KURATSUBO PHENOMENON OF THE FOURIER SERIES OF SOME RADIAL FUNCTIONS IN FOUR DIMENSIONS

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Abstract. On the Fourier series the Gibbs-Wilbraham phenomenon is well known. In 1993, Pinsky, Stanton and Trapa discovered the so called Pinsky phenomenon on the spherical partial sum for the Fourier series of the indicator function of a d-dimensional ball with $d \geq 3$. In 2010, Kuratsubo discovered the third phenomenon in dimension $d \geq 5$. Recently, Taylor found that the Pinsky phenomenon arises even in two-dimension. In this paper we prove that the Kuratsubo phenomenon arises even in four-dimension.

1 Introduction For the Fourier series of piecewise continuous functions, the Gibbs-Wilbraham phenomenon is well known. For example, let

$$\chi_a(x) = \begin{cases} 1, & |x| < a, \\ 0, & |x| > a, \end{cases} \quad x \in \mathbb{R}^d, \quad a > 0.$$

Let $d = 1$. Then the partial sums overshoot the jump at $x = \pm a$ by approx. 9% of the jump, while its partial sum $S_\lambda(\chi_a)(x)$ converges $\chi_a(x)$ as $\lambda \to \infty$ at $x \neq \pm a$. This phenomenon can

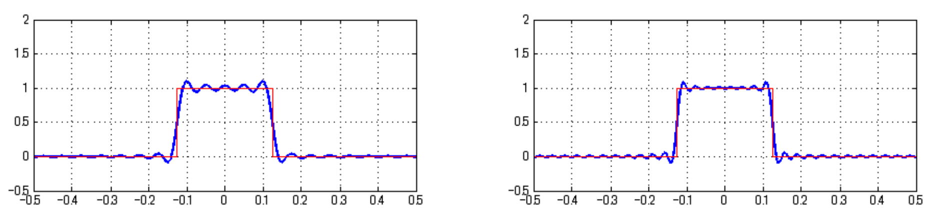

Figure 1: Gibbs-Wilbraham phenomenon $S_\lambda(\chi_{1/8})$ $(\lambda = 20, 30)$ [2]

be seen not only in one dimension but also in higher dimensions (see for example [1, 8, 12]).

In one dimension, it is also well known as the localization property that, if the function is zero on an interval, then the Fourier series converges to zero there. However, in higher dimensions this property is no longer valid. In 1993, Pinsky, Stanton and Trapa [10] showed that, for the Fourier series of the indicator function of a d-dimensional ball with $d \geq 3$, the spherical partial sum diverges at the center of the ball. This phenomenon is called the Pinsky phenomenon.

In 1996 Kuratsubo [3] conjectured that, if $d \geq 5$, then the third phenomenon would arise, see also [4]. After the numerical calculation by [7] (2006) he proved that his conjecture is true in [5] (2010). Namely, for the Fourier series of the indicator function of a d-dimensional ball with $d \geq 5$, the spherical partial sum diverges at all rational points, while it converges almost everywhere, see Figures 3–5. Figure 4 is the expansion of Figure 3 to the direction of the vertical axis for the interval $[0.2, 0.5]$. Figure 5 is created using 3D graphics.

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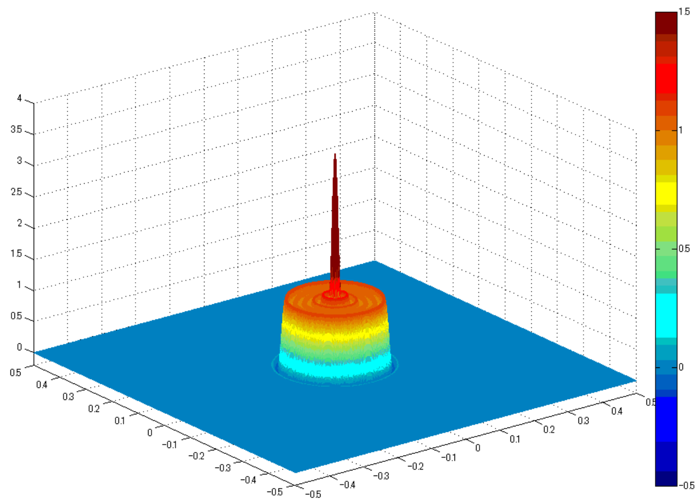

Figure 2: Pinsky phenomenon in 4 dim. $S_\lambda(\chi_{1/8})(x_1, x_2, 0, 0)$ $(\lambda = 47)$ [2]

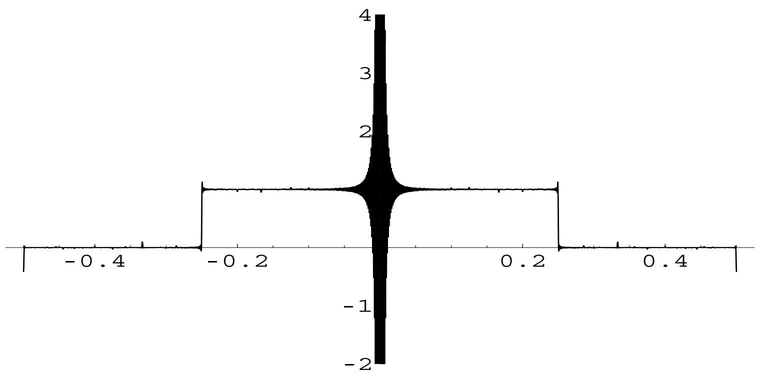

Figure 3: Kuratsubo phenomenon in 6 dim. $S_\lambda(\chi_{1/4})(x, 0, 0, 0, 0, 0)$ $(\lambda = 800)$ [7]

Recently, Taylor [13, 14] found that the Pinsky phenomenon arises even in two dimensions. He treated the radial function

$$U_a(x) = \begin{cases} 1/\sqrt{a^2 - |x|^2}, & |x| < a, \\ 0, & |x| \geq a, \end{cases} \quad x \in \mathbb{R}^2, \ a > 0.$$

See Figure 6.

Our aim in this paper is to prove the Kuratsubo phenomenon in four dimensions. We consider the Fourier series of the function

$$U_{\beta,a}(x) = \begin{cases} (a^2 - |x|^2)^\beta, & |x| < a \\ 0, & |x| \geq a, \end{cases} \quad x \in \mathbb{R}^4, \ a > 0, \ \beta > -1. \tag{1.1}$$

If $\beta = 0$, then $U_{\beta,a}(x)$ is the same as the indicator function of the ball centered at the origin and of radius a. If $\beta = -1/2$, then $U_{\beta,a}(x)$ is the function considered by Taylor [13, 14]. We consider the case $-1 < \beta < -1/2$.

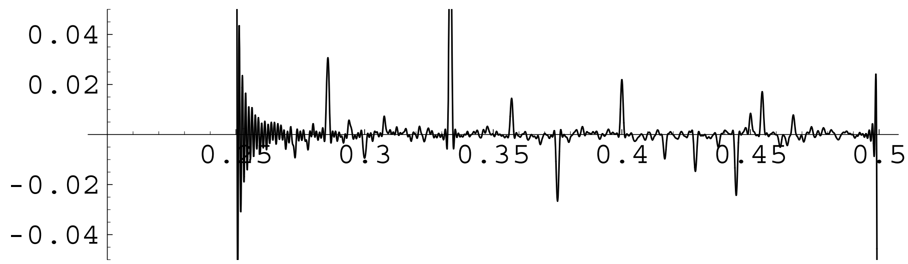

Figure 4: Kuratsubo phenomenon (expansion of Figure 3) [7]

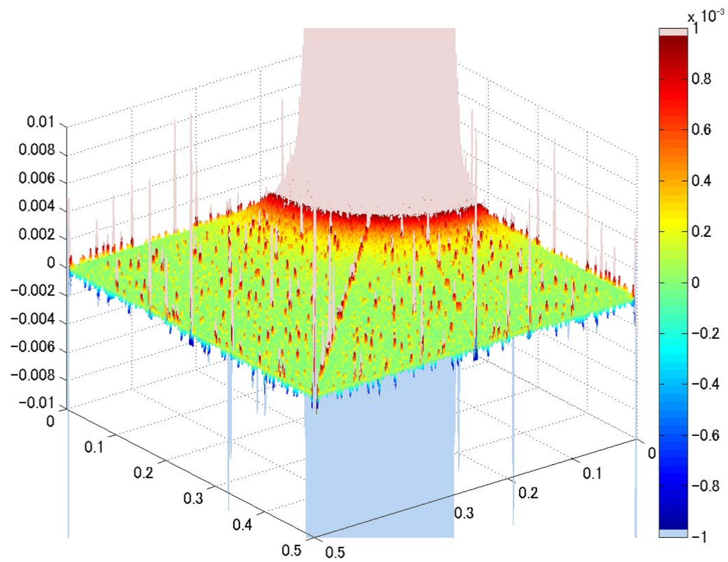

Figure 5: Kuratsubo phenomenon $S_\lambda(\chi_{1/8})(x_1, x_2, 0, 0, 0, 0)$ $(\lambda = 407)$ [2]

In the next section we give the definitions of the Fourier spherical partial sum and the Fourier spherical partial integral and state some known results on them. Then we state our main result in Section 3 and prove it in Section 4.

At the end of this section we note the sources of the figures. Figures 1, 2 and 5 were made by MATLAB in [2]. Figures 3 and 4 were made by Mathematica in [7]. In this time we made Figure 6 by Mathematica and Figures 7 and 8 by gnuplot with Java.

2 Definitions and known results

By $\mathbb{R}^d$, $\mathbb{Z}^d$ and $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ we denote the d-dimensional Euclidean space, integer lattice and torus, respectively. In this paper, however, we always identify $\mathbb{T}^d$ with $(-1/2, 1/2]^d$, that is, $x \in \mathbb{T}^d$ means $x \in (-1/2, 1/2]^d$ and $\mathbb{T}^d \subset \mathbb{R}^d$. Let $\mathbb{Q}$ be the set of all rational numbers, and let $\mathbb{Q}^d = \{(x_1, \ldots, x_d) : x_1, \ldots, x_d \in \mathbb{Q}\}$.

For an integrable function $F(x)$ on $\mathbb{R}^d$, its Fourier transform $\hat{F}(\xi)$ and its Fourier spher-

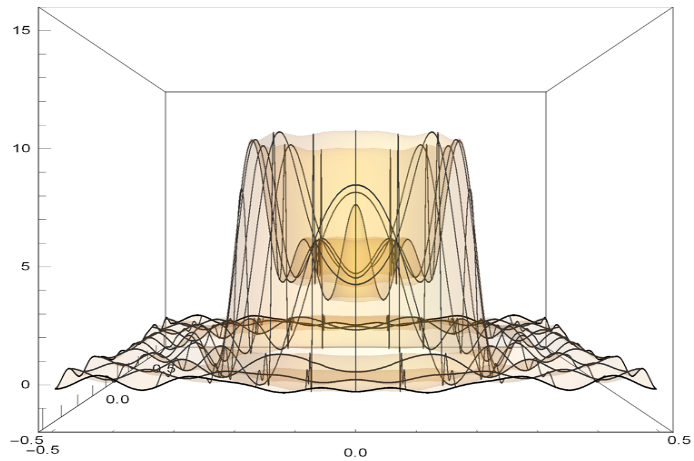

Figure 6: Pinsky phenomenon in 2 dim, $S_{10}(U_{1/4})$

ical partial integral $\sigma_\lambda(F)(x)$ of order $\lambda \ge 0$ are defined by

$$\hat{F}(\xi) = \int_{\mathbb{R}^d} F(x) e^{-2\pi i \xi x}\, dx, \quad \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d, \tag{2.1}$$

$$\sigma_\lambda(F)(x) = \int_{|\xi|<\lambda} \hat{F}(\xi) e^{2\pi i \xi x} d\,\xi, \quad |\xi| = \sqrt{\sum_{k=1}^{d} {\xi_k}^2}, \; x \in \mathbb{R}^d, \tag{2.2}$$

respectively, where ξx is the inner product $\sum_{k=1}^d \xi_k x_k$. Also, for an integrable function $f(x)$ on $\mathbb{T}^d$, its Fourier coefficients $\hat{f}(m)$ and its Fourier spherical partial sum $S_\lambda(f)(x)$ of order $\lambda \ge 0$ are defined by

$$\hat{f}(m) = \int_{\mathbb{T}^d} f(x) e^{-2\pi i m x}\, dx, \quad m = (m_1, \cdots, m_d) \in \mathbb{Z}^d, \tag{2.3}$$

$$S_\lambda(f)(x) = \sum_{|m|<\lambda} \hat{f}(m) e^{2\pi i m x}, \quad |m| = \sqrt{\sum_{k=1}^{d} {m_k}^2}, \; x \in \mathbb{T}^d, \tag{2.4}$$

respectively.

For an integrable function $F(x)$ on $\mathbb{R}^d$, we consider its periodization

$$f(x) = \sum_{m \in \mathbb{Z}^d} F(x+m), \quad x \in \mathbb{T}^d. \tag{2.5}$$

Note that in (2.5) the series converges with respect to the L^1-norm on $\mathbb{T}^d$ and then f is an integrable function on $\mathbb{T}^d$. Then it is known as the Poisson summation formula that the equation

$$\hat{f}(m) = \hat{F}(m), \quad m \in \mathbb{Z}^d \tag{2.6}$$

holds, see for example [11, Theorem 2.4 (page 251)]. The left hand side of (2.6) is defined by (2.3) and the right hand side of (2.6) is defined by (2.1) with $\xi = m$.

In particular, we denote by $u_{\beta,a}(x)$ the periodization of $U_{\beta,a}(x)$. That is,

$$u_{\beta,a}(x) = \sum_{m \in \mathbb{Z}^d} U_{\beta,a}(x+m), \quad x \in \mathbb{T}^d. \tag{2.7}$$

In this paper we always assume that $0 < a < 1/2$. Then

$$u_{\beta,a}(x) = U_{\beta,a}(x), \quad x \in \mathbb{T}^d. \tag{2.8}$$

The behavior of $\sigma_\lambda(U_{\beta,a})(x)$ as $\lambda \to \infty$ is known by [6]. Let Γ be the Gamma function and J_ν the Bessel function of order ν. Then the following theorem is known:

Theorem 2.1 ([6, Theorem 4.1]). *Let $d \geq 1$, $a > 0$ and $\beta > -1$. Then*

$$\sigma_\lambda(U_{\beta,a})(x) = 2^\beta \Gamma(\beta+1) a^{2\beta} \int_0^{2\pi a\lambda} \frac{J_{\frac{d}{2}-1}(\frac{|x|}{a}s) J_{\frac{d}{2}+\beta}(s)}{\left(\frac{|x|}{a}\right)^{\frac{d}{2}-1} s^\beta}\, ds, \tag{2.9}$$

for all $x \in \mathbb{R}^d$ and $\lambda > 0$. Moreover, $\sigma_\lambda(U_{\beta,a})$ has the following properties:

1. *At $x = 0$,*
   (a) *if $\beta > (d-3)/2$, then $\sigma_\lambda(U_{\beta,a})(0)$ converges to $U_{\beta,a}(0)$ as $\lambda \to \infty$,*
   (b) *if $-1 < \beta \leq (d-3)/2$, then $\sigma_\lambda(U_{\beta,a})$ reveals the Pinsky phenomenon.*
2. *For near $|x| = a$,*
   (a) *if $\beta > 0$, then $\sigma_\lambda(U_{\beta,a})(x)$ converges to $U_{\beta,a}(x)$ as $\lambda \to \infty$,*
   (b) *$-1 < \beta \leq 0$, then $\sigma_\lambda(U_{\beta,a})$ reveals the Gibbs-Wilbraham phenomenon.*
3. *If $x \neq 0$ and $|x| \neq a$, then $\sigma_\lambda(U_{\beta,a})(x)$ converges to $U_{\beta,a}(x)$ as $\lambda \to \infty$ and the convergence is uniform on any compact subset of $\mathbb{R}^d \setminus \{x \neq 0, |x| \neq a\}$.*

The difference between $\sigma_\lambda(U_{\beta,a})$ and $\sigma_\lambda(u_{\beta,a})$ is also known by [6]. For $j = 0, 1, 2, \ldots$, let

$$D_j(s:x) = \frac{1}{\Gamma(j+1)} \sum_{|m|^2 < s} (s - |m|^2)^j e^{2\pi i m x}, \quad s > 0, \quad x \in \mathbb{R}^d,$$
$$\mathcal{D}_j(s:x) = \frac{1}{\Gamma(j+1)} \int_{|\xi|^2 < s} (s - |\xi|^2)^j e^{2\pi i x\xi}\, d\xi, \quad s > 0, \quad x \in \mathbb{R}^d,$$

and

$$\Delta_j(s:x) = D_j(s:x) - \mathcal{D}_i(s:x), \quad s > 0,\ x \in \mathbb{R}^d. \tag{2.10}$$

Further, for $j = 0, 1, 2, \ldots$, $\beta > -1$ and $a > 0$, let

$$A^{(j)}_{\beta,a}(s) = (-1)^j \frac{\Gamma(\beta+1)}{\pi^{\beta-j}} a^{\frac{d}{2}+\beta+j} \frac{J_{\frac{d}{2}+\beta+j}(2\pi a\sqrt{s})}{s^{\frac{1}{2}(\frac{d}{2}+\beta+j)}}, \quad s > 0, \tag{2.11}$$

and let

$$\mathcal{K}_{\beta,a}(s:x) = \sum_{j=0}^{d_\sharp} (-1)^j \Delta_j(s:x) A^{(j)}_{\beta,a}(s), \tag{2.12}$$

where $d_\sharp$ is the integer part of $(d+1)/2$. Then the following theorem is known:

Theorem 2.2 ([6, Corollary 6.2])**.** *Let $d \geq 1, \beta > -1$ and $0 < a < 1/2$. Then*

$$S_\lambda(u_{\beta,a})(x) = \sigma_\lambda(U_{\beta,a})(x) + \mathcal{K}_{\beta,a}(\lambda^2 : x) + O(\lambda^{-\beta-1}) \quad \text{as} \quad \lambda \to \infty \tag{2.13}$$

for all $x \in \mathbb{T}^d$.

In the above O is Landau's symbol, that is, $f(s) = O(g(s))$ as $s \to \infty$ means that $\limsup_{s\to\infty} |f(s)|/g(s) < \infty$ for the positive valued function g. Similarly, $f(s) = o(g(s))$ as $s \to \infty$ means that $\lim_{s\to\infty} f(s)/g(s) = 0$.

Therefore, to investigate the behavior of $S_\lambda(u_{\beta,a})(x)$ as $\lambda \to \infty$ we need to estimate $\mathcal{K}_{\beta,a}(\lambda^2 : x)$.

3 Main result

Recall that

$$u_{\beta,a}(x) = U_{\beta,a}(x), \quad x \in \mathbb{T}^d, \tag{3.1}$$

under the assumption $0 < a < 1/2$. Let

$$E_a = \{x \in \mathbb{T}^d : x \neq 0, |x| \neq a\}.$$

Our main result is the following:

Theorem 3.1. *Let $d = 4$ and $0 < a < 1/2$. Fix a point $x \in E_a \cap \mathbb{Q}^4$ arbitrarily. If $S_\lambda(u_{\beta,a})(x)$ converges $u_{\beta,a}(x)$ as $\lambda \to \infty$ for some $\beta \in (-1, -1/2)$, then $S_\lambda(u_{\beta,a})(x)$ diverges for all other $\beta \in (-1, -1/2)$.*

This theorem shows that the Kuratsubo phenomenon arises even if $d = 4$. On the other hand, it is known by [6, Theorem 1.3] that, if $d = 4$ and $\beta > -1/10$, then $S_\lambda(u_{\beta,a})(x)$ converges to $u_{\beta,a}(x)$ as $\lambda \to \infty$ for all $x \in E_a$. Therefore, the case of $\beta \in [-1/2, -1/10]$ is an open problem. Note also that, if $\beta > -1/2$, then $S_\lambda(u_{\beta,a})(x)$ converges to $u_{\beta,a}(x)$ as $\lambda \to \infty$ a.e. $x \in \mathbb{T}^4$, see [6, Theorem 1.5].

Figures 7 and 8 are graphs of $S_\lambda(u_{\beta,a})(x, 0, 0, 0)$ for $\beta = -9/10$ and $a = 1/8$ in four dimensions. We can observe the Kuratsubo phenomenon in Figure 8.

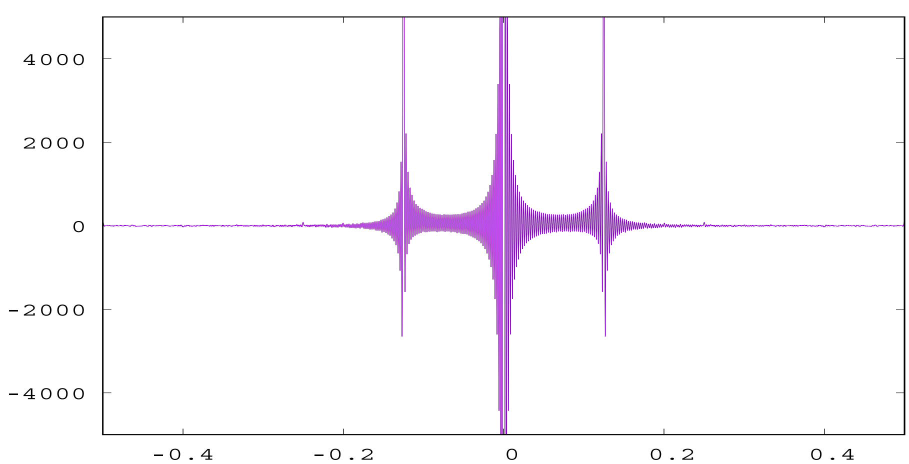

Figure 7: Kuratsubo phenomenon in 4 dim. $S_\lambda(u_{\beta,a})(x, 0, 0, 0)$ $(\lambda = 400)$

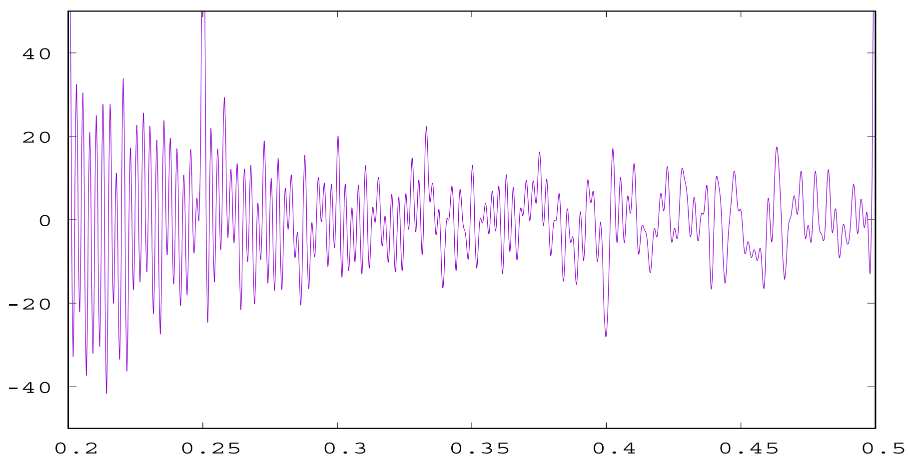

Figure 8: Kuratsubo phenomenon in 4 dim. (expansion of Figure 7)

4 Proof Let $x \in E_a$. By Theorems 2.1, 2.2 and (3.1), we see that $S_\lambda(u_{\beta,a})(x) \to u_{\beta,a}(x)$ if and only if $\mathcal{K}_{\beta,a}(\lambda^2 : x) \to 0$. To estimate

$$\mathcal{K}_{\beta,a}(\lambda^2 : x) = \sum_{j=0}^{d_\sharp} (-1)^j \Delta_j(\lambda^2 : x) A_{\beta,a}^{(j)}(\lambda^2),$$

we combine the estimates of $\Delta_j(\lambda^2 : x)$ and $A_{\beta,a}^{(j)}(\lambda^2)$.

Firstly, by the asymptotic behavior of Bessel functions

$$J_\nu(s) = \sqrt{\frac{2}{\pi s}} \cos\left(s - \frac{2\nu+1}{4}\pi\right) + O(s^{-3/2}) \quad \text{as} \quad s \to \infty, \tag{4.1}$$

we see that

$$\begin{aligned} A_{\beta,a}^{(j)}(s) &= (-1)^j \frac{\Gamma(\beta+1)}{\pi^{\beta-j}} a^{\frac{d}{2}+\beta+j} \frac{J_{\frac{d}{2}+\beta+j}(2\pi a\sqrt{s})}{s^{\frac{1}{2}(\frac{d}{2}+\beta+j)}} \\ &= (-1)^j \frac{\Gamma(\beta+1)}{\pi^{\beta-j+1}} \frac{a^{\frac{d}{2}+\beta+j-\frac{1}{2}}}{s^{\frac{1}{2}(\frac{d}{2}+\beta+j+\frac{1}{2})}} \cos\left(2\pi a\sqrt{s} - \frac{d+2\beta+2j+1}{4}\pi\right) \\ &\qquad + O(s^{-\frac{1}{2}(\frac{d}{2}+\beta+j+\frac{3}{2})}) \quad \text{as } s \to \infty, \end{aligned} \tag{4.2}$$

which shows

$$A_{\beta,a}^{(j)}(\lambda^2) = O(\lambda^{-(\frac{d}{2}+\beta+j+\frac{1}{2})}) \quad \text{as} \quad \lambda \to \infty. \tag{4.3}$$

In the above, for the asymptotic behavior (4.1) of Bessel functions, see [11, Lemma 3.11 on page 158] for example.

For the terms $\Delta_j(s : x)$, we use known results related to the lattice point problem.

Lemma 4.1 ([6, Lemma 5.1])**.** *Let $d \geq 1$. Then, as $s \to \infty$,*

$$\Delta_\alpha(s : x) = \begin{cases} O(s^{\frac{d}{2}-\frac{d}{d+1}}), & \text{if } \alpha = 0, \\ O(s^{\frac{d}{2}-\frac{d}{d+1}+\frac{\alpha}{d+1}+\varepsilon}) \text{ for every } \varepsilon > 0, & \text{if } 0 < \alpha \leq \frac{d-1}{2}, \\ O(s^{\frac{d-1}{4}+\frac{\alpha}{2}}), & \text{if } \alpha > \frac{d-1}{2}, \end{cases} \tag{4.4}$$

uniformly with respected to $x \in \mathbb{T}^d$.

For $\alpha \geq 0$, let

$$P_\alpha(s:x) = D_\alpha(s:x) - \frac{\pi^{\frac{d}{2}} s^{\frac{d}{2}+\alpha}}{\Gamma(\frac{d}{2}+\alpha+1)}\delta(x), \quad x \in \mathbb{R}^n,\ s \geq 0, \tag{4.5}$$

where $\delta(x)$ is the indicator function of $\mathbb{Z}^d$.

Theorem 4.2 (Novák [9])**.** *Let $d \geq 3$. Then, for all $x \in \mathbb{Q}^d$, there exists a positive constant $K_d(x)$ such that*

$$\int_0^s |P_0(t:x)|^2\, dt = \begin{cases} K_d(x)s^2 \log s + O(s^2 \log^{1/2} s), & \text{if } d=3, \\ K_d(x)s^3 + O(s^{5/2}\log s), & \text{if } d=4, \\ K_d(x)s^4 + O(s^3 \log^2 s), & \text{if } d=5, \\ K_d(x)s^{d-1} + O(s^{d-2}), & \text{if } d \geq 6. \end{cases} \tag{4.6}$$

Remark 4.1. In Theorem 4.2 the positive constant $K_d(x)$ is given explicitly for each $x \in \mathbb{Q}^d$, see [5].

Remark 4.2. Theorem 4.2 valids for $\Delta_\alpha(s:x)$ instead of $P_\alpha(s:x)$, if $x \in \mathbb{T}^d \cap \mathbb{Q}^d$. Actually,

$$\Delta_\alpha(s:x) - P_\alpha(s:x) = \begin{cases} 0, & \text{if } x = 0, \\ -\mathcal{D}_\alpha(s:x) = O(s^{\frac{d-1}{4}+\frac{\alpha}{2}}), & \text{if } x \in \mathbb{T}^d \setminus \{0\}, \end{cases}$$

see [6, Remark 5.2].

Lemma 4.3. *Let $d=4$. Then, for all $x \in \mathbb{Q}^d$ and all $\mu > 0$, there exists a positive constant $K(x)$ and a sequence $\{\lambda_k\}_k$ such that $\lambda_k \to \infty$ as $k \to \infty$ and*

$$|\Delta_0({\lambda_k}^2, x)| \geq K(x){\lambda_k}^{2-\mu}. \tag{4.7}$$

Proof. By Theorem 4.2 and Remark 4.2 we have

$$\int_0^s |\Delta_0(t,x)|^2\, dt = K_4(x)s^3 + O(s^{5/2}\log s) \quad \text{as} \quad s \to \infty. \tag{4.8}$$

We may assume that $0 < \mu < 1$. If there exists a positive constant t_0 such that, for all $t > t_0$,

$$|\Delta_0(t,x)| \leq K_4(x)t^{1-\mu/2}, \quad \text{i.e.} \quad |\Delta_0(t^2,x)| \leq K_4(x)t^{2-\mu},$$

then

$$\int_0^s |\Delta_0(t,x)|^2\, dt \leq {K_4}^2(x)s^{3-\mu} \quad \text{for large } s,$$

which contradicts (4.8). □

Proof of Theorem 3.1. Let $d=4$ and $0 < a < 1/2$. Then $d_\sharp = 2$ and

$$\mathcal{K}_{\beta,a}(\lambda^2:x) = \sum_{j=0}^{2} (-1)^j \Delta_j(\lambda^2:x) A^{(j)}_{\beta,a}(\lambda^2).$$

By Lemma 4.1 and (4.3) we have

$$\begin{cases} \Delta_0(\lambda^2,x) = O(\lambda^{12/5}), \\ \Delta_1(\lambda^2,x) = O(\lambda^{14/5+\epsilon}), \\ \Delta_2(\lambda^2,x) = O(\lambda^{7/2}), \end{cases} \qquad \begin{cases} A^{(0)}_{\beta,a}(\lambda^2) = O(\lambda^{-\beta-5/2}), \\ A^{(1)}_{\beta,a}(\lambda^2) = O(\lambda^{-\beta-7/2}), \\ A^{(2)}_{\beta,a}(\lambda^2) = O(\lambda^{-\beta-9/2}), \end{cases} \quad \text{as} \quad \lambda \to \infty,$$

which imply

$$\text{(4.9)}\qquad \begin{cases} \Delta_0(\lambda^2 : x)A^{(0)}_{\beta,a}(\lambda^2) = O(\lambda^{-\beta-1/10}), \\ \Delta_1(\lambda^2 : x)A^{(1)}_{\beta,a}(\lambda^2) = O(\lambda^{-\beta-7/10+\varepsilon}), \\ \Delta_2(\lambda^2 : x)A^{(2)}_{\beta,a}(\lambda^2) = O(\lambda^{-\beta-1}), \end{cases} \qquad \text{as} \quad \lambda \to \infty.$$

It follows that, if $\beta > -1/10$, then $\mathcal{K}_{\beta,a}(\lambda^2 : x) \to 0$ as $\lambda \to \infty$, that is, $S_\lambda(u_{\beta,a})(x) \to u_{\beta,a}(x)$ for all $x \in E_a$, which is a known result as mentioned after Theorem 3.1.

We shall consider the main term $\Delta_0(\lambda^2 : x)A^{(0)}_{\beta,a}(\lambda^2)$ more precisely. From (4.2) it follows that

$$\text{(4.10)}\qquad A^{(0)}_{\beta,a}(\lambda^2) = C_0\lambda^{-\beta-5/2}\cos\left(2\pi a\lambda - \frac{2\beta+5}{4}\pi\right) + O(\lambda^{-\beta-7/2}),$$

where $C_0 = \Gamma(\beta+1)a^{3/2+\beta}/\pi^{\beta+1}$. Let $x \in E_a \cap \mathbb{Q}^4$. For any small $\mu > 0$, take $\{\lambda_k\}_k$ as in Lemma 4.3. If there exists $\beta_0 \in (-1, -1/2-\mu)$ such that $S_\lambda(u_{\beta_0,a})(x) \to u_{\beta_0,a}(x)$ as $\lambda \to \infty$, then $\mathcal{K}_{\beta_0,a}(\lambda^2 : x) \to 0$ as $\lambda \to \infty$, which implies

$$\text{(4.11)}\qquad \lim_{k\to\infty} \cos\left(2\pi a\lambda_k - \frac{2\beta_0+5}{4}\pi\right) = 0.$$

Actually, if

$$\limsup_{k\to\infty} \left|\cos\left(2\pi a\lambda_k - \frac{2\beta_0+5}{4}\pi\right)\right| = 2\delta > 0,$$

then by (4.7) and (4.10) we have

$$|\Delta_0({\lambda_k}^2 : x)A^{(0)}_{\beta_0,a}({\lambda_k}^2)| \geq C_0K(x){\lambda_k}^{-\beta_0-1/2-\mu}\delta,$$

for infinitely many k, which means that $\mathcal{K}_{\beta_0,a}({\lambda_k}^2 : x)$ diverges, since the other terms are smaller, see (4.9).

Now, (4.11) is equivalent to

$$\lim_{k\to\infty}\left(2\pi a\lambda_k - \frac{2\beta_0+5}{4}\pi\right) = \frac{\pi}{2} \mod. \pi.$$

In this case, for all $\beta \in (-1, -1/2-\mu) \setminus \{\beta_0\}$,

$$\lim_{k\to\infty}\left(2\pi a\lambda_k - \frac{2\beta+5}{4}\pi\right) = \frac{\pi}{2} - \frac{(\beta-\beta_0)\pi}{2} \mod. \pi,$$

which shows

$$\lim_{k\to\infty}\left|\cos\left(2\pi a\lambda_k - \frac{2\beta+5}{4}\pi\right)\right| > 0.$$

This means that $\mathcal{K}_{\beta,a}({\lambda_k}^2 : x)$ diverges as seen before. Since $\mu > 0$ is arbitrary, we have the desired conclusion. □

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