A CHARACTERIZATION OF FULLY BOUNDED DUBROVIN VALUATION RINGS

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Abstract. Let $R$ be a Dubrovin valuation ring. It is shown that $R$ is fully bounded iff for any prime ideal $P$ of $R$ which is different from the Jacobson radical of $R$, $P$ is Goldie prime and either it is lower limit or there is a Goldie prime ideal $P_1$ such that the prime segment $P_1 \supset P$ is Archimedean.

1. Introduction. A ring is called right bounded if any essential right ideal contains a non-zero (two-sided) ideal. Similarly, we can define a left bounded ring. A ring is just called bounded if it is both right bounded and left bounded.

Let $S$ be a ring. We say that $S$ is fully bounded if $S/P$ is bounded for any prime ideal $P$ of $S$. We write $J(S)$ for the Jacobson radical of $S$ and Spec$(S)$ for the set of all prime ideals of $S$.

Let $R$ be a Dubrovin valuation ring of a simple Artinian ring $Q$ (see [7, chap. II] for the definition and elementary properties of Dubrovin valuation rings). A prime ideal $P$ of $R$ is called Goldie prime if $R/P$ is a prime Goldie ring.

We denote by G-Spec$(R)$ the set of all Goldie prime ideals of $R$. Now let $P_1, P \in$ G-Spec$(R)$ with $P_1 \supset P$. The pair $P_1 \supset P$ is called a prime segment if there are no Goldie primes properly between $P_1$ and $P$.

Let $P \in$ G-Spec$(R)$ with $P \neq J(R)$ and set $P_1 = \cap \{P_\lambda \mid P_\lambda \in$ G-Spec$(R)$ with $P_\lambda \supset P \}$. Then, in [2], they have shown that the following four cases only occur:

1. $P$ is lower limit, i.e., $P = P_1$. Otherwise, $P_1 \supset P$ is a prime segment.
2. $P_1 \supset P$ is Archimedean ([1, Theorem 6(a)]).
3. $P_1 \supset P$ is simple ([1, Theorem 6(b)]).
4. $P_1 \supset P$ is exceptional, i.e., there exists a non-Goldie prime ideal $C$ such that $P_1 \supset C \supset P$ ([1, Theorem 6(c)]).

With this classification, we shall prove that $R$ is fully bounded iff (1) and (2) only hold (Theorem 2.5). (Note that $R/J(R)$ is bounded, because it is a simple Artinian ring). For any regular element $c$ in $J(R)$, we define $P(c) = \cap \{P_\lambda \mid P_\lambda \in$ G-Spec$(R)$ with $c \in P_\lambda \}$, a Goldie prime ideal ([1, Proposition 1]). $R$ is called locally invariant if $cP(c) = P(c)c$ for any regular element $c$ in $J(R)$. This concept was defined by Grätzer [5] in order to study the approximation theorem in the case where $R$ is a total valuation ring. We shall show that $R$ is fully bounded if and only if it is locally invariant, by using Theorem 2.5 (Proposition 2.6).

If $Q$ is of finite dimensional over its center, then $R$ is always fully bounded. In the end of the paper, we shall give several examples of fully bounded Dubrovin valuation rings of $Q$ with infinite dimension over the center.
2. Fully bounded Dubrovin valuation rings.

Throughout this section, \( R \) will denote a Dubrovin valuation ring of a simple Artinian ring \( Q \). For any \( P \in \text{Spec}(R) \), set \( C(P) = \{ c \in R \mid c \text{ is regular mod } P \} \). If \( P \in \text{G-Spec}(R) \), then \( C(P) \) is localizable and we denote by \( R_P \) the localization of \( R \) at \( P \). Before starting the lemmas, we note the following: There is a one-to-one correspondence between \( \text{G-Spec}(R) \) and the set of all overrings of \( R \), which is given by \( P \rightarrow R_P \) with \( P = J(R_P) \) and \( S \rightarrow J(S) \) \((P \in \text{G-Spec}(R) \text{ and } S \text{ is an overring of } R) \). Furthermore, for any \( P, P_1 \in \text{G-Spec}(R) \), \( P \supset P_1 \) iff \( R_P \subset R_{P_1} \) ([7, §6] and [1, §2]). We will freely use these properties throughout the paper.

**Lemma 2.1.** Let \( S \) be an order in \( Q \) and \( A \) be an \( S \)-ideal such that \( O_r(A) = T = O_t(A) \), where \( O_r(A) = \{ q \in Q \mid AQ \subseteq A \} \) and \( O_t(A) = \{ g \in Q \mid gA \subseteq A \} \). Suppose that \( A = aT \) for some \( a \in A \). Then \( A = Ta \).

**Proof.** \( T = O_t(A) = aTa^{-1} \) implies \( A = Ta \).

**Lemma 2.2.** Let \( R \) be a Dubrovin valuation ring of \( Q \) and \( P \in \text{G-Spec}(R) \). Suppose that \( P \) is lower limit, i.e., \( P = \cap \{ P_\lambda \mid P_\lambda \in \text{G-Spec}(R) \text{ with } P_\lambda \supset P \} \). Then \( R_P = \cup R_{P_\lambda} \) and \( C(P) = \cup C(P_\lambda) \).

**Proof.** Since \( P_\lambda \supset P \), it follows that \( R_P \supset R_{P_\lambda} \) so that \( R_P \supseteq S = \cup R_{P_\lambda} \). Suppose that \( R_P \supset S \). Then for any \( P_\lambda, P_\lambda = J(R_{P_\lambda}) \supseteq J(S) \supset J(R_P) = P \) implies \( P = \cap P_\lambda \supseteq J(S) \supset P \), a contradiction. Hence \( R_P = \cup R_{P_\lambda} \) and so \( C(P) = \cup C(P_\lambda) \) follows.

**Lemma 2.3.** Let \( R \) be a Dubrovin valuation ring of \( Q \) and \( P \in \text{G-Spec}(R) \). Then

1. \( \text{Spec}(R_P) = \{ P_i \mid P_i \in \text{Spec}(R) \text{ with } P \supset P_i \} \).
2. Let \( P_1 \) and \( P_2 \) be in \( \text{Spec}(R) \) with \( P \supseteq P_1 \supset P_2 \). Then \( P_1 \supset P_2 \) is a prime segment of \( R \) if and only if it is a prime segment of \( R_P \).

**Proof.** (1) Let \( P_i \in \text{Spec}(R_P) \).
Case 1. If \( P_i \) is Goldie prime, then \( (R_P)_P \) is an overring of \( R_P \) (and so of \( R \)) with \( J((R_P)_P) = P_i \), i.e., \( P_i \in \text{Spec}(R) \) and \( P = J(R_P) \supseteq P_i \).
Case 2. If \( P_i \) is non-Goldie prime, then we can construct an exceptional prime segment of \( R_P \), say \( P_2 \supset P_1 \supset P_0 \) by [1, Theorem 6]. By case 1, \( P \supseteq P_2 \) and \( P_2 \) is in \( \text{G-Spec}(R) \). It easily follows from note before Lemma 2.1 that there are no Goldie primes properly between \( P_2 \) and \( P_0 \), which implies \( P_2 \supset P_0 \) is a prime segment of \( R \). As in [1], let \( K(P_2) = \{ a \in P_2 \mid P_2aP_2 \subset P_2 \} \). Then \( K(P_2) = P_1 \) by [1, Corollary 7] and so \( P_2 \supset P_0 \) is an exceptional prime segment of \( R \) with \( K(P_2) = P_1 \), i.e., \( P_1 \) is non-Goldie prime of \( R \) with \( P \supset P_1 \). Conversely, let \( P_i \in \text{Spec}(R) \) with \( P \supseteq P_i \). Then from note before Lemma 2.1 and the method we have just done, we can easily see that \( P_i \in \text{Spec}(R_P) \) and that \( P_i \in \text{G-Spec}(R) \) iff \( P_i \in \text{G-Spec}(R_P) \).

(2) This is clear from (1).

**Lemma 2.4.** Let \( R \) be a Dubrovin valuation ring of \( Q \) and \( P \supset P_1 \) be an Archimedean prime segment. Then for any \( c \in P \setminus P_1 \), the following hold:

1. \( R_P cR_{P_1} = aR_{P_1} \) for some \( a \in P_1 \).
2. If \( c \) is a regular element, then \( cR_{P_1} = R_{P_1}c \) and \( cP_1 = P_1c \).


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**Proof.** Firstly note that $P_1 \supset P$ is an Archimedean prime segment of $R P_1$ by Lemma 2.3 and [1, Corollary 7].

(1) Let $\tilde{R}_P = R_P / P$, a Dubrovin valuation ring of $\overline{R_P} = R_P / P$ (see [7, (6.6)]) such that $J(\tilde{R}_P) = \tilde{P}_1 = P_1 / P$ and $\tilde{P}_1 \supset (0)$ is Archimedean. Here for any $a \in R_P$, we write $\tilde{a}$ for the image of $a$ in $\tilde{R}_P$. If $\tilde{P}_1 = \tilde{P}_1^2$, then $\tilde{0} \neq \tilde{R}_P \subset \tilde{P}_1 \subset \tilde{R}_P$ for some $a \in P_1$ by [2, (2.1)]. If $\tilde{P}_1 \supset \tilde{P}_1^2$, then $\tilde{R}_P$ is a Noetherian Dubrovin valuation ring and so any ideal of $\tilde{R}_P$ is power of $\tilde{P}_1$. Thus $\tilde{R}_P \subset \tilde{P}_1 = \tilde{a} \tilde{R}_P = \tilde{P}_1 \tilde{a}$ for some $a \in P_1$, because $\tilde{P}_1$ is principal. Hence, in both cases, $R_P \subset \tilde{P}_1 \subset \tilde{R}_P$ because there are no ideals properly between $\tilde{P}_1$. We have shown that $\tilde{a} \subset \tilde{R}_P$.

(2) Let $\tilde{P}_1 = \tilde{P}_1^2$, then $\tilde{R}_P$ is an Archimedean prime $P$-valuation ring. Hence, in both cases, $\tilde{a} \subset \tilde{R}_P$. Thus we have $\tilde{a} \tilde{R}_P = \tilde{P}_1 \tilde{a}$ for some $a \in P_1$ by [7, (22.6)]. Thus we have $\tilde{a} \tilde{R}_P = \tilde{P}_1 \tilde{a}$.

Thus, by [7, (6.9)], there is a Dubrovin valuation ring of a simple Artinian ring $Q$. Then, by [7, (6.3)], there is a $P_1 \subset \tilde{P}_1$ such that $\tilde{c} \tilde{R}_P = \tilde{c} \tilde{P}_1 = \tilde{c} \tilde{R}_P$. So $\tilde{c} \tilde{R}_P = \tilde{1} \subset \tilde{R}_P$. On the other hand, $\tilde{1} \subset \tilde{R}_P$ implies that $\tilde{c} \tilde{R}_P = \tilde{1}$.

Hence, $\tilde{c} \tilde{R}_P = \tilde{P}_1 \tilde{c} \tilde{R}_P$, a contradiction. Hence, $\tilde{c} \tilde{R}_P = \tilde{P}_1 \tilde{c} \tilde{R}_P$ and similarly $\tilde{R}_P \subset \tilde{P}_1$. Since $\tilde{P}_1 \subset \tilde{P}_1 \subset \tilde{R}_P$, we have $\tilde{P}_1 \subset \tilde{P}_1 \subset \tilde{P}_1 \subset \tilde{R}_P$.

We are now ready to prove the main result of the paper.

**Theorem 2.5.** Let $R$ be a Dubrovin valuation ring of a simple Artinian ring $Q$. Then $R$ is fully bounded if and only if for any $P \in \text{Spec}(R)$, $P \neq J(P)$, the following hold:

1. $P \in \text{G-Spec}(R)$.
2. $P$ is either lower limit or there is a $P_1 \in \text{Spec}(R)$ such that $P_1 \supset P$ is an Archimedean prime segment.

**Proof.** Suppose that $R$ is fully bounded.

(1) Assume that there is a non-Goldie prime ideal $C$. Then we have an exceptional prime segment, say, $P_1 \supset C \supset P_2$ by [1, Theorem 6]. $R$ is an $n$-chain ring by [7, (5.11)] and so is $R = R / C$. This implies that $\overline{C}$ has a finite Goldie dimension, say, $m(\equiv n)$. Thus there are non-zero uniform right ideals $U_i$ of $\overline{R}$ such that $U_1 \oplus \ldots \oplus U_m$ is an essential right ideal of $\overline{R}$.

Since $\overline{R}$ is a prime ring, $U_i \cap \overline{R} = \overline{U_i} \cap \overline{R} = \overline{U_i}$ and so there are non-zero $\overline{u_i} \in \overline{U_i} \cap \overline{R}$, where $u_i \in P_1$. Set $I = u_1 R + \ldots + u_m R$. Then $I = a R$ for some $a \in I$, because $R$ is Bezout (cf. [7, (5.11)]) and $\overline{I} = \pi_1 \overline{R} + \ldots + \pi_m \overline{R} = \overline{\pi R}$ is an essential right ideal of $\overline{R}$. We claim that $\overline{I} \supset \overline{I}$. On the contrary, suppose that $\overline{I} \supset \overline{I}$, i.e., $P_1 = a R + C$. Note that $O(C) = R_P = O_C (C)$ by [2, (2.2)] so that $C$ is an ideal of $R_P$. If $C$ is a principal right ideal of $R_P$, say, $C = c R_P$, for some $c \in C$, then $P_1 = a R_P + c R_P = b R_P$ for some $b \in P_1$. It follows from Lemma 2.1 that $P_1 = b R_P = R_P b$ and so $P_1 \supset P_1^2 \supset C$, which contradicts the fact that there are no ideals properly between $P_1$ and $C$ (cf. [1, Theorem 6]). If $C$ is not a principal right ideal of $R_P$, then $C_P = C$ by [7, (6.9)] and so $P_1 = P_1^2 = a P_1 + C P_1$. Thus we have $a = a p + d$ for some $p \in P_1$ and $d \in C$ and $a(1 - p) = d \in C$. It follows that $a \in C$, because $1 - p$ is a unit of $R_P$, which shows $\overline{I} = \overline{U}$, a contradiction.

We have shown that $\overline{I} \supset \overline{I}$ and $\overline{I}$ is an essential right ideal of $\overline{R}$. Hence $\overline{R}$ is not bounded, because there are no ideals properly between $P_1$ and $C$. Therefore, any prime ideal of $R$ is Goldie prime.

(2) Let $P \in \text{G-Spec}(R)$ and suppose that $P$ is not lower limit. Then there is a $P_1 \in \text{G-Spec}(R)$ such that $P_1 \supset P$ is a prime segment, which is not exceptional by (1). Suppose
that this is simple. For any $c \in P_1 \cap C(P)$, it follows that $\overline{\pi P_1}$ is an essential right ideal of $\overline{R} = R/P$, which is a Dubrovin valuation ring of $R_P/P$ (cf. [7, (5.12)]). Suppose that $\overline{\pi P_1} = \overline{P_1}$, i.e., $cP_1 + P = P_1$. Since $cP_1$ and $P$ are both left $cR_P, c^{-1}$ and right $R_P$-ideals (note $cR_P, c^{-1} \subseteq cR_P c^{-1} = R_P$), we have either $cP_1 \supset P$ or $cP_1 \subseteq P$ by [7, (6.4)]. The latter case is impossible and so $cP_1 \supset P$. Thus $cP_1 = P_1$ and $c^{-1} \in O(P_1) = R_{P_1}$ follows.

This is a contradiction, because $c \in P_1$. Hence we have shown that $\overline{P_1} \supset \overline{\pi P_1}$ and $\overline{\pi P_1}$ is an essential right ideal. Therefore, $\overline{R}$ is not bounded, because there are no ideals properly between $\overline{P_1}$ and $\overline{P}$. Hence either $P$ is lower limit or there is a $P_1 \in G\text{-}Spec(R)$ such that $P_1 \supset P$ is an Archimedean prime segment.

Conversely, suppose that the conditions (1) and (2) hold and let $P \in \text{Spec}(R)$. Then $P$ is Goldie prime by (1). Firstly, assume that $P$ is lower prime, i.e., $P = \cap\{P_{\lambda} \mid P_{\lambda} \in G\text{-}Spec(R) \text{ with } P_{\lambda} \supset P\}$. Then $C(P) = \cup C(P_{\lambda})$ by Lemma 2.2. So, for any $c \in C(P)$, we have $c \in C(P_{\lambda})$ for some $\lambda$. Then $cR \supset P_{\lambda}$, because $cR$ and $P_{\lambda}$ are both left $cRc^{-1}$ and right $R$-ideals. Hence $\pi R \supset P_{\lambda} \neq \emptyset$ in $\overline{R} = R/P$, showing that $\overline{R}$ is bounded. Secondly, suppose that the prime segment $P_1 \supset P$ is Archimedean and let $c \in C(P)$. Then, as before, $\pi P_1$ is an essential right ideal of $\overline{R} = R/P$ and so $cP_1 \cap C(P) \neq \emptyset$. Let $d \in cP_1 \cap C(P)$. Then, by Lemma 2.4 (2) and [7, (22.7)], $cR \supset cP_1 \supset dR_{P_1} = R_{P_1}d$ and $dR_{P_1} \supset P$ follows.

Therefore, $\overline{R} = R/P$ is bounded and hence $R$ is fully bounded.

As an application of Theorem 2.5, we have the following:

**Proposition 2.6.** Let $R$ be a Dubrovin valuation ring of a simple Artinian ring $Q$. Then $R$ is locally invariant if and only if it is fully bounded.

**Proof.** Suppose that $R$ is locally invariant. In order to prove that it is fully bounded, on the contrary, assume that $R$ is not fully bounded. Then there are prime ideals $P, P_1$ such that either the prime segment $P_1 \supset P$ is simple or $P_1 \in G\text{-}Spec(R)$, $P$ is a non-Goldie prime ideal and there are no ideals properly between $P_1$ and $P$. In either case, we shall prove that there is a regular element $c \in P_1 \setminus P$. Let $c_1$ be any element in $P_1 \setminus P$. If $c_1R$ is an essential right ideal. Then $c = c_1$ is regular. If $c_1R$ is not an essential right ideal, then there is a right ideal $I$ such that $cR \oplus I$ is essential. So it follows from Goldie’s theorem that $(cR \oplus I)P_1$ is also an essential right ideal which is contained in $P_1$ but not in $P$. So there is a regular element $c \in (c_1R \oplus I)P_1$ but not in $P$ by [8, (3.3.7), Corollary]. Now let $c \in P_1 \setminus P$ such that $c$ is regular. Then $cP_1 = P_1c$, because $P_1 = P(c)$. Since $P_1 \supset cP_1 = P_1c \supset P$, we have $cP_1 = P_1$, which implies $c^{-1} \in O(P_1) = R_{P_1}$. Hence $R_{P_1} = cR_{P_1} \subseteq P_1$, a contradiction. Therefore, $R$ is fully bounded.

Suppose that $R$ is fully bounded. Let $c \in J(R)$ such that $c$ is regular. By the assumption and Theorem 2.5, $P(c) = \cap \{P_{\lambda} \mid P_{\lambda} \in \text{Spec}(R) \text{ such that } P_{\lambda} \supset c \}$, which is Goldie prime by [1, Proposition 1]. Suppose that $P(c)$ is upper limit, i.e., $P(c) = \cup \{P_{\mu} \mid P_{\mu} \in G\text{-}Spec(R) \text{ with } P_{\mu} \subset P(c) \}$. Then there is a $P_\mu$ with $P_\mu \supset c$. This contradicts the choice of $P(c)$. Hence $P(c) \supset P = \cup \{P_{\mu} \mid P(c) \supset P_{\mu} \}$ is a prime segment which must be Archimedean by Theorem 2.5. Since $c \in P(c) \setminus P$ and $c$ is regular, we have $cP(c) = P(c)c$ by Lemma 2.4. Hence $R$ is locally invariant.

We say that $R$ is *invariant* if $cRc^{-1} = R$ for any regular element $c$ in $R$ and that it is of *rank* $n$ if there are exactly $n$ Goldie prime ideals. From Lemma 2.4, we have

**Proposition 2.7.** Suppose that $R$ is Archimedean and is of rank one. Then it is invariant.
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Proof. Let c be any regular element and let c₁ be any regular element in \( J(R) \). Then we have \( cRc^{-1} = cc_1R(cc_1)^{-1} = R \) by Lemma 2.4, because \( c_1, cc_1 \in J(R) \).

We will give several examples of fully bounded Dubrovin valuation rings.

Example 2.8. Any Dubrovin valuation ring of a simple Artinian ring with finite dimension over its center is fully bounded.

Example 2.9. Any invariant valuation ring of a division ring is fully bounded (see [9, Remarks to examples 2.1 and 2.4] for invariant valuation rings of division rings with infinite dimensions over the centers).

In order to give more general examples, we recall the skew polynomial ring \( Q[x, \sigma] \) over \( Q \) in an indeterminate \( x \), where \( \sigma \in \text{Aut}(Q) \). Since \( Q[x, \sigma] \) is a principal ideal ring, the maximal ideal \( P = xQ[x, \sigma] \) is localizable, i.e., \( T = Q[x, \sigma]_P = \{ f(x)c(x)^{-1} \mid f(x) \in Q[x, \sigma] \} \) and \( c(x) \in C(P) \), the localization of \( Q[x, \sigma] \) at \( P \), is a Noetherian Dubrovin valuation ring with \( J(T) = xT \). Since \( Q \) is a simple Artinian ring, \( C(P) = \{ c(x) \in Q[x, \sigma] \mid c(x) = c_0 + c_1x + \ldots + c_\alpha x^\alpha \text{ such that } c_0 \text{ is a unit in } Q \} \). For any \( t = f(x)c(x)^{-1} \in T \), where \( f(x) = f_0 + f_1x + \ldots + f_\alpha x^\alpha \) and \( c(x) = c_0 + c_1x + \ldots + c_\alpha x^\alpha \), the map \( \varphi \colon T \to Q \) defined by \( \varphi(t) = f_0c_0^{-1} \) is an ring epimorphism. Now let \( R \) be a Dubrovin valuation ring of \( Q \). Then, by [9, (1.6)], \( \tilde{R} = \varphi^{-1}(R) \), the complete inverse image of \( R \) by \( \varphi \), is a Dubrovin valuation ring of \( Q(x, \sigma) \) \((Q(x, \sigma) \text{ stands for the quotient ring of } Q[x, \sigma]) \). Furthermore, let \( P = p\tilde{R} \) \((p \in \text{Spec}(R)) \). Then \( P \subseteq \text{Spec}(\tilde{R}) \) and \( \tilde{R}/P \cong R/p \) by [9, (1.6)] and its proof. Thus it follows from [9, (1.6)] that \( \tilde{R} \) is fully bounded iff \( R \) is fully bounded. Hence we have

Example 2.10. With notation above, suppose that \( R \) is a fully bounded Dubrovin valuation ring of \( Q \) and that \( \sigma \) is of infinite order ([9, Examples 2.1 ~ 2.6, 2.7 and 2.8]). Then \( \tilde{R} \) is a fully bounded Dubrovin valuation ring of \( Q(x, \sigma) \) and \( Q(x, \sigma) \) is of infinite dimensional over the center.

Finally, we give a few remarks on non-fully bounded total valuation rings: An example of a total valuation ring with a simple segment was first constructed by Mathiak ([6]). See [3] for other examples of total valuation rings with simple segments. Dubrovin constructed an example of a total valuation ring with an exceptional prime segment ([4]).

References


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