RICCI CURVATURES AND SCALAR CURVATURES OF HOMOGENEOUS MINIMAL REAL HYPERSURFACES IN NONFLAT COMPLEX SPACE FORMS

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Abstract. We compute Ricci curvatures and scalar curvatures of minimal homogeneous real hypersurfaces in nonflat complex space forms $\widetilde{M}_n(c)$.

1. Introduction

Standard examples play an important role in geometry. We denote by $\widetilde{M}_n(c)$ a complex $n\,(\geqq 2)$-dimensional complete and simply connected nonflat complex space form of constant holomorphic sectional curvature $c(\neq 0)$, namely a complex projective space $\mathbb{C}P^n(c)(c > 0)$ or a complex hyperbolic space $\mathbb{C}H^n(c)(c < 0)$. In the theory of real hypersurfaces in $\widetilde{M}_n(c)$ it is interesting to investigate geometric properties of *homogeneous* examples. Here, a real hypersurface M^{2n-1} in $\widetilde{M}_n(c)$ is said to be homogeneous if M is an orbit of some subgroup of the isometry group $\mathrm{I}(\widetilde{M}_n(c))$ of the ambient space. For example, we recall the following fact. In $\mathbb{C}H^n(c)$ there exist homogeneous real hypersurfaces with positive sectional curvature and also ones with negative sectional curvature. On the other hand, $\mathbb{C}P^n(c)$ admits homogeneous real hypersurfaces with positive sectional curvature, but does not admit those with nonpositive curvatures (*cf.* [9]).

Thus it is natural to study Ricci curvatures and scalar curvatures of homogeneous real hypersurfaces in $\widetilde{M}_n(c)$. In this paper, we pay particular attention to the case that M^{2n-1} is minimal in such ambient spaces. Our aim here is to compute Ricci curvatures and scalar curvatures of *minimal* homogeneous real hypersurfaces in nonflat complex space forms $\widetilde{M}_n(c)$.

2. homogeneous minimal real hypersurfaces in $\widetilde{M}_n(c)$

First of all we recall some fundamental notions on real hypersurfaces in a complete and simply connected nonflat complex space form. Let M be a real hypersurface of $\widetilde{M}_n(c)$ through an isometric immersion with a unit normal local vector field $\mathcal{N}$. Denote by g the standard Riemannian metric and by J the canonical Kähler structure of $\widetilde{M}_n(c)$. Then the hypersurface M can be equipped with an *almost contact metric structure* (ϕ, ξ, η, g) which consists of a tensor field

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ϕ of type $(1,1)$, a vector field ξ, a 1-form η and the induced Riemannian metric g. That is, we define ϕ, ξ and η on M by

$$\xi = -J\mathcal{N}, \quad \eta(X) = g(X,\xi) = g(JX,\mathcal{N}) \quad \text{and} \quad \phi X = JX - \eta(X)\mathcal{N} \tag{2.1}$$

for each tangent vector $X \in TM$. The structure satisfies

$$\phi^2 X = -X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(\xi) = 1, \quad \phi\xi = 0 \quad \text{and} \quad \eta(\phi X) = 0 \tag{2.2}$$

for all vectors $X, Y \in TM$. We call the vector field ξ the *characteristic vector field* on M.

The Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M_n}(c)$ and ∇ of M are related by $\widetilde{\nabla}_X Y = \nabla_X Y + g(AX,Y)\mathcal{N}$ and $\widetilde{\nabla}_X \mathcal{N} = -AX$ for vector fields X and Y tangent to M, where A is the shape operator of M in $\widetilde{M_n}(c)$. Moreover, we have the following equations.

$$\nabla_X \xi = \phi AX, \tag{2.3}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX,Y)\xi, \tag{2.4}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = (c/4)\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}. \tag{2.5}$$

The last one is known as the equation of Codazzi.

Eigenvalues and eigenvectors of the shape operator A of M are called *principal curvatures* and *principal curvature vectors* of M in $\widetilde{M_n}(c)$, respectively. We set $V_\lambda = \{X \in TM \mid AX = \lambda X\}$, which is called the *principal distribution* associated to the principal curvature λ. We call M a *Hopf hypersurface* if the characteristic vector ξ is a principal curvature vector at each point of M.

Next, we review the classification of homogeneous real hypersurfaces in $\widetilde{M_n}(c)$. Takagi ([12, 13]) classified homogeneous real hypersurfaces in $\mathbb{C}P^n(c)(c > 0)$ in an algebraic style. By virtue of the works of Cecil and Ryan ([4]) and Kimura ([6]), we can state geometrically that a homogeneous real hypersurface in $\mathbb{C}P^n(c)$ with $n \geqq 2$ is locally congruent to one of the following Hopf hypersurfaces all of whose principal curvatures are constant:

- (A_1) A geodesic sphere $G(r)$ of radius r, where $0 < r < \pi/\sqrt{c}$;
- (A_2) A tube of radius r around a totally geodesic $\mathbb{C}P^\ell(c)$ with $1 \leqq \ell \leqq n-2$, where $0 < r < \pi/\sqrt{c}$;
- (B) A tube of radius r around a complex hyperquadric $\mathbb{C}Q^{n-1}$, where $0 < r < \pi/(2\sqrt{c}\,)$;
- (C) A tube of radius r around the Segre embedding of $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$, where $0 < r < \pi/(2\sqrt{c}\,)$ and n ($\geqq 5$) is odd;
- (D) A tube of radius r around the Plüker embedding of a complex Grassmannian $\mathbb{C}G_{2,5}$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 9$;
- (E) A tube of radius r around a Hermitian symmetric space SO(10)/U(5), where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 15$.

Unifying types (A_1) and (A_2), we call them of type (A).

In Tables 1 and 2, we denote by δ the principal curvature associated with the characteristic vector ξ, that is, $A\xi = \delta\xi$. We also put $\tilde{r} := (\sqrt{|c|}\, r)/2$.

Hereinafter, we use these notations for simplicity. The principal curvatures of homogeneous real hypersurfaces in $\mathbb{C}P^n(c)$ are given as follows (*cf.* [13]):

TABLE 1. The principal curvatures of homogeneous real hypersurfaces in $\mathbb{C}P^n(c)$

Type	Principal curvatures	Multiplicities
(A_1)	$\delta = \sqrt{c}\cot 2\tilde{r}$	1
	$\lambda_1 = (\sqrt{c}/2)\cot\tilde{r}$	$2n-2$
(A_2)	$\delta = \sqrt{c}\cot 2\tilde{r}$	1
	$\lambda_1 = (\sqrt{c}/2)\cot\tilde{r}$	$2n-2\ell-2$
	$\lambda_2 = -(\sqrt{c}/2)\tan\tilde{r}$	2ℓ
(B)	$\delta = \sqrt{c}\cot 2\tilde{r}$	1
	$\lambda_1 = (\sqrt{c}/2)\cot\{\tilde{r}-(\pi/4)\}$	$n-1$
	$\lambda_2 = (\sqrt{c}/2)\cot\{\tilde{r}+(\pi/4)\}$	$n-1$
(C)	$\delta = \sqrt{c}\cot 2\tilde{r}$	1
	$\lambda_1 = (\sqrt{c}/2)\cot\{\tilde{r}-(\pi/4)\}$	2
	$\lambda_2 = (\sqrt{c}/2)\cot\{\tilde{r}+(\pi/4)\}$	2
	$\lambda_3 = (\sqrt{c}/2)\cot\tilde{r}$	$n-3$
	$\lambda_4 = -(\sqrt{c}/2)\tan\tilde{r}$	$n-3$
(D)	$\delta = \sqrt{c}\cot 2\tilde{r}$	1
	$\lambda_1 = (\sqrt{c}/2)\cot\{\tilde{r}-(\pi/4)\}$	4
	$\lambda_2 = (\sqrt{c}/2)\cot\{\tilde{r}+(\pi/4)\}$	4
	$\lambda_3 = (\sqrt{c}/2)\cot\tilde{r}$	4
	$\lambda_4 = -(\sqrt{c}/2)\tan\tilde{r}$	4
(E)	$\delta = \sqrt{c}\cot 2\tilde{r}$	1
	$\lambda_1 = (\sqrt{c}/2)\cot\{\tilde{r}-(\pi/4)\}$	6
	$\lambda_2 = (\sqrt{c}/2)\cot\{\tilde{r}+(\pi/4)\}$	6
	$\lambda_3 = (\sqrt{c}/2)\cot\tilde{r}$	8
	$\lambda_4 = -(\sqrt{c}/2)\tan\tilde{r}$	8

Note that by putting $\ell = 0$ in the case of homogeneous real hypersurfaces of type (A_2) we can obtain the case of type (A_1).

We describe the case of $\mathbb{C}H^n(c)$ ($n \geqq 2$). Let M be a homogeneous real hypersurface in such an ambient space. Then, thanks to [3], we know that M is locally congruent to one of the following:

(A_0) A horosphere in $\mathbb{C}H^n(c)$;
($A_{1,0}$) A geodesic sphere $G(r)$ of radius r, where $0 < r < \infty$;
($A_{1,1}$) A tube of radius r around a totally geodesic $\mathbb{C}H^{n-1}(c)$, where $0 < r < \infty$;
(A_2) A tube of radius r around a totally geodesic $\mathbb{C}H^{\ell}(c)$ with $1 \leqq \ell \leqq n-2$, where $0 < r < \infty$;
(B) A tube of radius r around a totally real totally geodesic $\mathbb{R}H^n(c/4)$, where $0 < r < \infty$;
(S) The homogeneous ruled real hypersurface *HR* determined by a horocycle in a totally geodesic $\mathbb{R}H^2(c/4)$ in $\mathbb{C}H^n(c)$, or an equidistant hypersurface from *HR* at distance r, where $0 < r < \infty$;
(W_1) A tube of radius r around the minimal ruled submanifold W^{2n-k} with $k \in \{2, \ldots, n-1\}$, where $0 < r < \infty$;

(W_2) A tube of radius r around the minimal ruled submanifold W_{φ}^{2n-k} for some $\varphi \in (0, \pi/2)$ and $k \in \{2, \dots, n-1\}$, where k is even and where $0 < r < \infty$.

Unifying real hypersurfaces of types (A_0), ($A_{1,0}$), ($A_{1,1}$) and (A_2), we call them real hypersurfaces of type (A). In the above list, all examples of types (A) and (B) are Hopf hypersurfaces and others are non-Hopf. The principal curvatures of homogeneous hypersurfaces are given in Table 2 ([1, 2]).

TABLE 2. The principal curvatures of homogeneous real hypersurfaces in $\mathbb{C}H^n(c)$

Type	Principal curvatures	Multiplicities
(A_0)	$\delta = \sqrt{\lvert c\rvert}$	1
	$\lambda_1 = \sqrt{\lvert c\rvert}/2$	$2n-2$
($A_{1,0}$)	$\delta = \sqrt{\lvert c\rvert}\coth 2\tilde{r}$	1
	$\lambda_1 = (\sqrt{\lvert c\rvert}/2)\coth\tilde{r}$	$2n-2$
($A_{1,1}$)	$\delta = \sqrt{\lvert c\rvert}\coth 2\tilde{r}$	1
	$\lambda_1 = (\sqrt{\lvert c\rvert}/2)\tanh\tilde{r}$	$2n-2$
(A_2)	$\delta = \sqrt{\lvert c\rvert}\coth 2\tilde{r}$	1
	$\lambda_1 = (\sqrt{\lvert c\rvert}/2)\coth\tilde{r}$	$2n-2\ell-2$
	$\lambda_2 = (\sqrt{\lvert c\rvert}/2)\tanh\tilde{r}$	2ℓ
(B)	$\delta = \sqrt{\lvert c\rvert}\tanh 2\tilde{r}$	1
	$\lambda_1 = (\sqrt{\lvert c\rvert}/2)\coth\tilde{r}$	$n-1$
	$\lambda_2 = (\sqrt{\lvert c\rvert}/2)\tanh\tilde{r}$	$n-1$
(S)	$\lambda_1 = (3\sqrt{\lvert c\rvert}/4)\tanh\tilde{r} + (\sqrt{\lvert c\rvert}/2)\sqrt{1-(3/4)\tanh^2\tilde{r}}$	1
	$\lambda_2 = (3\sqrt{\lvert c\rvert}/4)\tanh\tilde{r} - (\sqrt{\lvert c\rvert}/2)\sqrt{1-(3/4)\tanh^2\tilde{r}}$	1
	$\lambda_3 = (\sqrt{\lvert c\rvert}/2)\tanh\tilde{r}$	$2n-3$
(W_1)	$\lambda_1 = (3\sqrt{\lvert c\rvert}/4)\tanh\tilde{r} - (\sqrt{\lvert c\rvert}/2)\sqrt{1-(3/4)\tanh^2\tilde{r}}$	1
	$\lambda_2 = (3\sqrt{\lvert c\rvert}/4)\tanh\tilde{r} + (\sqrt{\lvert c\rvert}/2)\sqrt{1-(3/4)\tanh^2\tilde{r}}$	1
	$\lambda_3 = (\sqrt{\lvert c\rvert}/2)\tanh\tilde{r}$	$2n-k-2$
	$\lambda_4 = (\sqrt{\lvert c\rvert}/2)\coth\tilde{r}$	$k-1$
(W_2)	$\lambda_i = -(\sqrt{\lvert c\rvert}/6)\left\{\coth\tilde{r}\left(u^i_{\tilde{r},\varphi} + \frac{1}{u^i_{\tilde{r},\varphi}}\right) - \operatorname{csch}\tilde{r}\operatorname{sech}\tilde{r} - 4\tanh\tilde{r}\right\}$ for $i = 1, 2, 3$. The number $u^i_{\tilde{r},\varphi}$ is the i-th cubic root of $\left(\beta_{\tilde{r},\varphi} + \sqrt{\beta^2_{\tilde{r},\varphi} - 4}\right)/2$, where $\beta_{\tilde{r},\varphi} = 27\sin^2\varphi\tanh^2\tilde{r}\operatorname{sech}^4\tilde{r} - 2$.	1
	$\lambda_4 = (\sqrt{\lvert c\rvert}/2)\tanh\tilde{r}$	$2n-k-2$
	$\lambda_5 = (\sqrt{\lvert c\rvert}/2)\coth\tilde{r}$	$k-2$

Note that a real hypersurface of type (B) with radius $r = (1/\sqrt{|c|})\log(2+\sqrt{3})$ has two distinct principal curvatures $\lambda_1 = \delta = \sqrt{3|c|}/2$ and $\lambda_2 = \sqrt{|c|}/(2\sqrt{3})$. It has three distinct principal curvatures for other case. Moreover, the principal curvatures of the homogeneous ruled real hypersurface *HR* can be obtained as limits of those of hypersurfaces of type (S) given in Table 2 by taking $r \to 0$.

3. Ricci curvatures of minimal homogeneous real hypersurfaces

In this section, we investigate Ricci curvatures of minimal homogeneous real hypersurfaces M in nonflat complex space forms $\widetilde{M}_n(c)$ $(n \geqq 2)$. We first recall that the Ricci tensor S of an arbitrary real hypersurface M in $\widetilde{M}_n(c)$ is expressed as

$$SX = (c/4)\{(2n+1)X - 3\eta(X)\xi\} + (\text{trace } A)AX - A^2X \tag{3.1}$$

for all $X \in TM$. The Ricci curvature $Ric(X,X) = g(SX,X)$ in the direction of unit vector $X \in TM$ is given as

$$Ric(X,X) = (c/4)\{(2n+1) - 3\eta(X)^2\} + (\text{trace } A)g(AX,X) - \|AX\|^2. \tag{3.2}$$

On the other hand, by solving the equation Trace $A = 0$ one can find easily a minimal homogeneous real hypersurface in $\widetilde{M}_n(c)$. In fact, a homogeneous real hypersurface M in $\mathbb{C}P^n(c)$ $(n \geqq 2)$ is minimal if and only if it is congruent to either of type (A_1), (A_2), (B), (C), (D) or (E), and the radius r satisfies the following cases, respectively:

(A_1) $\cot\tilde{r} = 1/\sqrt{2n-1}$;
(A_2) $\cot\tilde{r} = \sqrt{(2\ell+1)/(2n-2\ell-1)}$;
(B) $\cot\tilde{r} = \sqrt{n} + \sqrt{n-1}$;
(C) $\cot\tilde{r} = (\sqrt{n} + \sqrt{2}\,)/\sqrt{n-2}$;
(D) $\cot\tilde{r} = \sqrt{5}$;
(E) $\cot\tilde{r} = (\sqrt{15} + \sqrt{6}\,)/3$.

In the case of $\mathbb{C}H^n(c)$ $(n \geqq 2)$, a homogeneous real hypersurface M is minimal if and only if it is congruent to the homogeneous ruled real hypersurface HR determined by a horocycle in a totally geodesic $\mathbb{R}H^2(c/4)$ in $\mathbb{C}H^n(c)$.

Since every homogeneous real hypersurface in $\mathbb{C}P^n(c)$ is Hopf, it is enough to check $Ric(\xi,\xi)$ and $Ric(X,X)$ for each unit principal curvature vector X orthogonal to ξ in order to compute Ricci curvatures. For the homogeneous ruled real hypersurface HR in $\mathbb{C}H^n(c)$ we note that the characteristic vector ξ is a eigenvector of the Ricci tensor S, although the real hypersurface HR is not Hopf. Then, a straightforward computation shows the following:

Theorem 1. (1) *The Ricci curvature Ric of a minimal homogeneous real hypersurface in complex projective space $\mathbb{C}P^n(c)$ $(n \geqq 2)$ satisfies the following sharp inequalities*:

(A_1) $(c/4)(2n-2)/(2n-1) \leqq Ric \leqq (c/4)(4n^2-2)/(2n-1)$;
(A_2) $(c/4)\{2n - (2n-2\ell-1)/(2\ell+1) - (2\ell+1)/(2n-2\ell-1)\} \leqq Ric$ $\leqq (c/4)\{2n+1-(2n-2\ell-1)/(2\ell+1)\}$;
(B) $(c/4)(-2n+2) \leqq Ric \leqq (c/4)\{2n+1-(\sqrt{n}-1)/(\sqrt{n}+1)\}$;
(C) $(c/4)\{n+2-\sqrt{n(n-2)}\} \leqq Ric \leqq (c/4)\{n+2+\sqrt{n(n-2)}\}$;
(D) $(c/4)(31-3\sqrt{5})/2 \leqq Ric \leqq (c/4)(31+3\sqrt{5})/2$;
(E) $(c/4)(27-\sqrt{15}) \leqq Ric \leqq (c/4)(27+\sqrt{15})$.

(2) *The Ricci curvature Ric of the minimal homogeneous real hypersurface HR in complex hyperbolic space $\mathbb{C}H^n(c)$ $(n \geqq 2)$ satisfies the following*

sharp inequalities:

$$(c/4)(2n+2) \leqq Ric \leqq (c/4)(2n-1).$$

For a minimal real hypersurface in $\mathbb{C}P^n(c)$ the following theorem is known.

Theorem A ([8]). *Let M be a minimal real hypersurface in $\mathbb{C}P^n(c)$ $(n \geqq 3)$. Suppose that the Ricci curvature Ric of M satisfies $c(n-1)/2 \leqq Ric \leqq cn/2$. Then M is locally congruent to the minimal homogeneous real hypersurface of type* (A_2) *with $2\ell = n-1$. In this case, M is a tube of radius $\pi/(2\sqrt{c})$ around a totally geodesic $\mathbb{C}P^\ell(c)$.*

Related to the above theorem, we pose the following.

Problem 1. Let M be a compact orientable minimal real hypersurface of $\mathbb{C}P^n(c)$ $(n \geqq 3)$. If every Ricci curvature of M is not less than $c(n-1)/2$, is M congruent to the tube of radius $\pi/(2\sqrt{c})$ around a totally geodesic $\mathbb{C}P^\ell(c)$ with $2\ell = n-1$?

4. Ricci curvatures of homogeneous real hypersurfaces of types (A) and (B)

In this section, we investigate Ricci curvatures $Ric(X,X)$ with $\|X\| = 1$ of homogeneous real hypersurfaces M of types (A) and (B) in a complex projective space $\mathbb{C}P^n(c)$ $(n \geqq 2)$. By a direct computation we have the following propositions.

Proposition 1. *Let M be a real hypersurface of type* (A_1) *in complex projective space $\mathbb{C}P^n(c)$ $(n \geqq 2)$. Denote by Ric the Ricci curvature of M and put $\tilde{r} = (\sqrt{c}\, r)/2$. Then the maximum and the minimum values of Ric are given as follows*:

$$\max Ric = (c/2)\{n + (n-1)\cot^2\tilde{r}\},$$
$$\min Ric = (c/2)(n-1)\cot^2\tilde{r}$$

Proposition 2. *Let M be a real hypersurface of type* (A_2) *in complex projective space $\mathbb{C}P^n(c)$ $(n \geqq 3)$. Denote by Ric the Ricci curvature of M and by X_i a unit principal curvature vector with corresponding principal curvature λ_i $(i = 1, 2)$. Then, we have*

$$Ric(\xi,\xi) = (c/2)\{(n-\ell-1)\cot^2\tilde{r} + \ell\tan^2\tilde{r}\},$$
$$Ric(X_1,X_1) = (c/2)\{n-\ell + (n-\ell-1)\cot^2\tilde{r}\},$$
$$Ric(X_2,X_2) = (c/2)(1 + \ell\sec^2\tilde{r}),$$

where $\tilde{r} = (\sqrt{c}\, r)/2$. *Moreover, the maximum and the minimum values of Ric are given as follows*:

$$\max Ric = \begin{cases} Ric(X_1, X_1) & \text{if } 0 < r \leqq (2/\sqrt{c}) \tan^{-1} \sqrt{(n-\ell-1)/\ell}\,, \\ Ric(X_2, X_2) & \text{if } (2/\sqrt{c}) \tan^{-1} \sqrt{(n-\ell-1)/\ell} < r, \end{cases}$$

$$\min Ric = \begin{cases} Ric(X_2, X_2) & \text{if } 0 < r \leqq (2/\sqrt{c}) \tan^{-1} \sqrt{(n-\ell-1)/(\ell+1)}\,, \\ R(\xi, \xi) & \text{if } (2/\sqrt{c}) \tan^{-1} \sqrt{(n-\ell-1)/(\ell+1)} < r \\ & \qquad \leqq (2/\sqrt{c}) \tan^{-1} \sqrt{(n-\ell)/\ell}\,, \\ Ric(X_1, X_1) & \text{if } (2/\sqrt{c}) \tan^{-1} \sqrt{(n-\ell)/\ell} < r. \end{cases}$$

Proposition 3. *Let M be a real hypersurface M of type* (B) *in complex projective space $\mathbb{C}P^n(c)$ $(n \geqq 2)$. Denote by Ric the Ricci curvature of M and by X_i a unit principal curvature vector with corresponding principal curvature λ_i $(i = 1, 2)$. Then, we have*

$$\begin{aligned} Ric(\xi, \xi) =& -(c/2)(n-1), \\ Ric(X_1, X_1) =& (c/4)\{2n - 2 - \tan\tilde{r} - \cot\tilde{r} \\ & + 4(n-2)/(\tan\tilde{r} + \cot\tilde{r} - 2)\}, \\ Ric(X_2, X_2) =& (c/4)\{2n - 2 + \tan\tilde{r} + \cot\tilde{r} \\ & - 4(n-2)/(\tan\tilde{r} + \cot\tilde{r} + 2)\}, \end{aligned}$$

where $\tilde{r} = (\sqrt{c}\, r)/2$. *Moreover, we set* $T = \tan\tilde{r} + \cot\tilde{r}\,(> 2)$. *Then the maximum and the minimum values of Ric are given as follows*:

$$\max Ric = \begin{cases} Ric(X_1, X_1) & \text{if } 2 < T < 2\sqrt{n-1}, \\ Ric(X_2, X_2) & \text{if } 2\sqrt{n-1} \leqq T, \end{cases}$$

$$\min Ric = \begin{cases} Ric(\xi, \xi) & \text{if } 2 < T < 2n - 1 + \sqrt{4n^2 - 8n + 1}, \\ Ric(X_1, X_1) & \text{if } 2n - 1 + \sqrt{4n^2 - 8n + 1} \leqq T. \end{cases}$$

5. The derivative of the Ricci tensor on real hypersurfaces of type (A)

In this section, we calculate the length of the derivative of the Ricci tensor of real hypersurfaces of type (A) in a complex projective space $\mathbb{C}P^n(c)$ $(n \geqq 2)$. It is known that there exist no real hypersurfaces with parallel Ricci tensor in nonflat complex space forms $\widetilde{M}_n(c)$ with $n \geqq 3$ (see [11]). We establish the following.

Proposition 4. *Let M be a real hypersurface M of type (A) in complex projective space $\mathbb{C}P^n(c)$ $(n \geqq 2)$, that is to say, M is a tube of radius r around a totally geodesic $\mathbb{C}P^\ell(c)$ with $0 \leqq \ell \leqq n-2$ and $0 < r < \pi/\sqrt{c}$. Denote by S the Ricci tensor of M in $\mathbb{C}P^n(c)$ and put $\tilde{r} = (\sqrt{c}\, r)/2$. Then we have*

$$\begin{aligned} \|\nabla S\|^2 =& (c^3/4)(n-\ell-1)\{(n-\ell)\cot\tilde{r} - \ell\tan\tilde{r}\}^2 \\ & + (c^3/4)\ell\{(n-\ell-1)\cot\tilde{r} - (\ell+1)\tan\tilde{r}\}^2. \end{aligned} \tag{5.1}$$

Set $x = \cot\tilde{r}$ and denote the right-hand side of (5.1) *by $F(x)$. Then the function $F(x)$ takes its minimum at* $x = \{\ell(n\ell + \ell + 1)/(n-\ell-1)(n^2 - n\ell - \ell)\}^{1/4}$.

Proof. Since trace A is constant, we have from (3.1) that

$$\begin{aligned}(\nabla_X S)Y =& \nabla_X(SY) - S\nabla_X Y \\ =& -(3c/4)\{g(\phi AX, Y)\xi + \eta(Y)\phi AX\} \\ &+ (\text{trace } A)(\nabla_X A)Y - (\nabla_X A)AY - A(\nabla_X A)Y.\end{aligned} \tag{5.2}$$

We recall the fact that a connected real hypersurface M in a nonflat complex space form is locally congruent to a hypersurface of type (A) if and only if the shape operator A of M satisfies

$$(\nabla_X A)Y = -(c/4)\{g(\phi X, Y)\xi + \eta(Y)\phi X\} \tag{5.3}$$

for $X, Y \in TM$ ([10, 11]). The equation (5.2), together with (5.3), yields

$$\begin{aligned}(\nabla_X S)Y = -(c/4)\{&3g(\phi AX, Y)\xi + 3\eta(Y)\phi AX + (\text{trace } A)g(\phi X, Y)\xi \\ &+ (\text{trace } A)\eta(Y)\phi X - g(\phi X, AY)\xi - \eta(AY)\phi X \\ &- g(\phi X, Y)A\xi - \eta(Y)A\phi X\}.\end{aligned} \tag{5.4}$$

We decompose the tangent bundle TM of M as the direct sum of principal distributions: $TM = V_\delta \oplus V_{\lambda_1} \oplus V_{\lambda_2}$, where $\delta = \sqrt{c}\cot 2\tilde{r}$, $\lambda_1 = (\sqrt{c}/2)\cot\tilde{r}$, $\lambda_2 = -(\sqrt{c}/2)\tan\tilde{r}$ and $\dim V_\delta = 1$, $\dim V_{\lambda_1} = 2n - 2\ell - 2$,$\dim V_{\lambda_2} = 2\ell$ $(0 \leqq \ell \leqq n-2)$ (see Table 1). Needless to say that $V_\delta = \mathbb{R}\xi$.

Now, we have the following lemma.

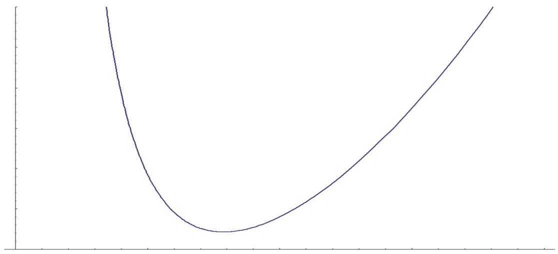
Lemma 1 ([5, 10])**.** *Let M be a Hopf hypersurface of a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$. If a nonzero vector $X \in TM$ orthogonal to ξ satisfies $AX = \lambda X$, then $(2\lambda - \delta)A\phi X = (\delta\lambda + (c/2))\phi X$ holds, where δ is the principal curvature associated with ξ.*

By virtue of Lemma 1 we find that $\phi V_{\lambda_i} = V_{\lambda_i}$ $(i = 1, 2)$ for a hypersurface of type (A). Equation (5.4), combined with this fact, yields the following:

$$\begin{aligned}&(\nabla_X S)Y = -(c/4)(\text{trace } A + \lambda_1 - \lambda_2)g(\phi X, Y)\xi \quad \text{if } X, Y \in V_{\lambda_1}, \\ &(\nabla_X S)Y = -(c/4)(\text{trace } A + \lambda_2 - \lambda_1)g(\phi X, Y)\xi \quad \text{if } X, Y \in V_{\lambda_2}, \\ &(\nabla_X S)\xi = -(c/4)(\text{trace } A + \lambda_1 - \lambda_2)\phi X \quad \text{if } X \in V_{\lambda_1}, \\ &(\nabla_X S)\xi = -(c/4)(\text{trace } A + \lambda_2 - \lambda_1)\phi X \quad \text{if } X \in V_{\lambda_2}, \\ &(\nabla_X S)Y = 0 \quad \text{if } X \in V_{\lambda_i}, Y \in V_{\lambda_j} (i \neq j), \\ &(\nabla_\xi S)Z = 0 \quad \text{for any } Z \in TM.\end{aligned} \tag{5.5}$$

Let $\{\xi, X_1, \phi X_1, \ldots, X_{n-\ell-1}, \phi X_{n-\ell-1}, Y_1, \phi Y_1, \ldots, Y_\ell, \phi Y_\ell\}$ be an orthonormal basis of TM with $V_{\lambda_1} = Span\{X_1, \phi X_1, \ldots, X_{n-\ell-1}, \phi X_{n-\ell-1}\}$ and $V_{\lambda_2} = Span\{Y_1, \phi Y_1, \ldots, Y_\ell, \phi Y_\ell\}$. We apply equations in (5.5) to this basis. Then we can get (5.1) after a computation. The last statement can also be obtained by elementary calculation. □

Remark 1. The following is the graph of the function of $F(x)$ given in Proposition 4.



Remark 2. We put $x_0 = \cot \tilde{r}_0 = \{\ell(n\ell+\ell+1)/(n-\ell-1)(n^2-n\ell-\ell)\}^{1/4}$ $(0 \leqq \ell \leqq n-2,\ 0 \leqq r_0 < \pi/\sqrt{c})$, which is the value where $F(x) = \|\nabla S\|^2$ takes its minimum. On the other hand, a homogeneous real hypersurface of type (A) is minimal if and only if $\cot \tilde{r} = \sqrt{(2\ell+1)/(2n-2\ell-1)}$. Denote this value by m_0. Then we have the following: $m_0 < x_0 \Leftrightarrow n-1 < 2\ell$; $m_0 = x_0 \Leftrightarrow n-1 = 2\ell$; $m_0 > x_0 \Leftrightarrow n-1 > 2\ell$.

6. Scalar curvatures of minimal homogeneous real hypersurfaces

We study all minimal homogeneous real hypersurfaces in nonflat complex space forms $\widetilde{M}_n(c)$ $(n \geqq 2)$ by their scalar curvatures. Let M be an arbitrary real hypersurface in $\widetilde{M}_n(c)$. Then the scalar curvature $\rho = \text{trace } Ric$ is given by

$$\rho = c(n^2-1) + (\text{trace } A)^2 - \text{trace } (A^2). \tag{6.1}$$

Theorem 2. *Let M be a minimal homogeneous real hypersurface in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$. Then the scalar curvature ρ of M satisfies the following*:

(1) $\rho = (c/2)(2n^2-n-1)$ *when M is of type* (A) *in* $\mathbb{C}P^n(c)$;
(2) $\rho = (c/2)(2n^2-3n-1)$ *when M is of either type* (B), (C), (D) *or* (E) *in* $\mathbb{C}P^n(c)$;
(3) $\rho = (c/2)(2n^2-1)$ *when M is the minimal homogeneous real hypersurface HR in* $\mathbb{C}H^n(c)$.

Remark 3. Theorem 2 shows that we cannot distinguish minimal homogeneous real hypersurfaces of types (B), (C), (D) and (E) in $\mathbb{C}P^n(c)$ by their scalar curvatures.

The following theorem is known.

Theorem B ([7])**.** *Let M be a compact orientable minimal real hypersurface in $\mathbb{C}P^n(c)$ $(n \geqq 2)$. Suppose that the scalar curvature ρ of M satisfies $\rho \geqq (c/2)(2n^2-n-1)$. Then $\rho = (c/2)(2n^2-n-1)$ and M is congruent to a homogeneous real hypersurfaces of type* (A).

The following problem is still open.

Problem 2. Let M be a minimal non-homogeneous real hypersurface in $\mathbb{C}P^n(c)$ $(n \geqq 2)$. Does there exist M (even in local) with the scalar curvature ρ satisfying the following inequalities?

$$(c/2)(2n^2-3n-1) \leqq \rho \leqq (c/2)(2n^2-n-1).$$

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