NEW SPECTRAL MAPPING THEOREM OF THE TAYLOR SPECTRUM

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Abstract. We show new spectral mapping theorem of the Taylor spectrum for doubly commuting pairs of p-hyponormal operators and log-hyponormal operators. And we give Putnam inequality for log-hyponormal tuples.

1 Introduction Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. For $T \in B(\mathcal{H})$, let $\sigma(T), \sigma_p(T)$ and $\sigma_a(T)$ denote the spectrum, the point spectrum and the approximate point spectrum of T, respectively. Let $\lambda \in \mathbb{C}$ belong to the residual spectrum $\sigma_r(T)$ of T if there exists $c > 0$ such that $\|(T-\lambda)x\| \geq c\|x\|$ for all $x \in \mathcal{H}$ and $(T-\lambda)\mathcal{H} \neq \mathcal{H}$. It is easy to see that if $\lambda \in \sigma_r(T)$, then $0 \in \sigma_p((T-\lambda)^*)$. It is well known that $\sigma(T) = \sigma_a(T) \cup \sigma_r(T)$. For an Hermitian operator $A \in B(\mathcal{H})$, we denote $A \geq 0$ if $(Ax, x) \geq 0$ for every $x \in \mathcal{H}$ and $A \geq B$ if $A - B \geq 0$. When $(Ax, x) > 0$ for every non-zero $x \in \mathcal{H}$, then we denote $T > 0$. For a given $p > 0$, $T \in B(\mathcal{H})$ is said to be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$. When $p = 1/2$, T is said to be semi-hyponormal. It means that T is semi-hyponormal if and only if $|T| \geq |T^*|$. T is said to be log-hyponormal if T is invertible and $\log|T| \geq \log|T^*|$. It is well known that if T is invertible p-hyponormal for some $p > 0$, then T is log-hyponormal. If $\mathcal{M}$ is a reducing subspace for a p-hyponormal or log-hyponormal operator T, then so is $T|_{\mathcal{M}}$, respectively.

For a commuting n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$, we explain the Taylor spectrum $\sigma(\mathbf{T})$ of $\mathbf{T}$ shortly. Let E^n be the exterior algebra on n generators, that is, E^n is the complex algebra with identity e generated by indeterminates $e_1, ..., e_n$. Let $E_k^n(\mathcal{H}) = \mathcal{H} \otimes E_k^n$. Define $d_k^n : E_k^n(\mathcal{H}) \longrightarrow E_{k-1}^n(\mathcal{H})$ by

$$d_k^n(x \otimes e_{j_1} \wedge \cdots \wedge e_{j_k}) := \sum_{i=1}^{k} (-1)^{i-1} T_{j_i} x \otimes e_{j_1} \wedge \cdots \wedge \check{e}_{j_i} \wedge \cdots \wedge e_{j_k},$$

where $\check{e}_{j_i}$ means deletion. We denote d_k^n by d_k simply. We think Koszul complex $E(\mathbf{T})$ of $\mathbf{T}$ as follows:

$$E(\mathbf{T}) \ : \ 0 \longrightarrow E_n^n(\mathcal{H}) \xrightarrow{d_n} E_{n-1}^n(\mathcal{H}) \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} E_1^n(\mathcal{H}) \xrightarrow{d_1} E_0^n(\mathcal{H}) \longrightarrow 0.$$

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It is easy to see that $E^n_k(\mathcal{H}) \cong \overbrace{\mathcal{H}\oplus\cdots\oplus\mathcal{H}}^{\frac{n!}{(n-k)!\,k!}}$ $(k=1,...,n)$.

Definition 1.1. *A commuting n-tuple* $\mathbf{T}=(T_1,...,T_n)\in B(\mathcal{H})^n$ *is said to be singular if and only if the Koszul complex* $E(\mathbf{T})$ *of* $\mathbf{T}$ *is not exact.*

Definition 1.2. *For a commuting n-tuple* $\mathbf{T}=(T_1,...,T_n)\in B(\mathcal{H})^n$*, the Taylor spectrum* $\sigma_T(\mathbf{T})$ *of* $\mathbf{T}$ *is the set of all* $z=(z_1,...,z_n)\in\mathbb{C}^n$ *such that* $\mathbf{T}-z=(T_1-z_1,...,T_n-z_n)$ *is singular.*

About the definition of the Taylor spectrum, see details J. L. Taylor [11] and [12].

For a commuting pair $\mathbf{T}=(T_1,T_2)\in B(\mathcal{H})^2$, it is well known that, for polynomials $f_1,...,f_n$ of 2 variables, if $f(z_1,z_2)=(f_1(z_1,z_2),...,f_n(z_1,z_2))$, then it holds

$$\sigma_T(f(T_1,T_2))=f(\sigma_T(T_1,T_2)),$$

where $\sigma_T(T_1,T_2)$ is the Taylor spectrum of $\mathbf{T}=(T_1,T_2)$. See Theorem 4.7 in [12].

In this paper, we study other spectral mapping theorem, that is, let $T_j=U_j|T_j|$ $(j=1,2)$ be the polar decomposition of T_j and $f(t)$ be a continuous function on the non-negative real line. Let $S_j=U_jf(|T_j|)$ $(j=1,2)$ and $\mathbf{S}=(S_1,S_2)$. Then under some assumption does it hold

$$\sigma_T(\mathbf{S})=\{(e^{i\theta_1}f(r_1),e^{i\theta_2}f(r_2))\;:\;(r_1e^{i\theta_1},r_2e^{i\theta_2})\in\sigma_T(\mathbf{T})\,\}\ ?$$

For a single operator, it holds for some classes of operators. For example, if $T=U|T|$ is a p-hyponormal operator or a log-hyponormal operator with $\log|T|>0$ and $f(t)=t^{2p}$ or $f(t)=\log t$, then

$$\sigma(Uf(|T|))=\{e^{i\theta}f(r)\;:\;re^{i\theta}\in\sigma(T)\,\}, \tag{1}$$

respectively by [7, 10].

Let $\mathbf{T}=(T_1,T_2)$ be a commuting pair of operators on $\mathcal{H}$, $\mathbf{z}=(z_1,z_2)\in\mathbb{C}^2$ and let

$$\alpha(\mathbf{T}-\mathbf{z}):=\begin{pmatrix} T_1-z_1 & T_2-z_2\\ -(T_2-z_2)^* & (T_1-z_1)^*\end{pmatrix}\quad\text{on}\quad\mathcal{H}\oplus\mathcal{H}.$$

Then Vasilescu proved the following result.

Proposition 1.3. (Theorem 1.1, Vasilescu [13]) *Let* $\mathbf{T} = (T_1, T_2) \in B(\mathcal{H})^2$ *be a commuting pair. Then*

$$\mathbf{z} = (z_1, z_2) \in \sigma_T(\mathbf{T}) \textit{ if and only if } \alpha(\mathbf{T} - \mathbf{z}) \textit{ is not invertible.}$$

Therefore, we have

$$\mathbf{z} = (z_1, z_2) \in \sigma_T(\mathbf{T}) \text{ if and only if } \quad 0 \in \sigma(\alpha(\mathbf{T} - \mathbf{z})).$$

For an n-tuple $\mathbf{T} = (T_1, ..., T_n)$, the joint point spectrum $\sigma_{jp}(\mathbf{T})$ is the set of all numbers $\mathbf{z} = (z_1, ..., z_n) \in \mathbb{C}^n$ such that there exists a non-zero vector $x \in \mathcal{H}$ which satisfies $T_j x = z_j x$ $(\forall j = 1, ..., n)$ and the joint approximate point spectrum $\sigma_{ja}(\mathbf{T})$ is the set of all numbers $\mathbf{z} = (z_1, ..., z_n) \in \mathbb{C}^n$ such that there exists a sequence $\{x_k\}$ of unit vectors of $\mathcal{H}$ which satisfies

$$(T_j - z_j)x_k \ \rightarrow \ 0 \text{ as } k \rightarrow \infty \ (\forall j = 1, ..., n).$$

Following proposition is due to Berberian [1] for a single operator case. It is easy to see a proof for n-tuples. See Berberian [1] and Chō [2].

Proposition 1.4. *Let* $B(\mathcal{H})$ *be the set of all bounded linear operators on* $\mathcal{H}$. *Then there exist an extension space* $\mathcal{K}$ *of* $\mathcal{H}$ *and a faithful* $*$*-representation of* $B(\mathcal{H})$ *into* $B(\mathcal{K})$: $T \ \rightarrow \ T^\circ$ *such that*

$$\sigma_{ja}(T_1, ..., T_n) = \sigma_{jp}(T^\circ, ..., T_n^\circ) = \sigma_p(T_1^\circ, ..., T_n^\circ).$$

We have Putnam inequalities of hyponormal tuples, semi-hyponormal tuples, and p-hyponormal tuples. See [2], [3], [4], [5], [8]. Finally we give Putnam inequality of log-hyponormal tuple.

2 New spectral mapping theorem

Following results are well known.

Proposition 2.1. *Let* $T = U|T|$ *be the polar decomposition of* T *and* f *be a continuous function on the non-negative real line which contains* $\sigma(|T|)$. *For a sequence* $\{x_n\}$ *of unit vectors, if* $(T - re^{i\theta})x_n \rightarrow 0$ *and* $(T - re^{i\theta})^* x_n \rightarrow 0$, *then* $(U - e^{i\theta})x_n \rightarrow 0, (|T| - r)x_n \rightarrow 0$ *and* $(f(|T|) - f(r))x_n \rightarrow 0$.

See Lemma 1.2.4 in [15].

Proposition 2.2. *Let* T *be semi-hyponormal. Then* $\sigma(T) = \{\overline{z} : z \in \sigma_a(T^*)\}$.

See Theorem 1.2.6 in [15].

Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition of T with unitary U and f be a continuous function on the non-negative real line which contains $\sigma(|T|)$. Let $\mathcal{K}$ be Berberian extension of $\mathcal{H}$ and $\circ : B(\mathcal{H}) \ni T \to T^\circ \in B(\mathcal{K})$ be a faithful $*$-representation. We set the following conditions (2) and (3):

(2) For a sequence $\{x_n\}$ of unit vectors, if $(T-z)x_n \to 0$, then $(T-z)^* x_n \to 0$.

(3) If a closed subspace $\mathcal{M}$ of $\mathcal{K}$ reduces T° and $re^{i\theta} \in \sigma(T^\circ|_{\mathcal{M}})$,

then $\mathcal{M}$ reduces $U^\circ, |T|^\circ$ and $e^{-i\theta} f(r) \in \sigma_p\left((U^\circ|_{\mathcal{M}} f(|T|)^\circ|_{\mathcal{M}})^*\right)$.

Remark. If T is p-hyponormal and $f(t) = t^{2p}$, then (2) holds by Theorem 4 of [5]. If T is log-hyponormal and $f(t) = \log t$, then (2) holds by Lemma 3 of [10]. About (3), since the mapping $\circ$ of Berberian method is a faithful $*$-representation, so is T° if T is p-hyponormal or log-hyponormal, respectively. Let $\mathcal{M}$ be a reducing subspace for T. It is clear that if T is p-hyponormal or log-hyponormal, then so is $T|_{\mathcal{M}}$, respectively.
(i) Let T be p-hyponormal and $T = U|T|$ be the polar decomposition of T and $f(t) = t^{2p}$. Then $S = U|T|^{2p}$ is semi-hyponormal and $\sigma(U|T|^{2p}) = \{r^{2p}e^{i\theta} : re^{i\theta} \in \sigma(T)\}$ by Theorem 3 of [7]. Hence (3) holds by Proposition 2.2.
(ii) Let $T = U|T|$ be log-hyponormal and $f(t) = \log t$. Then $S = U \log |T|$ is semi-hyponormal and $\sigma(U \log |T|) = \{e^{i\theta} \log r : re^{i\theta} \in \sigma(T)\}$ by Lemma 8 of [10]. Hence (3) holds by Proposition 2.2.
Therefore, if T is p-hyponormal or log-hyponormal and $f(t) = t^{2p}$ or $f(t) = \log t$, respectively, then T satisfies (2) and (3) for this f.

Theorem 2.3. *Let $\mathbf{T} = (T_1, T_2)$ be a doubly commuting pair of operators and $T_j = U_j|T_j|$ $(j = 1, 2)$ be the polar decomposition. Let $f(t)$ be a continuous function on a open interval in the non-negative real line which contains $\sigma(|T_1|) \cup \sigma(|T_2|)$. Let $S_j = U_j f(|T_j|)$ $(j = 1, 2)$ and $\mathbf{S} = (S_1, S_2)$. Let T_1, T_2 and f satisfy (2) and (3). If $(r_1e^{i\theta_1}, r_2e^{i\theta_2}) \in \sigma_T(\mathbf{T})$, then $(e^{i\theta_1} f(r_1), e^{i\theta_2} f(r_2)) \in \sigma_T(\mathbf{S})$.*

Proof. Let $\mathbf{z} = (z_1, z_2) = (r_1e^{\theta_1}, r_2e^{i\theta_2}) \in \sigma_T(\mathbf{T})$. Then $0 \in \sigma(\alpha(\mathbf{T} - \mathbf{z}))$ by Proposition 1.1.

Case 1. If $0 \in \sigma_a(\alpha(\mathbf{T} - \mathbf{z}))$, then there exists a sequence $\{x_n \oplus y_n\}$ of unit vectors of $\mathcal{H} \oplus \mathcal{H}$ such that

$$\alpha(\mathbf{T} - \mathbf{z})(x_n \oplus y_n) = \begin{pmatrix} (T_1 - z_1)x_n + (T_2 - z_2)y_n \\ -(T_2 - z_2)^* x_n + (T_1 - z_1)^* y_n \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since T_1, T_2 are doubly commuting, we have

$$(T_1 - z_1)^*(T_1 - z_1)x_n + (T_2 - z_2)(T_2 - z_2)^* x_n \to 0$$

and

$$(T_1 - z_1)(T_1 - z_1)^* y_n + (T_2 - z_2)^*(T_2 - z_2)y_n \to 0.$$

If $x_n \not\to 0$, then $(z_1, \overline{z_2}) \in \sigma_{ja}(T_1, T_2^*)$, and if $y_n \not\to 0$, then $(\overline{z_1}, z_2) \in \sigma_{ja}(T_1^*, T_2)$.

Case 2. If $0 \in \sigma_r(\alpha_2(\mathbf{T}-\mathbf{z})) \subset \sigma_p(\alpha(\mathbf{T}-\mathbf{z})^*)$, then there exists a non-zero vector $x \oplus y$ such that

$$\alpha(\mathbf{T}-\mathbf{z})^*(x \oplus y) = \begin{pmatrix} (T_1-z_1)^*x-(T_2-z_2)y \\ (T_2-z_2)^*x+(T_1-z_1)y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, we have

$$(T_1-z_1)(T_1-z_1)^*x+(T_2-z_2)^*(T_2-z_2)x=0$$

and

$$(T_1-z_1)^*(T_1-z_1)y+(T_2-z_2)(T_2-z_2)^*y=0.$$

If $x \neq 0$, then we have $(T_1-z_1)^*x=(T_2-z_2)x=0$ and if $y \neq 0$, then $(T_1-z_1)y = (T_2-z_2)^*y=0$.

Therefore, if necessarily by changing order, we may assume that there exists a sequence $\{x_n\}$ of unit vectors such that (the proof of case $(T_1-z_1)^*x_n \to 0$ and $(T_2-z_2)x_n \to 0$ is similar.)

$$(T_1-z_1)x_n \to 0 \text{ and } (T_2-z_2)^*x_n \to 0.$$

Hence

$$(U_1-e^{i\theta_1})x_n \to 0, \ (|T_1|-r_1)x_n \to 0, (S_1-e^{i\theta_1}f(r_1))x_n \to 0$$

by the assumption. Let $\mathcal{K}$ be the Berberian extension of $\mathcal{H}$. Then there exists $0 \neq x^\circ \in \mathcal{K}$ such that

$$(S_1^\circ - e^{i\theta_1}f(r_1))x^\circ = (T_2^\circ - z_2)^*x^\circ = 0.$$

Let $\mathcal{M} = \ker(S_1^\circ - e^{i\theta_1}f(r_1))$. Since (S_1°, T_2°) are doubly commuting pair, $\mathcal{M}$ is a reducing subspace for T_2°. Since $x^\circ \in \mathcal{M}$, we have $z_2 = r_2e^{i\theta_2} \in \sigma(T_2^\circ|_{\mathcal{M}})$. Let $S_2 = U_2f(|T_2|)$. Then by the assumption (2), we have $T_2^\circ|_{\mathcal{M}} = U_2^\circ|_{\mathcal{M}}|T_2|^\circ|_{\mathcal{M}}$ and $e^{-i\theta_2}f(r_2) \in \sigma_p(S_2^{\circ *}|_{\mathcal{M}})$. Hence there exists non-zero $y^\circ \in \mathcal{M}$ such that $(S_2^\circ - e^{i\theta_2}f(r_2))^*y^\circ = 0$. Since $y^\circ \in \mathcal{M}$, we have $(S_1^\circ - e^{i\theta_1}f(r_1))y^\circ = 0$. Therefor there exists a sequence $\{y_n\}$ of unit vectors such that

$$(S_1-e^{i\theta_1}f(r_1))y_n \to 0 \text{ and } (S_2-e^{i\theta_2}f(r_2))^*y_n \to 0.$$

Then

$$\alpha(\mathbf{S}-(e^{i\theta_1}f(r_1), e^{i\theta_2}f(r_2))) \begin{pmatrix} y_n \\ 0 \end{pmatrix} = \begin{pmatrix} (S_1-e^{i\theta_1}f(r_1))y_n \\ -(S_2-e^{i\theta_2}f(r_2))^*y_n \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence $0 \in \sigma\left(\alpha(\mathbf{S}-(e^{i\theta_1}f(r_1), e^{i\theta_2}f(r_2)))\right)$ and $\left(e^{i\theta_1}f(r_1), e^{i\theta_2}f(r_2)\right) \in \sigma_T(\mathbf{S})$. This completes the proof.

□

Corollary 2.4. *Let $\mathbf{T}=(T_1,T_2)$ be a doubly commuting pair of p-hyponormal operators $(0<p<1)$. Let U_j be unitary for the polar decomposition of $T_j=U_j|T_j|$ $(j=1,2)$ and $\mathbf{S}=(U_1|T_1|^{2p}, U_2|T_2|^{2p})$. Then*

$$\sigma_T(\mathbf{S}) = \{(r_1^{2p}e^{i\theta_1}, r_2^{2p}e^{i\theta_2}) : (r_1e^{i\theta_1}, r_2e^{i\theta_2}) \in \sigma_T(\mathbf{T})\}.$$

Proof. Let $f(t) = t^{2p}$ on the non-negative real line. Since $\mathbf{T}$ is a doubly commuting pair of p-hyponormal operators and $f(t) = t^{2p}$, T_1, T_2 and f satisfy (2) and (3). Hence, by Theorem 2.3 we have

$$\sigma_T(\mathbf{S}) \supset \{(r_1^{2p}e^{i\theta_1}, r_2^{2p}e^{i\theta_2}) : (r_1e^{i\theta_1}, r_2e^{i\theta_2}) \in \sigma_T(\mathbf{T})\}.$$

Conversely, put $g(t) = t^{\frac{1}{2p}}$ on the non-negative real line. Since $\mathbf{S}$ is a doubly commuting pair of semi-hyponormal operators, S_1, S_2 and g satisfy (2) and (3). Then we have the converse inclusion by Theorem 2.3 and similar argument. □

Corollary 2.5. *Let* $\mathbf{T} = (T_1, T_2)$ *be a doubly commuting pair of* log-*hyponormal operators with* $\log|T_j| > 0$. *Let* U_j *be unitary for the polar decomposition of* $T_j = U_j|T_j|$ $(j = 1, 2)$ *and* $\mathbf{S} = (U_1 \log|T_1|, U_2 \log|T_2|)$. *Then*

$$\sigma_T(\mathbf{S}) = \{e^{i\theta_1}\log r_1, e^{i\theta_2}\log r_2) : (r_1e^{i\theta_1}, r_2e^{i\theta_2}) \in \sigma_T(\mathbf{T})\}.$$

Proof. Let $f(t) = \log t$ on $(0, \infty)$. Since $\mathbf{T}$ is a doubly commuting pair of log-hyponormal operators and $f(t) = \log t$, T_1, T_2 and f satisfy (2) and (3). So by Theorem 2.3 we have

$$\sigma_T(\mathbf{S}) \supset \{e^{i\theta_1}\log r_1, e^{i\theta_2}\log r_2) : (r_1e^{i\theta_1}, r_2e^{i\theta_2}) \in \sigma_T(\mathbf{T})\}.$$

Conversely, let $g(t) = e^t$ on the non-negative real line. Since $\mathbf{S}$ is a doubly commuting pair of semi-hyponormal operators, S_1, S_2 and g satisfy (2) and (3). Hence, we have the converse inclusion by similar argument. □

3 Putnam inequality

In this section we study for Putnam inequality of log-hyponormal tuples. Let $\mathbf{U} = (U_1, ..., U_n)$ be an n-tuple of unitary operators. For $T \in B(\mathcal{H})$, an operator $\mathbf{Q}_j$ $(j = 1, ..., n)$ on $B(\mathcal{H})$ is defined by

$$\mathbf{Q}_jT := T - U_jTU_j^*.$$

Definition 3.1. *Let* $\mathbf{U} = (U_1, \cdots, U_n)$ *be a commuting* n*-tuple of unitary operators and* $A \geq 0$. *An* $(n+1)$*-tuple* $(\mathbf{U}, A)$ *is said to be a semi-hyponormal tuple if*

$$\mathbf{Q}_{j_1} \cdots \mathbf{Q}_{j_m} A \geq 0 \quad \text{for all} \quad 1 \leq j_1 < \cdots < j_m \leq n.$$

Definition 3.2. *Let* $\mathbf{U} = (U_1, \cdots, U_n)$ *be a commuting* n*-tuple of unitary operators and* $A > 0$ *with* $\log A \geq 0$. *An* $(n+1)$*-tuple* $(\mathbf{U}, A)$ *is said to be a* log-*hyponormal tuple if* $(\mathbf{U}, \log A)$ *is a semi-hyponormal tuple.*

Let $\mathbf{U} = (U_1, \cdots, U_n)$ be an n-tuple of unitary operators and $T \in B(\mathcal{H})$. If

$$\mathcal{S}_j^{\pm}(T) := \text{s}-\lim_{n\to\pm\infty}(U_j^{-n}TU_j^n)$$

exist, then the operators $\mathcal{S}_j^{\pm}(T)$ are called the polar symbols of T. If $U_j|A|$ is semi-hyponormal, then the polar symbols $\mathcal{S}_j^{\pm}(T)$ exist.

For $k \in [0,1]$ and $A \geq 0$, we denote

$$(k\mathcal{S}_j^{+} + (1-k)\mathcal{S}_j^{-})A := k\mathcal{S}_j^{+}(A) + (1-k)\mathcal{S}_j^{-}(A).$$

Let $\mathbf{k} = (k_1, ..., k_n) \in [0,1]^n$ and $(\mathbf{U}, A)$ be a semi-hyponormal tuple. Then the generalized polar symbols $A_{\mathbf{k}}$ of A are defined by

$$A_{\mathbf{k}} := \Pi_{j=1}^{n}\big(k_j\mathcal{S}_j^{+} + (1-k_j)\mathcal{S}_j^{-}\big)A.$$

Since $A \geq 0$, then $A_{\mathbf{k}} \geq 0$. Hence it is clear that $(\mathbf{U}, A_{\mathbf{k}})$ is a commuting $(n+1)$-tuple of normal operators for every $\mathbf{k} \in [0,1]^n$.

Definition 3.3.
(1) *Let $(\mathbf{U}, A)$ be a semi-hyponormal tuple. The the Xia spectrum $\sigma_X(\mathbf{U}, A)$ is defined by*

$$\sigma_X(\mathbf{U}, A) := \bigcup_{\mathbf{k}\in[0,1]^n} \sigma_{ja}(\mathbf{U}, A_{\mathbf{k}}).$$

(2) *Let $(\mathbf{U}, A)$ be a* log*-hyponormal tuple. Since $(\mathbf{U}, \log A)$ is a semi-hyponormal tuple, Xia spectrum $\sigma_X(\mathbf{U}, A)$ of $(\mathbf{U}, A)$ is defined by*

$$\sigma_X(\mathbf{U}, A) := \{(z_1, ..., z_n, e^r) \, : \, (z_1, ..., z_n, r) \in \sigma_X(\mathbf{U}, \log A)\}.$$

Proposition 3.4. (Theorem 5, Xia [14]) *Let $(\mathbf{U}, A)$ be a semi-hyponormal tuple. Then*

$$||\mathbf{Q}_1 \cdots \mathbf{Q}_n A|| \leq \frac{1}{(2\pi)^n} \int \cdots \int_{\sigma_X(\mathbf{U},A)} d\theta_1 \cdots d\theta_n \, dr.$$

Let $(\mathbf{U}, A)$ be a semi-hyponormal tuple and $\mathbf{k} = (k_1, \cdots, k_n) \in [0,1]^n$. We define

$$A_{\log,\mathbf{k}} := \exp\{\Pi_{j=1}^{n}\big(k_j\mathcal{S}_j^{+}(\log A) + (1-k_j)\mathcal{S}_j^{-}(\log A)\big)\}.$$

Then $(\mathbf{U}, A_{\log,\mathbf{k}})$ is a commuting tuple and $(\mathbf{U}, \log A)$ is a semi-hyponormal tuple. Let

$$(\log A)_{\mathbf{k}} = \Pi_{j=1}^{n}\big(k_j\mathcal{S}_j^{+}(\log A) + (1-k_j)\mathcal{S}_j^{-}(\log A)\big).$$

Then $A_{\log,\mathbf{k}} = \exp\{(\log A)_{\mathbf{k}}\}$ and we have the following lemma.

Lemma 3.5. *Let $(\mathbf{U}, A)$ be a* log*-hyponormal tuple and $\mathbf{k} \in [0,1]^n$. Then*

$$(z_1, \cdots, z_n, \log r) \in \sigma_{ja}(\mathbf{U}, (\log A)_{\mathbf{k}}) \text{ if and only if } (z_1, \cdots, z_n, r) \in \sigma_{ja}(\mathbf{U}, A_{\log,\mathbf{k}}).$$

Proof. It is easy from $A_{\log,\mathbf{k}} = \exp\{(\log A)_{\mathbf{k}}\}$.

□

Theorem 3.6. *Let* $(\mathbf{U}, A)$ *be a* log*-hyponormal tuple. Then*

$$\sigma_X(\mathbf{U}, A) = \bigcup_{\mathbf{k}\in[0,1]^n} \sigma_{ja}(\mathbf{U}, A_{\log,\mathbf{k}}).$$

Proof. Since $\sigma_X(\mathbf{U}, A) = \{(z_1, \cdots, z_n, e^r) : (z_1, \cdots, z_n, r) \in \sigma_X(\mathbf{U}, \log A)\}$ by the definition 3.3 (2), we have

$$\sigma_X(\mathbf{U}, \log A) = \bigcup_{\mathbf{k}\in[0,1]^n} \sigma_{ja}(\mathbf{U}, (\log A)_{\mathbf{k}}).$$

Hence we have

$$\sigma_X(\mathbf{U}, A) = \bigcup_{\mathbf{k}\in[0,1]^n} \sigma_{ja}(\mathbf{U}, A_{\log,\mathbf{k}})$$

by Lemma 3.5.

□

Theorem 3.7. *Let* $(\mathbf{U}, A)$ *be a* log*-hyponormal tuple. Then*

$$\|\mathbf{Q}_1 \cdots \mathbf{Q}_n \log A\| \leq \frac{1}{(2\pi)^n} \int \cdots \int_{\sigma(\mathbf{U},A)} \frac{1}{r}\, d\theta_1 \cdots d\theta_n dr.$$

Proof. Since $(\mathbf{U}, \log A)$ is a semi-hyponormal tuple, it holds

$$\|\mathbf{Q}_1 \cdots \mathbf{Q}_n \log A\| \leq \frac{1}{(2\pi)^n} \int \cdots \int_{\sigma(\mathbf{U},\log A)} d\theta_1 \cdots d\theta_n dr$$

by proposition 3.4. Since

$$\sigma_X(\mathbf{U}, \log A) = \{(z_1, \cdots, z_n, \log s) : (z_1, \cdots, z_n, s) \in \sigma_X(\mathbf{U}, A)\}$$

by definition, we have

$$(z_1, \cdots, z_n, r) \in \sigma_X(\mathbf{U}, \log A) \Longleftrightarrow (z_1, \cdots, z_n, e^r) \in \sigma_X(\mathbf{U}, A).$$

Let $s = e^r$. Then $ds = e^r dr$ and $dr = \frac{1}{s} ds$. Hence

$$\frac{1}{(2\pi)^n} \int \cdots \int_{\sigma(\mathbf{U},\log A)} d\theta_1 \cdots d\theta_n dr = \frac{1}{(2\pi)^n} \int \cdots \int_{\sigma(\mathbf{U},A)} \frac{1}{s}\, d\theta_1 \cdots d\theta_n ds.$$

□

Let $\mathbf{T} = (T_1, ..., T_n)$ be a doubly commuting n-tuple of log-hyponormal operators and $T_j = U_j|T_j|$ be the polar decomposition of T_j with $\log|T_j| \geq 0$ $(j = 1, ..., n)$. Let $\mathbf{U} = (U_1, ..., U_n)$ and $A = \exp\big(\log|T_1| \cdots \log|T_n|\big)$. Then $\mathbf{U}$ is a commuting n-tuple of unitary operators and $A \geq 0$. By the definition of the operator $\mathbf{Q}_j$, it is easy to see that

$$\mathbf{Q}_j \log A = \big(\Pi_{k\neq j} \log|T_k|\big)(\log|T_j| - \log|T_j^*|)$$

for all $j = 1, ..., n$. Therefore $(\mathbf{U}, A)$ is a log-hyponormal tuple and

$$\mathbf{Q}_1 \cdots \mathbf{Q}_n \log A = \Pi_{j=1}^n \big(\log|T_j| - \log|T_j^*|\big) \geq 0.$$

Hence we have the following theorem.

Theorem 3.8. *Let $\mathbf{T} = (T_1, ..., T_n)$ be a doubly commuting n-tuple of* log*-hyponormal operators with $\log|T_j| \geq 0$. Let $T_j = U_j|T_j|$ $(j = 1, ..., n)$ be the polar decomposition and $A = \exp\big(\log|T_1| \cdots \log|T_n|\big)$. Then*

$$||\Pi_{j=1}^n(\log|T_j| - \log|T_j^*|)|| \leq \frac{1}{(2\pi)^n} \int \cdots \int_{\sigma(\mathbf{U},A)} \frac{1}{r}\, d\theta_1 \cdots d\theta_n dr.$$

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