LIFE PERIOD ESTIMATION BY CELLULAR AUTOMATION IN CYCLE GRAPHS

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Abstract

Cellular automata have a configuration consisting of cells which may become a state "live (infected)" and "dead (non-infected)", and a configuration evolves according to some rules with respect to time. Cellular automata also have been used for simulations of spreading some disease. We often have difficulty to estimate the evolution of configurations. In this manuscript, we focus on a cycle graph with $2^k (k > 1)$ cells and 1D cellular automaton rule 90. We first show that any initial configuration becomes a null configuration which consists of all "non-infected" cells with a time period of a finite number. Furthermore, some theorems give an estimation for the time period of an initial configuration until the null configuration by the position of the cells without any simulation or numerical computations.

1 Introduction

The system of cellular automata has been originally proposed by Stanislaw Ulam and John von Neumann for studying the growth of crystals [3] and building self-replicating robots [8]. Cellular automata have been used for representing some epidemic models [11, 7, 2, 9] with 2D models as well. Recently, we have faced some epidemic diseases such as influenza, MERS, SARS, and COVID-19, and it is important to estimate how diseases are spread with respect to time in real applications. Moreover, we often want to obtain an upper bound of the time period until disappearing infected patients or epidemics.

Before considering 2D models, we focus on the cellular automata in cycle graphs with 2^k cells ($k \in \mathbb{N}$, $\mathbb{N}$ is the set of natural numbers). We have a cell whose stage can be either "live (1)" or "dead (0)". We could consider the two stages which are "infected" and "cured (non-infected)" in some graphs whose edges represent the connection between people. A disease is often spread with respected to time by the interaction with people, i.e. an infected person interacts

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Table 1: Cellular automaton rule 90

state	111	110	101	100	011	010	001	000
future state	0	1	0	1	1	0	1	0

with a non-infected person. Therefore, we need to define a rule how a disease is spread in human society or some graphs.

In 1D cellular automata, there are 256 rules in total, and many rules have been studied in [5, 10, 13]. This manuscript focuses on rule 90 which has known as a generator of Sierpinski triangle [6, 12] (see Table 1). In the top of Table 1, we have three consecutive cells (state) at time t. The center of the three becomes 0 or 1 at future time $t+1$ (future state) according to the neighbor cells. We can interpret the state 101 at time t and the center cell 0 stays 0 at the future state as follows; when two infected people around a non-infected person, the non-infected person pays an attention of a disease carefully for not getting infected. This rule is named rule 90 because $0*2^7+1*2^6+0*2^5+1*2^4+1*2^3+0*2^2+1*2^1+0*2^0 = 90$.

We prepare an *initial configuration* which consists of cells in cycle graphs and observe how the initial configuration evolves with respect to time according to rule 90. This manuscript first shows that any initial configurations become a *null configuration* which consists of "dead" cells with a time period of t, $t \leq 2^{k-1}$. In real life applications, it is important to know the time period until all people get cured from some disease. However, we often have some difficulty of predicting the behavior of configuration in general cellular automata because the evolution of configurations often acts "randomly" or "repeatedly" [12]. That is because an initial configuration with a small number of "live" cells does not usually mean that it becomes a null space quickly. Therefore, we detect a set of initial configurations which become the null configuration with a time period of t, where $t = 2^{k-1}, t \leq 2^{k-2}$, etc. The objective of this manuscript is to estimate an upper bound of a time period of a given initial configuration until the null configuration without any simulations.

The remainder of this manuscript is organized as follows. First in Section 2, we introduce cycle graphs and show any initial configuration becomes a null configuration with a time period of some finite number. In Section 3, we detect a set of initial configurations which becomes a null configuration with a time period of t, $t = 2^{k-1}$, $t \leq 2^{k-2}$. In Section 4, we give simulations with some instances following our theorems. Finally, the manuscript is concluded in Section 5.

2 Cycle graphs

We let C_n be a cycle graph consisting n vertices (cells) and n edges. We let a vector of n consecutive cells in the cycle graph $B = [b_0, \ldots, b_{n-1}]$, where a cell at position i, $b_i \in \{0,1\}$. The cells b_i $(i = 0,\ldots,n-1)$ at the position i are located clockwise. If a cell $b_i = 0,1$ holds, then a cell i is "dead", "live", respectively. We note that b_0 and b_{n-1} are connected with an edge. If all cells b_i $(i = 0,\ldots,n-1)$ are 0, then the configuration is called a null configuration B_{null}. We also describe a configuration $B^{(t)} = [b_0^{(t)},\ldots,b_{n-1}^{(t)}]$ at time t.

We again introduce rule 90 of 1D cell automaton (see Table 1). A future state $b_i^{(t+1)}$ at time $t+1$ and position i can be defined by two cells $b_{i-1}^{(t)}, b_{i+1}^{(t)}$ at previous time t and position $i-1$, $i+1$ as follows, $b_i^{(t+1)} = b_{i-1}^{(t)} + b_{i+1}^{(t)}$ (mod 2). Therefore, the future configuration according to rule 90 can be obtained by the following matrix multiplication in modulo 2.

$$B^{(t+1)T} = Adj(C_n)B^{(t)T} \text{ (mod 2)}, \tag{1}$$

where B^T represents the transpose of a vector B and $Adj(C_n)$ is an adjacency matrix of C_n.

For instance, if an initial configuration at time 0 in a cycle graph with eight cells is $B^{(0)} = [1,1,0,0,1,0,1,1]$, then the next configurations at time 1, 2, 3 are $B^{(1)} = [0,1,1,1,0,0,1,0]$, $B^{(2)} = [1,1,0,1,1,1,0,1]$, $B^{(3)} = [0,1,0,1,0,1,0,1]$, $B^{(4)} = [0,0,0,0,0,0,0,0]$ according to rule 90. This instance never becomes a null configuration because of "repeatedly".

Although this manuscript focuses on rule 90, rule 60, 102, and 150 can be expressed with some modifications for equation (1) (see details in Appendix A). We also obtain the following equation by induction;

$$B^{(t)T} = Adj(C_n)^{t-1}B^{(0)T} \text{ (mod 2)}. \tag{2}$$

This manuscript focuses on circle graphs C_n, $n = 2^k$, a natural number $k > 1$. We show a theorem that any initial configurations $B^{(0)}$ become a null configuration B_{null} with a time period of t, $t \leq 2^{k-1}$. We prove the theorem combinatorically although the theorem has been proved polynomially [13]. For the theorem, we first prepare some lemmas. We first introduce the feature of an adjacency matrix. We note that the (i,j)th entry a_{ij} of $Adj(C_n)^m$ counts the number of walks (ways) of length m having start and end cells b_i and b_j, respectively [4].

Next, we consider Pascal's triangle and values with rows $2^k, 2^k-1$ of the triangle. At first, we consider values with 2^kth row. We show that all values $\binom{2^k}{n}$ except the two ends $(n = 0, 2^k)$ are even by using mathematical induction.

Lemma 2.1. *If $n,k \in \mathbb{N}$ and $0 < n < 2^k$ holds, then $\binom{2^k}{n}$ is even.*

Proof. By induction, we assume that $(x+1)^{2^k} = x^{2^k} + 1$ (mod 2). We obtain

$$(x+1)^{2^{k+1}} = ((x+1)^{2^k})^2 = (x^{2^k}+1)^2 = x^{2^{k+1}} + 2x^{2^k} + 1 = x^{2^{k+1}} + 1 \text{ (mod 2)}.$$

□

We consider the values with 2^k-1th row of Pascal's triangle. We show that all values $\binom{2^k-1}{n}$ are odd by using lemma 2.1.

Lemma 2.2. *If $n, k \in \mathbb{N}$ and $1 \leq k$ holds, then $\binom{2^k-1}{n}$ is odd.*

Proof. All values $\binom{2^k-1}{n}$ in row $2^k - 1$ of Pascal's triangle are odd since all values in row 2^k is even except for the two ends $\binom{2^k}{0}$, $\binom{2^k}{2^k}$ by lemma 2.1. □

Theorem 2.1. *For a natural number $k > 1$, any initial configurations $B^{(0)}$ of C_{2^k} become B_{null} with a time period t, $t \leq 2^{k-1}$.*

Proof. We first consider $Adj(C_{2^k})^{2^{k-1}}$, and (i, j)th entry of $Adj(C_{2^k})^{2^{k-1}}$ represents the number of walks (ways) to reach a cell b_i from a cell b_j with length 2^{k-1}. Without loss of generality, in cycle graphs we consider a cell b_0. For the convenience of proof, we consider the number of walks of length 2^{k-1} from b_0 to b_j, $0 \leq j \leq 2^{k-1}$ in a half circle.

The distance between b_j and b_0 is j, and we consider all possible walks to get to b_j from b_0 with length 2^{k-1}.

Let x and y, $x, y \in \mathbb{N}$, be the number of lengths to clockwise and counterclockwise, respectively. Then, we obtain the following equations;

$$x + y = 2^{k-1},\ x - y = j. \tag{3}$$

We next obtain

$$x = \frac{j + 2^{k-1}}{2},\ y = \frac{-j + 2^{k-1}}{2}. \tag{4}$$

When $j \in \mathbb{N}$ is odd, $x, y \notin \mathbb{N}$, which means that the number of walks from b_0 to b_j is 0. When $j \in \mathbb{N}$ is even, the number of walks from b_0 to b_j can be obtained as follow;

$$\binom{x+y}{x} = \binom{2^{k-1}}{x}. \tag{5}$$

In the case $j = 2^{k-1}$, $\binom{2^{k-1}}{2^{k-1}} = 1$, and there is another way from the other half circle, therefore there are two walks to $b_{2^{k-1}}$.

In the other case $0 \leq j \leq 2^{k-1} - 2$, we obtain $2^{k-2} \leq x \leq 2^{k-1} - 1$ and $\binom{2^{k-1}}{x} = 0$, (mod 2) by lemma 2.1. Finally, we obtain $Adj(C_{2^k})^{2^{k-1}} = O$ in modulo 2. We obtain $Adj(C_{2^k})^{2^{k-1}} B^{(0)T} = OB^{(0)T} = B^T_{null}$ (mod 2). □

By a similar manner of Theorem 2.1, when $B^{(t+1)}$ becomes a null configuration at time $t + 1$, the configuration $B^{(t)}$ has only two configurations. The first case is a configuration with all "live" cells. The second case is a configuration which locates "live" and "dead" cells alternately. We can confirm that with some instances in Section 4.

3 Survival period

We consider a set of initial configurations which takes a time period of exactly 2^{k-1} until a null configuration. We first obtain a set of initial configurations which a time period of t, $t \leq 2^{k-1} - 1$. The compliment of the obtained set is a set that we want to obtain.

After that, we consider a set of initial configurations which takes a time period of t, $t \leq 2^{k-2}$ until a null configuration.

Theorem 3.1. *If $\sum_{i=0}^{2^{k-1}-1} b_{2i}^{(0)}$ or $\sum_{i=0}^{2^{k-1}-1} b_{2i+1}^{(0)}$ is odd, then an initial configuration $B^{(0)} = [b_0^{(0)}, \ldots, b_{2^k-1}^{(0)}]$ becomes B_{null} with a time period of exactly 2^{k-1}.*

Proof. We again consider a half circle between a cell b_0 and a cell b_j, $0 \leq j \leq 2^{k-1}$. Let x and y, $x, y \in \mathbb{N}$ be the number of lengths to clockwise and counter-clockwise, respectively. The distance between b_j and b_0 is j, where $0 \leq j \leq 2^{k-1} - 1$. We obtain the following equations:

$$x + y = 2^{k-1} - 1, \ x - y = j. \tag{6}$$

We obtain

$$x = \frac{j + 2^{k-1} - 1}{2}, \ y = \frac{-j + 2^{k-1} - 1}{2}. \tag{7}$$

When j is even, $x, y \notin \mathbb{Z}$, which means that the number of walks from b_0 to b_j is 0. When j is odd, the number of walks from b_0 to b_j obtained as follows:

$$\binom{x+y}{x} = \binom{2^{k-1}-1}{x} \tag{8}$$

is odd by the lemma 2.2. Therefore, when we obtain the entry (i, j)th entry a_{ij} of $Adj(C_{2^k})^{2^{k-1}-1}$, $a_{ij} = 0$ if $i - j$ is even, otherwise $a_{ij} = 1$.

If $\sum_{i=0}^{2^{k-1}-1} b_{2i}^{(0)}$ and $\sum_{i=0}^{2^{k-1}-1} b_{2i+1}^{(0)}$ are even, its initial configuration $B^{(0)}$ becomes B_{null} with a time period of t, $t \leq 2^k - 1$. The compliment of the set of the initial configurations is a set which we want to obtain. □

Next, a set of initial configurations which become B_{null} with a time period of t, $t \leq 2^{k-2}$ is considered with the similar technique of Theorem 3.1.

Theorem 3.2. *If $|i - j| = 2^{k-1}$ and $b_i^{(0)} + b_j^{(0)}$ is even for any i, j, then an initial configuration $B^{(0)}$ becomes B_{null} with a time period of t, $t \leq 2^{k-2}$.*

Proof. This proof is obtained with a similar manner as the proof of Theorem 3.1. □

We believe that we can obtain the statement that an initial configuration becomes B_{null} with a time period of t, $t \leq 2^{k-3}$ with the similar manner and some modifications.

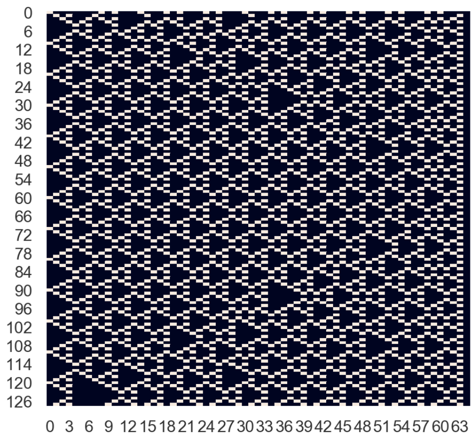
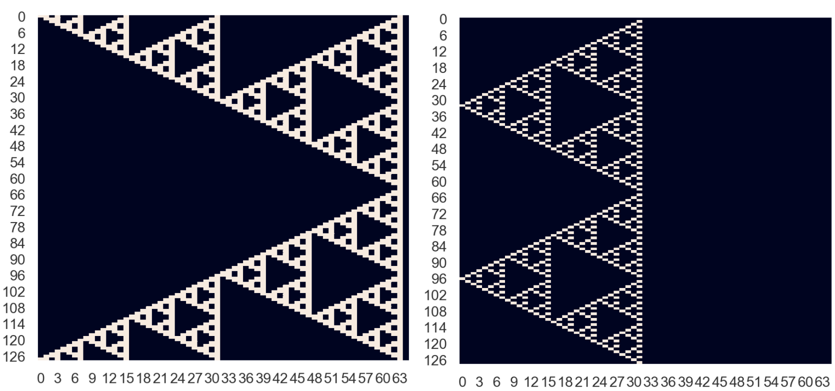


Figure 1: An initial configuration with thirteen “live” cells at {0, 10, 20, ..., 120}.



(a)

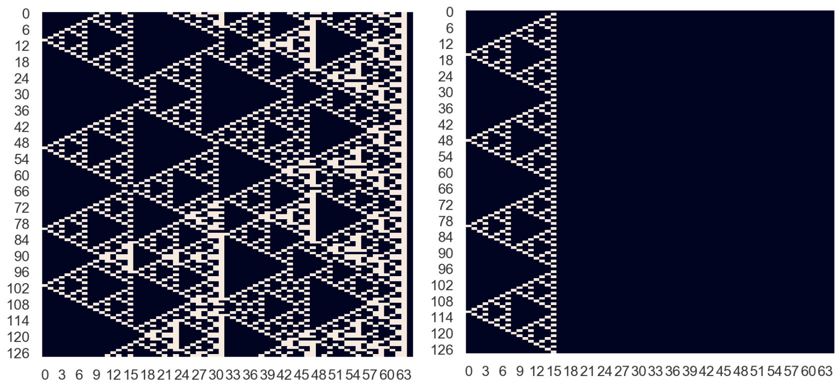
(b)

Figure 2: (a) An initial configuration with two “live” cells at {0, 127}. (b) An initial configuration with two “live” cells at {32, 96}.

4 Experiments

To confirm our theorems, we prepare a cycle model $C_{2^7} = C_{128}$ and review our theorems with some initial configurations $B^{(0)}$. In Figure 1, 2, and 3, the ith column represents $B^{(i)}$ and the jth row represents b_j. Therefore, the value at ith column and jth row represents $b_j^{(i)}$. A white cell $b_j^{(i)}$ represents “live” or 1, and a black cell represents “dead” or 0.

At first, we prepare an initial configuration $B^{(0)}$ following Theorem 3.1,



(a) (b)

Figure 3: (a) An initial configuration with four "live" cells at {10, 50, 80, 101}. (b) An initial configuration with four "live" cells at {16, 48, 80, 112}.

which is $B^{(0)}$ with thirteen "live" cells; $b_j^{(0)} = 1, j \in \{0, 10, \ldots, 120\}$, otherwise $b_j^{(0)} = 0$. Since $\sum_{i=0}^{2^{k-1}-1} b_{2i}^{(0)} = 13$ (odd), we estimate the initial configuration $B^{(0)}$ becomes B_{null} with a time period of exactly $2^{7-1} = 64$ (see Figure 1). We also prepare an initial configuration $B^{(0)}$ following the Theorem 3.1. $B^{(0)}$ with two "live" cells $b_j^{(0)} = 1, j \in \{0, 127\}$, otherwise $b_j^{(0)} = 0$. Since $\sum_{i=0}^{2^{k-1}-1} b_{2i}^{(0)} = 1$ and $\sum_{i=0}^{2^{k-1}-1} b_{2i+1}^{(0)} = 1$, we also estimate the initial configuration becomes the null configuration with a time period of exactly 64 (see Figure 2(a)).

In Figure 2, we prepare two initial configurations with two "live" cells, which have different time periods until the null configuration. We prepare an initial configuration $B^{(0)}$ following the Theorem 3.2. $B^{(0)}$ with two "live" cells; $b_j^{(0)} = 1, j \in \{32, 96\}$, otherwise $b_j^{(0)} = 0$. Since $b_{32}^{(0)} + b_{96}^{(0)} = 2$ and the rest of $b_i^{(0)} + b_j^{(0)} = 0$, where $|i-j| = 2^{k-1} = 64$, the initial configuration becomes the null configuration with a time period of t, $t \leq 2^{7-2} = 32$ (see Figure 2(b)).

In Figure 3, we prepare two initial configurations with four "live" cells, which have different time periods until the null configuration. We prepare an initial configuration $B^{(0)}$; $b_j^{(0)} = 1, j \in \{10, 50, 80, 101\}$, otherwise $b_j^{(0)} = 0$ (see Figure 3(a)). We prepare another initial configuration $B^{(0)}$ with four "live" cells; $b_j^{(0)} = 1, j \in \{16, 48, 80, 112\}$, otherwise $b_j^{(0)} = 0$. Its initial configuration becomes B_{null} with a time period of $2^{7-3} = 16$ (see Figure 3(b)). Although we have the two similar initial configurations, we can see that the evolution of one configuration differs from that of the other due to the position of "live" cells.

5 Conclusion

In this manuscript, we study the cellular automata (rule 90) in the cycle graphs with $2^k, k > 1$ cells. First, we show that any initial configuration becomes the null configuration with a time period of exactly 2^{k-1} combinatorically. Next, we investigate the condition for a set of initial configurations with a time period of t, $t \leq 2^{k-2}$. With some modifications, we believe that we can obtain a set of initial configurations with a time period of 2^m, where $m < k$. According to simulations with some instances, we confirmed our theorems hold and the position of "live" and "dead" cells matters for the time period. For each initial configuration, we can estimate the time period until the null configuration without any simulations. We really hope this study gives some contribution for epidemic diseases in real applications.

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Appendix A Matrix Representations

Table 2: Cellular automaton rule 150

state	111	110	101	100	011	010	001	000
future state	1	0	0	1	0	1	1	0

With a given cycle graph C_n, we prepare a matrix A with an identity matrix I with size $n \times n$ such as $A = Adj(C_n) + I$. We can express rule 150 in Table 2 by using the matrix A. We obtain $B^{(t+1)}$ by matrix multiplication as follows;

$$B^{(t+1)T} = AB^{(t)T}. \tag{9}$$

By some modifications of the matrix A, we can express rule 60 and rule 102 by matrix multiplication of A.

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