrho^{k-n} q_{k,n}$ for all $k,n \in S$.

**Remark 2.2** If $X(t)$ has a central symmetry and possesses a stationary distribution $\{\pi_n; n \in S\}$, with $\lim_{t \to +\infty} p_{k,n}(t) = \pi_n > 0$ for all $k,n \in S$, then the following statements hold:

(a) Sequence $\{x_n\}$ is constant, so that $p_{N-k,N-n}(t) = p_{k,n}(t)$ for all $k,n \in S$ and $t \geq 0$.

(b) The stationary distribution is symmetric with respect to $N/2$, i.e.

$$
\pi_{N-n} = \pi_n \quad \text{for all } n \in S.
$$
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(c) Let $X^*(t)$ be the reversed process of $X(t)$, obtained from $X(t)$ when time is reversed, and characterized by rates and transition probabilities

$$q_{k,n}^* = \frac{\pi_n}{\pi_k} q_{n,k}, \quad p_{k,n}^*(t) = \frac{\pi_n}{\pi_k} p_{n,k}(t), \quad k, n \in S, \quad t \geq 0.$$ 

Then, also $X^*(t)$ has a central symmetry, with $p_{N-k,N-n}^*(t) = p_{k,n}^*(t)$ for all $k, n \in S$ and $t \geq 0$.

(d) Let $D = \{d_{k,n}\}$ be the deviation matrix of $X(t)$, with elements (see Coolen-Schrijner and Van Doorn [2])

$$d_{k,n} = \int_0^{+\infty} [p_{k,n}(t) - \pi_n] dt, \quad k, n \in S.$$

Then, $D$ has a central symmetry, i.e. $d_{N-k,N-n} = d_{k,n}$ for all $k, n \in S$.

An example of a Markov chain satisfying the assumptions of Remark 2.2 is the birth-death process on $S$ with birth rate $\lambda_n = \alpha (N - n)$ and death rate $\mu_n = \alpha n$ (see Giorno et al. [11], or Section 4.1 of Di Crescenzo [6]).

3 First-passage-time densities

In this section we shall focus on the first-passage-time problem for Markov chains $X(t)$ that have a central symmetry and that satisfy the following assumptions:

(i) $N = 2s$, with $s$ a positive integer;
(ii) $q_{i,i} = q_{j,j} = 0, \sum_{i \in S_+} q_{i,i} > 0, \sum_{j \in S_-} q_{j,j} > 0, \sum_{i \in S_+} q_{i,s} > 0$ and $\sum_{j \in S_-} q_{s,j} > 0$ for all $i \in S_-$ and $j \in S_+$, where

$$S_- = \{n \in S; n < s\}, \quad S_+ = \{n \in S; n > s\};$$

(in other words, if states $i$ and $j$ are separated by $s$ then all sample-paths of $X(t)$ from $i$ to $j$, or from $j$ to $i$, must cross $s$);

(iii) the subchains defined on $S_-$ and $S_+$ are irreducibles.

In addition, we introduce the following non-negative random variables:

$$T_{i,s}^+ = \text{upward first-passage time of } X(t) \text{ from state } i \in S_- \text{ to state } s,$$

$$T_{j,s}^- = \text{downward first-passage time of } X(t) \text{ from state } j \in S_+ \text{ to state } s.$$

We shall denote by $g_{i,s}^+(t)$ and $g_{j,s}^-(t)$ the corresponding probability density functions. Due to assumptions (i)-(iii), for all $t > 0$ such densities satisfy the following renewal equations:

$$p_{i,j}(t) = \int_0^t g_{i,s}^+(\vartheta) p_{s,j}(t - \vartheta) d\vartheta, \quad i \in S_-, \quad j \in \{s\} \cup S_+, \quad (6)$$

$$p_{j,i}(t) = \int_0^t g_{j,s}^-(\vartheta) p_{s,i}(t - \vartheta) d\vartheta, \quad i \in S_+ \cup \{s\}, \quad j \in S_-, \quad (7)$$

For all $t > 0$ and $k \in S$ let us now introduce the probability currents

$$h_{k,s}^+(t) = \lim_{\tau \to 0} \frac{1}{\tau} P\{X(t + \tau) = s, X(t) < s | X(0) = k\} = \sum_{i \in S_-} p_{k,i}(t) g_{i,s}, \quad (8)$$

$$h_{k,s}^-(t) = \lim_{\tau \to 0} \frac{1}{\tau} P\{X(t + \tau) = s, X(t) > s | X(0) = k\} = \sum_{j \in S_+} p_{k,j}(t) g_{j,s}. \quad (9)$$
They represent respectively the upward and downward entrance probability fluxes at state \( s \) at time \( t \). Due to assumptions (i)-(iii) and Eqs. (6)-(9), for \( i \in \mathcal{S}_- \), \( j \in \mathcal{S}_+ \) and \( t > 0 \) they satisfy the following integral equations:

\[
\begin{align*}
\mathcal{H}^+_{i,s}(t) &= \int_0^t g^+_{i,s}(\vartheta) \mathcal{H}^+_{s,s}(t-\vartheta) \, d\vartheta, \\
\mathcal{H}^-_{j,s}(t) &= \int_0^t g^-_{j,s}(\vartheta) \mathcal{H}^+_{s,s}(t-\vartheta) \, d\vartheta.
\end{align*}
\]


**Proposition 3.1** Under assumptions (i)-(iii), for all \( i \in \mathcal{S}_- \), \( j \in \mathcal{S}_+ \) and \( t > 0 \) the following equations hold:

\[
\begin{align*}
\mathcal{H}^+_{i,s}(t) &= h^+_{i,s}(t) - \int_0^t g^+_{i,s}(\vartheta) h^+_{s,s}(t-\vartheta) \, d\vartheta, \\
\mathcal{H}^-_{j,s}(t) &= h^-_{j,s}(t) - \int_0^t g^-_{j,s}(\vartheta) h^+_{s,s}(t-\vartheta) \, d\vartheta.
\end{align*}
\]

**Proof.** For all \( t > 0 \) and \( i \in \mathcal{S}_- \), making use of assumptions (i)-(iii) and Eq. (8) we have

\[
\frac{d}{dt} p_{i,s}(t) = \sum_{n \in \mathcal{S}} p_{i,n}(t) q_{n,s} = h^+_{i,s}(t) + \sum_{n \in \{s\} \cup \mathcal{S}_+} p_{i,n}(t) q_{n,s}.
\]

Hence, recalling (6) we obtain

\[
\begin{align*}
\mathcal{H}^+_{i,s}(t) &= \frac{d}{dt} \left[ \int_0^t g^+_{i,s}(\vartheta) p_{s,s}(t-\vartheta) \, d\vartheta \right] - \sum_{n \in \{s\} \cup \mathcal{S}_+} \left[ \int_0^t g^+_{i,s}(\vartheta) p_{s,n}(t-\vartheta) \, d\vartheta \right] q_{n,s} \\
&= g^+_{i,s}(t) + \int_0^t g^+_{i,s}(\vartheta) \left[ \frac{\partial}{\partial \vartheta} p_{s,s}(t-\vartheta) - \sum_{n \in \{s\} \cup \mathcal{S}_+} p_{s,n}(t-\vartheta) q_{n,s} \right] \, d\vartheta,
\end{align*}
\]

where use of initial condition \( p_{s,s}(0) = 1 \) has been made. From Chapman-Kolmogorov forward equation we have

\[
\frac{\partial}{\partial \vartheta} p_{s,s}(t-\vartheta) - \sum_{n \in \{s\} \cup \mathcal{S}_+} p_{s,n}(t-\vartheta) q_{n,s} = h^+_{s,s}(t-\vartheta), \quad t > \vartheta,
\]

so that Eq. (14) gives (12). The proof of (13) goes along similar lines.

With reference to a Markov chain that has a central symmetry, we now come to the main result of this paper, expressing the first-passage-time densities through the symmetry state \( s \) as difference of probability currents (8) and (9).

**Theorem 3.1** For a Markov chain that has a central symmetry and satisfies assumptions (i)-(iii), for all \( t > 0 \) and \( k \in \mathcal{S} \) there results:

\[
\begin{align*}
\mathcal{H}^+_{2s-k,s}(t) &= \frac{x_s}{x_k} h^+_{k,s}(t), \\
g^+_{i,s}(t) &= h^+_{i,s}(t) - h^-_{i,s}(t), \quad g^-_{j,s}(t) = h^-_{j,s}(t) - h^+_{j,s}(t).
\end{align*}
\]

Moreover, for all \( i \in \mathcal{S}_- \), \( j \in \mathcal{S}_+ \) and \( t > 0 \) the upward and downward first-passage-time densities through state \( s \) are given by
Proof. Recalling that $N = 2s$, for $t > 0$ we have

$$h_{2s-k,s}^-(t) = \sum_{j \in S_+} p_{2s-k,j}(t) q_{j,s}, \quad \text{from (9)}$$

$$= \sum_{i \in S_-} p_{2s-k,2s-i}(t) q_{2s-i,s}, \quad \text{setting } j = 2s - i$$

$$= \frac{x_s}{x_k} \sum_{i \in S_-} p_{k,i}(t) q_{i,s}, \quad \text{from (4) and (5)}$$

$$= \frac{x_s}{x_k} h_{k,s}^+(t). \quad \text{from (8)}$$

Eq. (15) then holds. In particular, for $k = s$ it implies that $h_{2s,s}^-(t - \vartheta) = h_{2s,s}^+(t - \vartheta)$ for all $t > \vartheta$. Hence, relations (16) follow from Eqs. (10)-(13).

For a Markov chain $X(t)$ satisfying assumptions (i)-(iii) let us now introduce the $s$-avoiding transition probabilities:

$$p_{k,n}^{(s)}(t) = P \{ X(t) = n, X(\vartheta) \neq s \text{ for all } \vartheta \in (0,t) \mid X(0) = k \},$$

where $k, n \in S_- \cup S_+$. We note that $p_{k,n}^{(s)}(t)$ is related to $p_{k,n}(t)$ by

$$p_{k,n}^{(s)}(t) = \begin{cases} p_{k,n}(t) - \int_0^t g_{k,s}^+(\vartheta) p_{s,n}(t - \vartheta) \, d\vartheta, & k, n \in S_-, \\ p_{k,n}(t) - \int_0^t g_{k,s}^-(\vartheta) p_{s,n}(t - \vartheta) \, d\vartheta, & k, n \in S_+. \end{cases} \quad (17)$$

In the following theorem, for symmetric Markov chains two different expressions are given for $p_{k,n}^{(s)}(t)$ in terms of $p_{k,n}(t)$. It extends Theorem 2.4 of Di Crescenzo [6]; the proof is similar and therefore is omitted.

**Theorem 3.2** Under the assumptions of Theorem 3.1, for $t > 0$ and for $k, n \in S_- \cup S_+$ there holds:

$$p_{k,n}^{(s)}(t) = p_{k,n}(t) - \frac{x_k}{x_s} p_{2s-k,n}(t)$$

$$= p_{k,n}(t) - \frac{x_s}{x_n} p_{k,2s-n}(t).$$

We conclude this section by pointing out that for a Markov chain having a central symmetry, for all $t > 0$ the following relations hold:

$$g_{i,s}^+(t) = \frac{x_s}{x_i} g_{2s-i,s}^-(t), \quad g_{j,s}^-(t) = \frac{x_j}{x_s} g_{2s-j,s}^+(t), \quad i \in S_-, \ j \in S_+,$$

$$p_{2s-k,2s-n}^{(s)}(t) = \frac{x_n}{x_k} p_{k,n}^{(s)}(t), \quad k, n \in S_- \cup S_+.$$

4 A bilateral birth-death process with jumps In this section we shall apply the above results to a special symmetric Markov chain $X(t)$ with state-space $\mathbb{Z}$, characterized by the following transitions: (a) from $n \in \mathbb{Z}$ to $n + 1$ with rate $\lambda$, (b) from $n \in \mathbb{Z}$ to $n - 1$ with rate $\mu$, and (c) from $n \in \mathbb{Z} - \{0\}$ to 0 with rate $\alpha$. Hence, $X(t)$ is a bilateral
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The probability generating function
\[ H(z, t) = \sum_{n=-\infty}^{+\infty} p_k(t) z^n \]
is thus solution of
\[ \frac{\partial}{\partial t} H(z, t) = u(z) H(z, t) + \alpha, \]
where \( u(z) = -(\lambda + \mu + \alpha) + \lambda z + \frac{\mu}{z} \), with initial condition \( H(z, 0) = z^k \). The unique solution of (19) is
\[ H(z, t) = H(z, 0) e^{u(z)t} + \alpha \int_0^t e^{u(z)\tau} d\tau. \]
Hence, recalling that
\[ \exp \left\{ (\lambda z + \frac{\mu}{z}) t \right\} = \sum_{n=-\infty}^{+\infty} I_n(\gamma t) (\beta z)^n \]
for \( \gamma = 2\sqrt{\lambda \mu} \) and \( \beta = \sqrt{\lambda/\mu} \), from (20) we obtain:
\[ H(s, t) = e^{-(\lambda+\mu+\alpha)t} \sum_{n=-\infty}^{+\infty} I_{n-k}(\gamma t) \beta^{n-k} s^n + \alpha \int_0^t e^{-(\lambda+\mu+\alpha)\tau} \sum_{n=-\infty}^{+\infty} I_n(\gamma \tau) (\beta s)^n d\tau, \]
where \( I_n(x) \) denotes the modified Bessel function of the first kind. Equating the coefficients of \( z^n \) on both sides of (21) finally yields the transition probabilities
\[ p_k(t) = \left( \frac{\lambda}{\mu} \right)^{\frac{n-k}{2}} I_{n-k} \left( 2\sqrt{\lambda\mu} t \right) e^{-(\lambda+\mu+\alpha)t} + \alpha \left( \frac{\lambda}{\mu} \right)^{\frac{n}{2}} \int_0^t e^{-(\lambda+\mu+\alpha)\tau} I_n(2\sqrt{\lambda\mu} \tau) d\tau. \]
Note that (22) can be expressed as
\[ p_k(t) = e^{-\alpha t} \tilde{p}_k(t) + \alpha \int_0^t e^{-\alpha \tau} \tilde{p}_0(t) d\tau, \]
where, for all \( t \geq 0 \) and \( k, n \in \mathbb{Z} \),
\[ \tilde{p}_k(t) := \left( \frac{\lambda}{\mu} \right)^{\frac{n+k}{2}} I_{n-k} \left( 2\sqrt{\lambda\mu} t \right) e^{-(\lambda+\mu)t}, \]
is the transition probability of the Poisson bilateral birth-death process with birth rate \( \lambda \) and death rate \( \mu \) (see, for instance, Section 2.1 of Conolly [1]). Assuming that the stationary
Indeed, if

\[ \lim_{t \to +\infty} p_{k,n}(t) \]

exist for all \( n \in \mathbb{Z} \), from (19) we have

\[ \sum_{n=-\infty}^{+\infty} \pi_n z^n = \lim_{t \to +\infty} H(z, t) = -\frac{\alpha}{u(z)} = \frac{\alpha z}{\lambda(z - z_1)(z_2 - z)} \]

\[ = \frac{\alpha}{\lambda(z_2 - z_1)} \left[ \sum_{n=-\infty}^{-1} \left( \frac{z}{z_1} \right)^n + \sum_{n=0}^{+\infty} \left( \frac{z}{z_2} \right)^n \right], \]

where

\[ z_{1,2} = \frac{\lambda + \mu + \alpha \pm \sqrt{(\lambda + \mu + \alpha)^2 - 4\lambda\mu}}{2\lambda}, \quad 0 < z_1 < 1 < z_2. \]

Hence,

\[ \pi_n = \begin{cases} \alpha \frac{z_1^{-n}}{\lambda(z_2 - z_1)} & \text{for } n = -1, -2, \ldots, \\ \alpha \frac{z_2^{-n}}{\lambda(z_2 - z_1)} & \text{for } n = 0, 1, 2, \ldots. \end{cases} \]

(25)

It is not hard to see that if \( \lambda = \mu \) then \( X(t) \) has a central symmetry with respect to state 0, with \( x_k = 1 \) for all \( k \):

\[ p_{-k,-n}(t) = p_{k,n}(t), \quad q_{-k,-n} = q_{k,n} \]

for all \( t > 0 \) and \( k, n \in \mathbb{Z} \). Note that if \( \lambda = \mu \), then \( z_1 \) and \( z_2 \) are reciprocal zeroes of \( u(z) \), so that the stationary distribution (25) is symmetric, i.e. \( \pi_n = \pi_{-n} \) for all \( n \in \mathbb{Z} \). Since \( q_{i,j} = q_{j,i} = 0, q_{i,0} > 0 \) and \( q_{j,0} > 0 \) for all \( i, j \in \mathbb{Z} \) such that \( i < 0 < j \), and \( q_{0,1} > 0 \) and \( q_{0,-1} > 0 \), this Markov chain satisfies assumptions (i)-(iii) for which 0 is a symmetry state. In this case the first-passage-time densities through 0 can be obtained via Theorem 3.1. Indeed, if \( \lambda = \mu \), making use of (22) and of property \( I_n(x) = I_{-n}(x) \), for all \( t > 0 \) and \( k = 1, 2, \ldots \) we have:

\[ g_{k,0}^-(t) = h_{k,0}^-(t) - h_{k,0}^+(t) = \sum_{j=1}^{+\infty} p_{k,j}(t) q_{j,0} - \sum_{i=-\infty}^{-1} p_{k,i}(t) q_{i,0} \]

\[ = \lambda [p_{k,1}(t) - p_{k,-1}(t)] + \alpha \left[ \sum_{j=1}^{+\infty} p_{k,j}(t) - \sum_{i=-\infty}^{-1} p_{k,i}(t) \right] \]

(26)

\[ = e^{-2\lambda + \alpha t} \left\{ \lambda [I_{k-1}(2\lambda t) - I_{k+1}(2\lambda t)] + \alpha \sum_{j=1}^{+\infty} [I_{k-j}(2\lambda t) - I_{k+j}(2\lambda t)] \right\}. \]

Furthermore, recalling (18), in this special case for all \( t > 0 \) and \( k = 1, 2, \ldots \) there holds:

\[ g_{k,0}^+(t) = \hat{g}_{k,0}^-(t). \]

In analogy with Theorem 3.2 and by virtue of (22), when \( \lambda = \mu \), we have

\[ p_{k,n}^0(t) = p_{k,n}(t) - p_{-k,n}(t) \]

(27)

\[ = e^{-2\lambda + \alpha t} [I_{n-k}(2\lambda t) - I_{n+k}(2\lambda t)], \quad t > 0. \]

Note that

\[ p_{k,0}^0(t) = p_{0,k}^0(t), \]

(28)

\[ p_{k,n}^0(t) = e^{-\alpha t} \hat{p}_{k,n}^0(t), \]

(29)
where \( \hat{p}_{k,n}(t) \) is the transition probability of \( \hat{X}(t) \) when \( \lambda = \mu \). Functions (26) and (27) are shown in Figure 1 for some choices of the involved parameters.

We finally remark that Eqs. (23) and (29) are in agreement with similar results for birth-death processes with catastrophes obtained in Di Crescenzo et al. [9] and [10].

Figure 1: On the left-hand are the plots of the downward first-passage-time density (26) for \( k = 3, \lambda = 1 \) and \( \alpha = 0.1, 0.2, 0.3 \), from bottom to top near the origin. On the right the 0-avoiding transition probabilities (27) for \( k = 3, n = 1, \lambda = 1 \) and \( \alpha = 0.1, 0.2, 0.5, 1 \) (top to bottom) are indicated.

5 Strong similarity

The notion of similarity between stochastic processes has attracted the attention of several authors (see Giorno et al. [12], for time-homogeneous diffusion processes, Gutiérrez Jáimez et al. [14] for time-nonhomogeneous diffusion processes, Di Crescenzo [3], [4], and Lenin et al. [15], for birth-death processes, and Pollett [16], for Markov chains).

Two continuous-time Markov chains \( X(t) \) and \( \tilde{X}(t) \), with state-space \( \mathcal{S} \), are said to be strongly similar if their transition probabilities satisfy

\[
\tilde{p}_{k,n}(t) = \frac{\beta_n}{\beta_k} p_{k,n}(t), \quad \text{for all } t \geq 0 \text{ and } k,n \in \mathcal{S},
\]

where \( \{\beta_n, n \in \mathcal{S}\} \) is a suitable sequence of real positive numbers (we refer the reader to Pollett [16], for further details). In the following theorem we state that if a Markov chain has a central symmetry, then any of its similar chains has a central symmetry as well.

**Theorem 5.1** Let \( X(t) \) and \( \tilde{X}(t) \) be strongly similar continuous-time Markov chains with state-space \( \mathcal{S} \); if \( X(t) \) has a central symmetry, then for all \( t \geq 0 \) and \( k,n \in \mathcal{S} \) one has:

\[
\tilde{p}_{N-k,N-n}(t) = \frac{x_n}{x_k} \tilde{p}_{k,n}(t)
\]

with

\[
x_n = \frac{\beta_{N-n}}{\beta_n} x_n, \quad n \in \mathcal{S}.
\]

The proof is an immediate consequence of assumed symmetry and similarity properties.

Hereafter we show an application of Theorem 5.1 to a birth-death process having constant rates and state-space \( \mathbb{Z} \).

**Example 5.1** Let \( X(t) \) be the bilateral birth-death process with birth and death rates \( \lambda \) and \( \mu \), respectively. From transition probabilities (24) it is not hard to see \( X(t) \) has a central symmetry with respect to 0, i.e. for all \( t \geq 0 \) and \( k,n \in \mathbb{Z} \) there results

\[
p_{-k,-n}(t) = \frac{x_n}{x_k} p_{k,n}(t), \quad \text{with } x_n = \left( \frac{\lambda}{\mu} \right)^{-n}.
\]
The Markov chains that are strongly similar to \(X(t)\) constitute a family of bilateral birth-death processes characterized by birth and death rates (see Section 4 of Di Crescenzo [4], and Example 3 of Pollett [16])

\[
\tilde{\lambda}_n = \frac{\beta_{n+1}}{\beta_n} \lambda, \quad \tilde{\mu}_n = \frac{\beta_{n-1}}{\beta_n} \mu, \quad n \in \mathbb{Z},
\]

and by transition probabilities (30), with \(p_{k,n}(t)\) given in (24) and

\[
\beta_n = 1 + \eta \left( \frac{\lambda}{\mu} \right)^n, \quad n \in \mathbb{Z},
\]

for all \(\eta \geq 0\). Due to Theorem 5.1, the family of strongly similar processes has a central symmetry with respect to 0:

\[
\tilde{p}_{-k,-n}(t) = \tilde{x}_n \tilde{x}_k p_{k,n}(t), \quad \text{with} \quad \tilde{x}_n = \frac{\beta_{-n}}{\beta_n} x_n = \frac{1 + \eta \left( \frac{\lambda}{\mu} \right)^{-n}}{1 + \eta \left( \frac{\lambda}{\mu} \right)^n} \left( \frac{\lambda}{\mu} \right)^{-n}, \quad n \in \mathbb{Z}.
\]

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