One-dimensional cactoids and universality.

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Abstract. We present some properties of one-dimensional cactoids and construct a universal element Z for the family of one-dimensional cactoids X such that a simple cyclic chain between any two cut points of X is a cactus. One-dimensional cactoids are partial case of planar totally regular curves and are investigated by Whyburn [13] under the term “boundary curves”.

1 Introduction. In this paper under the term *continuum* is meant a nonempty, compact and connected metric space. A curve is a one-dimensional continuum.

A continuum Z is *universal* for a class $\mathcal{F}$ of continua provided that $Z \in \mathcal{F}$ and each member of $\mathcal{F}$ can be homeomorphically imbedded in Z. A space is *planar* if it is homeomorphic to a subset of the plane.

A *Peano continuum* is a locally connected continuum.

We will use the results of the papers of 1920s (see [2], [10], [11]) in which under the term *continuous curve* was meant a metric space X that is a continuous image of segment $[0, 1]$. According to Hahn–Mazurkiewicz Theorem (see [13, (4.1). p. 92]) the above condition for X is equivalent to the property of X to be a Peano continuum.

The order of a space X at the point $p \in X$, written $\mathrm{ord}(p, X)$, is the least cardinal or ordinal number $\mathfrak{m}$ such that p has an arbitrary small open neighborhood in X with boundary of cardinality $\leq \mathfrak{m}$. In particular, $\mathrm{ord}(p, X) = \omega$, where ω denotes the least infinite ordinal number, if p has arbitrary small open neighborhoods in X with finite boundaries but $\mathrm{ord}(p, X) > n$ for every natural number n [6, §51, I, p. 274].

The points of $B(X) = \{x \in X : \mathrm{ord}(p, X) \geq 3\}$ are called *branch points* of X and the points of $E(X) = \{x \in X : \mathrm{ord}(p, X) = 1\}$ are called *end points* of X.

A point p of a connected space X is *a cut point* if $X \setminus \{p\}$ is not connected. The set of all cut points of a connected space X will be denoted by $c(X)$.

A *simple closed curve* is a space homeomorphic to the circle. An *arc* is a space A homeomorphic with a segment $[0, 1]$. The arc A with end points p and q is written pq. An arc $pq \subseteq X$ is called *free* in X if the set $(pq) = pq \setminus \{p, q\}$ is an open subset of X.

A continuum X is said to be *cyclicly connected* provided that every two points of X lie together on some simple closed curve of X. By a *cyclic element* of Peano continuum X will be meant a cut point of X, an end point of X, or a nondegenerate cyclicly connected Peano subcontinuum M of X such that M is not a proper subset of any other cyclicly connected Peano subcontinuum of X. Any nondegenerate cyclic element of X is called *true cyclic element* of X.

A Peano continuum each true cyclic element of which is homeomorphic to a simple closed curve is called *a one-dimensional cactoid* [13]. The property of a Peano continuum M to be a one-dimensional cactoid is equivalent with any of following properties:

(i) No two simple closed curve of M have more than one point in common.

(ii) M contains no θ-curves.

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A *graph* is a continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their end points [7]. A *cactus* is a graph in which any two simple closed curves have at most one point in common [9]. Clearly, a cactus is a cactoid that is a graph.

A simple cyclic chain of Peano continuum X between two of its cyclic elements E_1 and E_2 is a connected subset S that is a union of some family $\mathcal{F}$ of cyclic elements of X such that $E_1, E_2 \in \mathcal{F}$ and no proper connected subset of S containing E_1 and E_2 is the sum of cyclic elements (see [11]). Note that a simple cyclic chain between any two cyclic elements of Peano continuum is uniquely determined [11, Theorem 3].

The main result of the paper is a construction of a universal cactoid Z for the class of all one-dimensional cactoids X such that a simple cyclic chain between any two cut points of X is a cactus.

2 One-dimensional cactoids as a boundary curves. Let X is a Peano continuum of the plane $\mathbf{P}$. Any component of $\mathbf{P} \setminus X$ is called *complementary domain* of X. The boundary of any complementary domain of X is a subcontinuum of X and is called a *boundary curve.*

Wilder in [10, Theorem 17] proved the following result:

Theorem 2.1. *If a Peano continuum M is a boundary of complementary domain of a Peano continuum, then M is the union of disjoint families of sets S_1, S_2 and P, where:*

(1) *S_1 is a countable set of all simple closed curves contained in M no two of which have more than one point in common,*

(2) *S_2 is a countable set of arcs no two of which have in common an interior point of both, and*

(3) *$P = M \setminus (S_1 \cup S_2)$ is a totally disconnected set of limit points of $S_1 \cup S_2$.*

From Theorem 2.1 it follows that:

Corollary 2.1.1. *Each boundary curve is a one-dimensional cactoid.*

The fact that any one-dimensional cactoid is planar follows from the result of Ayres [2, Theorem in page 92]:

Theorem 2.2. *In order that a Peano continuum M be homeomorphic with a plane Peano continuum which is the boundary of one of its complementary domains it is necessary and sufficient that every true cyclic element of M be a simple closed curve.*

From Theorem 2.2 it also follows that:

Corollary 2.2.1. *A Peano continuum M is a one-dimensional cactoid if and only if M is homeomorphic with a plane Peano continuum which is the boundary of one of its complementary domains.*

A continuum K is said to be *regular* if K has a basis of open sets with finite boundaries. Any regular continuum is hereditarily locally connected [6, §51, IV, Theorem 2, p. 283]. Since a one-dimensional cactoid contains no θ-curves, it follows that (see [6, §52, IV, Theorem 3, p. 329]):

Corollary 2.2.2. *Any one-dimensional cactoid X is regular and any connected subset of X is arcwise connected.*

A metric space (X,d) is *uniformly locally arcwise connected* provided that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in X$ and $d(x,y) \in (0,\delta)$, then x and y can be joined by an arc of diameter $< \varepsilon$. Any Peano continuum is uniformly locally arcwise connected [6, §50, II, Theorem 4, p. 257], hence:

Corollary 2.2.3. *Any one-dimensional cactoid is uniformly locally arcwise connected.*

3 Properties of one-dimensional cactoid. Let $\mathbb{N} = \{0, 1, \dots, n, \dots\}$. Given a subset G of a space X the closure and the boundary of G in X will be denoted by $cl_X(G)$ (or $cl(G)$) and $bd_X(G)$ (or $bd(G)$), respectively.

Proposition 3.1. *Each branch point of a one-dimensional cactoid X is a cut point.*

Proof. Let $r \in B(X)$. From the Menger n-Beinsatz (see [6, p. 277]), it follows that there exist arcs $A_1 = rx_1$, $A_2 = rx_2$, and $A_3 = rx_3$ of X having the unique point r in common.

Suppose, on the contrary, that r is not a cut point. Then the connected subset $X \setminus \{r\}$ of X is arcwise connected. Thus there exists an arc $A = x_1x_2 \subseteq X \setminus \{r\}$. Since $A \cap A_1$ is a compact subset of $A_1 \setminus r$, the component of $A_1 \setminus A$ containing r is a subarc $B_1 = rb_1$ of A_1 such that $B_1 \cap A = b_1$. Similarly, there exists a subarc $B_2 = rb_2$ of A_2 such that $B_2 \cap A = b_2$. Let $B = b_1b_2$ is a unique determined subarc of A joining the points b_1 and b_2. Then $B \cup B_1 \cup B_2$ is a closed curve containing the points r and b_1 .

Since $x_3, b_1 \in X \setminus \{r\}$ and $X \setminus \{r\}$ is arcwise connected, there is an arc $C = b_1x_3 \subseteq X \setminus \{r\}$. It is easy to see that the set $B \cup B_1 \cup B_2 \cup C \cup A_3$ contains a θ-curve. Hence, X is not a cactoid which is a contradiction. □

Proposition 3.2. *The set of branch points of one-dimensional cactoid is countable.*

Proof. Let X be a cactoid. Since all save possibly a countable number of cut points of X are of order 2 [13, (3.2), p. 49] in X, the cut points of X of order ≥ 3 are countable. Hence, $B(X)$ is countable from Theorem 3.1. □

Definition 3.1. A subcontinuum G of one-dimensional cactoid X is called *full* provided that each simple closed curve of X either is a subset of G, or does not intersect G, or intersects G in a single point.

Theorem 3.1. *If X is a one-dimensional cactoid, then for any full subcontinuum G of X and for any $x \in X \setminus G$ there exist a point $r_x \in G$ and an arc A_x from x to r_x such that:*

(1) *$A_x \cap G = \{r_x\}$ and r_x is a unique point that belongs any arc of X from x to any point of G.*

(2) *If G_x is a component of $X \setminus G$ containing x, then $G \cap cl(G_x) = \{r_x\}$.*

(3) *The map $r : X \to G$ by $r(x) = \begin{cases} x, & \text{if } x \in G \\ r_x, & \text{if } x \in X \setminus G \end{cases}$ is continuous.*

Proof. (1) Consider any $r_0 \in G$. Since X is arcwise connected there is an arc $A_0 = xr_0 \subseteq X$. Let S_x be a component of $A_0 \setminus G$ containing x. Clearly S_x is a half-open subarc $[xr_x)$ of A_0, where $r_x \in A_0 \cap G$. Hence, $A_x = cl(S_x)$ is an arc from x to r_x and $A_x \cap G = \{r_x\}$.

Let $A_1 = xr$ be an arc of X from x to $r \in G$ and $\widetilde{S}_x$ be a component of $A_1 \setminus G$ containing x. As above for a point r_0 we can find a point $g \in G$ and an arc $A_g = gx \subseteq A_1$ such that $A_g \cap G = \{g\}$. Suppose on the contrary that $r_x \notin A_1$. Then $r_x \neq g$. Let S_{r_x} be a component of $A_x \setminus A_g$ containing r_x. Then $A_2 = cl(S_{r_x})$ is an arc from r_x to $b \in A_g \cap A_x$. Since $b, g \in A_g$, there exists an arc $A_3 = bg \subseteq A_g$. Since $r_x, g \in G$ and G is arcwise connected, there is an arc $A_4 = gr_x \subseteq G$. From the above a simple closed curve $A_2 \cup A_3 \cup A_4$ of X intersects G in arc gr_x which is a contradiction, because G is full subcontinuum of X. Hence, $r_x \in A_1$.

Suppose that $r \in G$ and r belongs to any arc from x to any point of G. Then $r \in A_x$. Since $A_x \cap G = \{r_x\}$, $r = r_x$.

(2) Clearly, $r_x \in A_x \subseteq cl(G_x)$. Suppose that there exists $p \in G \cap cl(G_x)$ with $p \neq r_x$. Since $p \notin A_x$, there exists an open and connected subset V_p of X such that $p \in V_p \subseteq X \setminus A_x$. Since $p \in Cl(G_x)$, there exists $q \in V_p \cap G_x$. Since V_p is arcwise connected from Corollary 2.2.2, there exists an arc $qp \subseteq V_p$. Since $x, q \in G_x$ and G_x is arcwise connected, there exists arc $xq \in G_x$. Then $xq \cup qp$ contains an arc A from x to $p \in G$. Hence $r_x \in A$ from condition 1. On the other hand $r_x \notin qp \cup xq$. Hence $r_x \notin A$, which is a contradiction.

(3) Let $g \in G$ and W_g be an open and connected neighborhood of $r(g) = g$ in X. To prove that r is continuous at g it suffices to show that $r(W_g) \subseteq W_g$. Indeed, for $x \in W_g \cap G$ we have $r(x) = x \in W_g$. For $x \in W_g \setminus G$ there exists an arc $A \subseteq W_g$ from x to g. Since $r(x) = r_x \in A$ from 1, $r(x) \in W_g$.

Let $x \in X \setminus G$ and G_x be a component of $X \setminus G$ containing x. Since X is locally connected, G_x is open. To prove the continuity of r in x, it suffices to show that $r(G_x) = \{r(x)\}$. Indeed, if $p \in G_x \setminus \{x\}$, then G_x is a component of $X \setminus G$ containing p. From condition 2 of the Theorem it follows that $\{r_p\} = cl(G_x) \cap G = \{r_x\}$. Thus $r(p) = r_p = r_x = r(x)$. □

Remark 3.1. The map r defined in Theorem 3.1 is a retraction. We will call r the *first point map corresponding to full subcontinuum G of X*.

Lemma 3.1. *If a simple cyclic chain between any two cut points of one-dimensional cactoid X is a cactus, then any simple cyclic chain of X that is a subset of $X \setminus E(X)$ is a cactus.*

Proof. Let $C \subseteq X \setminus E(X)$ be a simple cyclic chain between cyclic elements E_1 and E_2 of X. Then each of E_1 and E_2 is either a cut point or a simple closed curve. Suppose that E_1 and E_2 are simple closed curves. Then $E_1 \cap E_2$ consists of at most one point. If $E_1 \cap E_2 = \{p\}$, then $C = E_1 \cup E_2$ is a cactus.

Suppose that $E_1 \cap E_2 = \emptyset$. Consider the first point maps $r_1 : X \to E_1$ and $r_2 : X \to E_2$. From Theorem 3.1 there are $p \in E_2$ and $q \in E_1$ such that $r_1(E_2) = r_1(p)$ and $r_2(E_1) = r_2(q)$. Obviously, $C^* = (C \setminus (E_1 \cup E_2)) \cup \{r_1(p), r_2(p)\}$ is a simple cyclic chain between cut points $r_1(p)$ and $r_2(q)$ of X. Hence C^* and, therefore, $C = C^* \cup E_1 \cup E_2$ are cactuses.

The proof is similar in the case that exactly one of E_1 and E_2 is a cut point. □

Lemma 3.2. *Let X be a one-dimensional cactoid, Y a full subcontinuum of X and $r : X \to Y$ a first point map.*

If $x \in X \setminus Y$, S is a cyclic element of X containing x, and C is a simple cyclic chain between $r(x)$ and S, then $Y \cup C$ is full.

Proof. Let L be a simple closed curve of X that intersects $Y \cup C$. If L intersects Y, then $L \cap Y = \{y\}$ because Y is full. If in addition L intersects C, then $y = r(y) = r(C) = r(x) \in C$. We conclude that $L \cap (Y \cup C) \subseteq L \cap C$.

Suppose, on the contrary, that $L \cap (Y \cup C)$ contains two points z and w. Then $z, w \in L \cap C$. Thus there exists an arc $A = zw \subseteq C$. Since X contains no θ-curves, $A \subseteq L$ and L is a unique simple closed curve containing A. Suppose that $q \in A$ with $\mathrm{ord}(q, X) = 2$. Since C is a union of cyclic elements, it follows that q is a cyclic element. Thus $q \in c(X)$. Hence, $X \setminus \{q\}$ contains at least two component. Since q does not separate L it follows that $L \setminus \{q\}$ is containing in some component W_1 of $X \setminus \{q\}$. Let w belongs to a component $W_2 \neq W_1$ of $X \setminus \{q\}$. Then there exists an arc $B = wq \subseteq W_2 \cup \{q\}$. Then $B \cap L = \{q\}$ and we conclude that $\mathrm{ord}(q, X) = 3$, which is a contradiction.

□

Lemma 3.3. *If X is a Peano continuum, then $X \setminus E(X)$ is dense in X.*

Proof. Let $U \neq \emptyset$ be an open subset of X. Since X is locally connected, there exists an open and connected set $V \neq \emptyset$ such that $V \subseteq U$. There exists an arc $ab \subseteq V$ [7, Theorem 8.26]. Then $ord(p, X) \geq ord(p, ab) = 2$ for $p \in (ab)$. Clearly, $p \in U \cap (X \setminus E(X))$. $\square$

It is easy to prove the following Lemma.

Lemma 3.4. *If a cactus K is a simple cyclic chain between two of its cyclic elements, then $K = \bigcup_{j=1}^{n} C_j$, where $n \in \mathbb{N} \setminus \{0\}$ and each C_j is either a simple closed curve or a maximal free arc of K. Moreover, if $n \geq 2$, then*

(*i*) *$C_j \cap C_{j+1} = \{b_j\}$ for $j = 1, ..., n-1$, where $b_j \in B(K)$, and*

(*ii*) *$C_j \cap C_i = \emptyset$ for $|i - j| > 2$.*

Theorem 3.2. *Let X be a one-dimensional cactoid such that a simple cyclic chain between any two cut points of X is a cactus.*

Then there exists a sequence $\{Y_k\}_{k=1}^{\infty}$ of full cactuses of X such that

(*i*) *$Y_1 = \{p_1\}$ or Y_1 is a simple closed curve;*

(*ii*) *$E(Y_k) \subseteq c(X)$ (including the case $E(Y_k) = \emptyset$);*

(*iii*) *$Y_k \subseteq Y_{k+1}$;*

(*iv*) *$cl(Y_{k+1} \setminus Y_k) \cap Y_k = \{p_k\}$ and $p_k \in c(X)$;*

(*v*) *$\lim Y_k = X$;*

(*vi*) *if $r_k : X \to Y_k$ is the first point map for $k = 1, 2, ...$, then the sequence of retractions $\{r_k\}_{k=1}^{\infty}$ converges uniformly to id_X.*

Proof. Since X is separable, from Lemma 3.3 it follows that there exists a dense subset $\{x_i\}_{i=1}^{\infty}$ of X such that $\{x_i\}_{i=1}^{\infty} \subseteq X \setminus E(X)$.

Let Y_1 be a maximal cyclic element of X containing x_1. From definition of cyclic element it follows that either Y_1 is a simple closed curve or $Y_1 = \{x_1\}$ and $x_1 \in c(X)$.

Consider the first point map $r_1 : X \to Y_1$. Put $m_1 = \min\{i : x_i \notin Y_1\}$ and $r_1(x_{m_1}) = \{p_1\}$. Then either $p_1 = x_1$ or Y_1 is a simple closed curve and $p_1 \in Y_1 \cap B(X)$. In any case $p_1 \in c(X)$.

Let S_1 be the maximal cyclic element of X containing x_{m_1}. Either S_1 is a simple closed curve or $S_1 = \{x_{m_1}\}$ and $x_{m_1} \in c(X)$. Let C_1 be a cyclic chain between cyclic elements p_1 and S_1. From Lemma 3.1 C_1 is a cactus. Let $Y_2 = Y_1 \cup C_1$. By Lemma 3.2, Y_2 is a full subcontinuum of X. Since Y_1 is full, $x_{m_1} \in Y_2 \setminus Y_1$ and $Y_2 \setminus Y_1$ is a connected subset (see [11, Theorem 6]) of $X \setminus Y_1$, from Theorem 3.1(4) $Y_1 \cap cl(Y_2 \setminus Y_1) = \{p_1\}$. Obviously, $E(Y_2) \subseteq \{x_1, x_{m_1}\} \subseteq c(X)$.

Suppose that cactuses $Y_1, ..., Y_k$ with properties $(i)-(iv)$ have been defined.

Consider the first point map $r_k : X \to Y_k$. Let $m_k = \min\{i : x_i \notin Y_k\}$ and $r_k(x_{m_k}) = p_k \in Y_k$. If $p_k \in E(Y_k)$, then $p_k \in c(X)$ by inductive assumption. Otherwise, p_k is a branch point and, therefore, $p_k \in c(X)$ from Theorem 3.1. Let S_k be a maximal cyclic element of X containing x_{m_k} and C_k be a cyclic chain between cyclic elements p_k and S_k. Similarly as for Y_2 it can be shown that Y_{k+1} is full and satisfies the properties $(i)-(iv)$ of the Theorem.

To prove (v), set $A_k = \{x_1, ..., x_k\}$. Since $A_k \subseteq A_{k+1}$ and $cl(\{x_i\}_{i=1}^{\infty}) = X$, it follows that $\lim A_k = X$. Since $x_k \leq x_{m_k}$ and $A_{m_k} \subseteq Y_{k+1}$, it follows that $A_k \subseteq Y_{k+1} \subseteq X$. Thus $\lim Y_k = \lim A_k = X$.

In order to prove (vi) we consider the Hausdorff metric H_d generated on the set of closed subsets of X by metric d of X. Then

$$H_d(X, Y_k) = \inf\{\varepsilon^* > 0 : X \subseteq \bigcup_{y \in Y_k} B_d(y, \varepsilon^*)\},$$

where $B_d(y, \varepsilon^*) = \{x \in X : d(y, p) < \varepsilon^*\}$. Let $\varepsilon > 0$. Since X is uniformly locally arcwise connected from Corollary 2.2.3, there exists $\delta > 0$ such that if $x, y \in X$, and $0 < d(x, y) < \delta$, then there exists an arc $A = xy$ with diameter $< \varepsilon$. Since $\lim Y_k = X$ from (v), there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have $H_d(X, Y_k) < \delta$. Thus

$$X \subseteq \bigcup_{y \in Y_k} B_d(y, \delta) \text{ for any } k \geq k_0.$$

Let $x \in X$ and $k \geq k_0$. Then there exists $y_k \in Y_k$ such that $x \in B_d(y_k, \delta)$. Hence, x and y_k can be joined by arc A_x^k of diameter $< \varepsilon$. Since $y_k \in Y_k$ and $r_k(x)$ belongs to any arc from x to any point of Y_k, $r_k(x) \in A_x^k$. Since $x, r_k(x) \in A_x^k$, we conclude that

$$d(id_X(x), r_k(x)) = d(x, r_k(x)) \leq \operatorname{diam}(A_x^k) \leq \varepsilon.$$

□

Theorem 3.3. *[7, 2.29] Let Y be a compact metric space, and let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of compact subsets of Y such that, for each $i = 1, 2, ...$, there are continuous and onto functions $\psi_i : Y_{i+1} \to Y_i$ and $r_i : Y \to Y_i$ such that $\psi_i \circ r_{i+1} = r_i$. If $\{r_i\}_{i=1}^{\infty}$ converges uniformly to the identity map id_Y on Y, then Y is homeomorphic to inverse limit $\varprojlim\{Y_i, \psi_i\}_{i=1}^{\infty}$.*

The following Theorem follows directly from Theorems 3.2 and 3.3

Theorem 3.4. *If X is a one-dimensional planar cactoid such that any two cut points of X can be joined by a simple cyclic chain that is a cactus and $\{Y_k\}_{k=1}^{\infty}$ is the sequence of cactuses satisfying Theorem 3.2, then X is homeomorphic to $X_\infty = \varprojlim\{Y_k, \psi_k\}$, where $\psi_k = r_k\big|_{Y_{k+1}} : Y_{k+1} \to Y_k$, $k = 1, 2, ...$.*

Theorem 3.5. *Let X be one-dimensional planar cactoid such that any two cut points can be joined by a simple cyclic chain that is a cactus.*

Then there exists an inverse sequence $\{X_i, g_i\}_{i=1}^{\infty}$ such that

- (i) *X_i is a full cactus and $g_i : X_{i+1} \to X_i$ is a monotone retraction;*
- (ii) *X_1 is a point or a simple closed curve;*
- (iii) *$X_i \subseteq X_{i+1}$ and there exists a unique point $t_i \in X_i$ such that $g_i^{-1}(t_i)$ is non degenerate and is either a simple closed curve or a free arc whose end points are in $c(X)$;*
- (iv) *X is homeomorphic to $\varprojlim\{X_i, g_i\}$.*

Proof. From Theorem 3.4, X is homeomorphic to $\varprojlim\{Y_k, \psi_k\}$, where $\{Y_k\}_{k=1}^{\infty}$ is the sequence of cactuses satisfying Theorem 3.2 and $\psi_k = r_k\big|_{Y_{k+1}}$.

Clearly, each $\psi_k : Y_{k+1} \to Y_k$ is a monotone retract.

From Theorem 3.2 there is a unique point $p_k \in Y_k$ for which $\psi_k^{-1}(p_k)$ is non degenerate. Also there exits $x_{m_k} \in X \setminus E(X)$ for which $\psi_k^{-1}(p_k) = cl(Y_{k+1} \setminus Y_k)$ is a cactus that is a simple cyclic chain from $p_k \in c(X)$ to the maximal cyclic element S_k of x_{m_k}. From Lemma

3.4 it follows that $\psi_k^{-1}(p_k) = \bigcup_{j=1}^{n_k} C_j^k$, where each C_j^k is either a simple closed curve or a maximal free arc of K. Moreover, if $n_k \geq 2$, then $C_j^k \cap C_{j+1}^k = \{b_j^k\}$ for $j = 1, ..., n_k - 1$ where $b_j \in B(X)$, and $C_j^k \cap C_i^k = \emptyset$ for $|i - j| > 2$.

For $k = 1$ we obtain $\psi_1^{-1}(p_1) = \bigcup_{j=1}^{n_1} C_j^1$. We define

$$X_1 = Y_1,\ X_2 = Y_1 \cup C_1^1,\ X_3 = X_2 \cup C_2^1, \ldots, X_{1+n_1} = X_{n_1} \cup C_{n_1}^1 = Y_2.$$

From Theorem 3.2 the set X_1 is a point or a simple closed curve.

Put $t_1 = p_1$ and $t_j = b_{j-1}^1$ for $j = 2, ..., n_1$. Let $g_j : X_{j+1} \to X_j$, $j = 1, \ldots, n_1$, be the first point map. Then $g_j^{-1}(t_j) = C_j^1$ for $j = 1, \ldots, n_1$.

Let $i > n_1 + 1$ be a positive integer. There exist a unique $k(i) \in \{1, 2, \ldots\}$ and a unique $m(i) \in \{1, \ldots, n_{k(i)}\}$ such that $i = 1 + n_1 + \cdots + n_{k(i)-1} + m(i)$. We define $X_i = Y_k \cup \left(\bigcup_{j=1}^{m(i)} C_j^{k(i)}\right)$. If $m(i) = 1$, then we define $t_i = p_{k(i)}$. Otherwise we define $t_i = b_{m(i)-1}^{k(i)}$. Let $g_{i-1} : X_i \to X_{i-1}$ be the first point map. Then $g_{i-1}^{-1}(t_i) = C_{m(i)}^{k(i)}$. Clearly, the condition $(i) - (iii)$ are satisfied.

To prove (iv) we observe that the inverse sequence $\{Y_k, \psi_k\}$ is confinal in the sequence $\{X_i, g_i\}$. Hence the inverse limits $\varprojlim\{X_i, g_i\}$ and $\varprojlim\{Y_k, \psi_k\}$ are homeomorphic [5, Corollary 2.5.11, page 102] . Since X is homeomorphic to $\varprojlim\{Y_k, \psi_k\}$, it follows that X is homeomorphic to $\varprojlim\{X_i, g_i\}$. $\square$

4 Construction of universal space Z.

Let $\mathbf{P}$ denote the plane with a system Oxy of orthogonal coordinates and a metric $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + y_1 - y_2)^2}$.

For any finite subset $\mathcal{V}$ of $\mathbf{P}$ we set

$$\mathrm{mesh}(\mathcal{V}) = \min\{d(x, y) : x, y \in \mathcal{V}, x \neq y\}.$$

For any finite family of subsets $\mathcal{G}$ of $\mathbf{P}$ we set

$$\mathrm{mesh}(\mathcal{G}) = \max\{\mathrm{diam}(G) : G \in \mathcal{G}\}.$$

Given a segment $E = \overline{pq}$ of $\mathbf{P}$ we denote by m_E the midpoint of E and define $\mathcal{E}(E) = \{\overline{pm}_E, \overline{qm}_E\}$.

Triangle of $\mathbf{P}$ with vertexes v_1, v_2, v_3 is the set $\overline{v_1v_2} \cup \overline{v_2v_3} \cup \overline{v_1v_3}$. For any triangle T of the plane we denote by $\mathcal{V}(T)$ the set of vertexes of T, by $\mathcal{E}(T)$ the set of sides of T, and by $\widehat{T}$ the 2-simplex of $\mathbf{P}$ with boundary T.

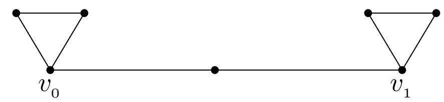
We will construct a sequence of cactuses $\{Z_i\}_{i=0}^{\infty}$ in $\mathbf{P}$ and monotone and surjective mappings $f_i : Z_{i+1} \to Z_i$ such that $Z_i \subseteq Z_{i+1}$ for each i. Our method is similar to construction of Ważewski's Universal Dendrite [7].

Consider the points $v_0 = (0, 0)$ and $v_1 = (1, 0)$ of $\mathbf{R}^2$. Set $Z_0 = \overline{v_0v_1}$, $\mathcal{E}_0 = \{\overline{v_0v_1}\}$, $\mathcal{V}_0 = \{v_0, v_1\}$, and $\varepsilon_0 = \frac{1}{2}$. Consider a family of disjoint triangles $\mathcal{T}_1 = \{T_v^1\}_{v \in V_0} \subseteq \mathbb{R}^2$ such that: v is a vertex of T_v^1, $T_v^1 \cap Z_0 = \{v\}$, and $T_v^1 \subseteq B(v, \frac{\varepsilon_0}{2})$. We define $Z_1 = Z_0 \cup \left(\bigcup_{v \in V_0} T_v^1\right)$ and $f_0 : Z_1 \to Z_0$ by

$$f_0(z) = \begin{cases} v, & \text{if } z \in T_v^1,\ v \in \mathcal{V}_0, \\ z, & \text{if } z \in Z_0. \end{cases}$$

Put

$$\begin{aligned}\mathcal{E}_1 &= \Big(\bigcup_{E\in\mathcal{E}_0}\mathcal{E}(E)\Big)\cup\Big(\bigcup_{v\in\mathcal{V}_0}\mathcal{E}(T^1_v)\Big)\\ \mathcal{V}_1 &= \{m_E\}_{E\in\mathcal{E}_0}\cup\Big(\bigcup_{v\in\mathcal{V}_0}\mathcal{V}(T^1_v)\Big)\\ \mu_1 &= \min\{d(v,E) : v\in\mathcal{V}_1, E\in\mathcal{E}_1, v\notin E\}\end{aligned}$$



Z_1 and the set of vertexes $\mathcal{V}_1$

We fix a positive real number $\varepsilon_1 < \frac{1}{4}\min\{\varepsilon_0, \mathrm{mesh}(\mathcal{V}_1), \mu_1\}$.

Suppose that for $1 \le i \le n$ there are defined:

(a_i) the cactus Z_i with set of vertexes $\mathcal{V}_i$ and set o edges (segments) $\mathcal{E}_i$;
(b_i) a finite family of disjoint triangles $\mathcal{T}_i = \{T^i_v\}_{v\in V_{i-1}} \subseteq \mathbb{R}^2$;
(c_i) the numbers $\varepsilon_i > 0$ and $\mu_i = \min\{d(v,E) : v\in\mathcal{V}_i, E\in\mathcal{E}_i, v\notin E\}$;
(d_i) a monotone surjective retraction $f_{i-1} : Z_i \to Z_{i-1}$;
such that
(1_i) $\mathcal{V}_{i-1} \subsetneqq \mathcal{V}_i$ and $Z_{i-1} \subsetneqq Z_i$;
(2_i) If $T^i_v \in \mathcal{T}_i$, then v is a vertex of T^i_v, $T^i_v \cap Z_{i-1} = \{v\}$, and $T^i_v \subseteq B(v, \frac{\varepsilon_{i-1}}{2})$;
(3_i) If $|f^{-1}_{i-1}(z)| > 1$, then $z \in \mathcal{V}_{i-1}$ and $f^{-1}_{i-1}(z) = T^i_z$;
(4_i) If $v \in \mathcal{V}_i \cap \mathcal{V}_j$ and $0 \le j < i$, then $\widehat{T}^i_v \cap \widehat{T}^j_v = \{v\}$.
(5_i) $\varepsilon_i < \frac{1}{4}\min\{\varepsilon_{i-1}, \mathrm{mesh}(\mathcal{V}_i), \mu_i\}$.

Since Z_n is a union of finite family of line segments and $\mathcal{V}_n$ is a finite subset of Z_n, there exists a finite family of disjoint triangles $\mathcal{T}_{n+1} = \{T^{n+1}_v\}_{v\in V_n} \subseteq \mathbb{R}^2$ such that: v is a vertex of T^{n+1}_v, $T^{n+1}_v \cap Z_n = \{v\}$, and $T^{n+1}_v \subseteq B(v, \frac{\varepsilon_n}{2})$.

We define $Z_{n+1} = Z_n \cup \big(\bigcup_{v\in V_n} T^{n+1}_v\big)$ and $f_n : Z_{n+1} \to Z_n$ by

$$f_n(z) = \begin{cases} v, & \text{if } z \in T^{n+1}_v,\ v \in \mathcal{V}_n,\\ z, & \text{if } z \in Z_n.\end{cases}$$

Put

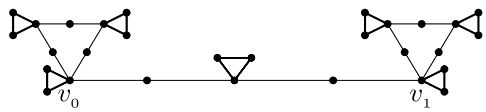
$$\begin{aligned}\mathcal{E}_{n+1} &= \Big(\bigcup_{E\in\mathcal{E}_n}\mathcal{E}(E)\Big)\cup\Big(\bigcup_{v\in\mathcal{V}_n}\mathcal{E}(T^{n+1}_v)\Big),\\ \mathcal{V}_{n+1} &= \{m_E\}_{E\in\mathcal{E}_n}\cup\Big(\bigcup_{v\in\mathcal{V}_n}\mathcal{V}(T^{n+1}_v)\Big),\\ \mu_{n+1} &= \min\{d(v,E) : v\in\mathcal{V}_{n+1}, E\in\mathcal{E}_{n+1}, v\notin E\},\end{aligned}$$

and fix a positive real number $\varepsilon_{n+1} < \frac{1}{4}\min\{\varepsilon_n, \mathrm{mesh}(\mathcal{V}_{n+1}), \mu_{n+1}\}$.

It is easy to see that the above properties $(1_i) - (5_i)$ are satisfied for $i = n+1$. Denote $f_{ji} = f_j \circ f_{j+1} \circ \cdots \circ f_{i-1} : Z_i \to Z_j$ for $j < i-1$, $f_{jj+1} = f_j$, and $f_{jj} = id_{Z_j}$. Then for $0 < i$ we have the following property:

(6_i) If $0 \le i_0 \le j \le i$, then $f_{i_0 i} = f_{i_0 j} \circ f_{ji}$.

We will prove an additional property that holds for $i > 0$:



Z_2 and the set of vertexes $\mathcal{V}_2$

(7_i) If $u \in \mathcal{V}_{i_0}$, $0 \leq i_0 < i$, then $f_{i_0 i}^{-1}(u) \subseteq B(u, \varepsilon_{i_0})$.

Let $z \in f_{i_0 i}^{-1}(u)$. Then $z \in T_v^i \in \mathcal{T}_i$, where $v \in \mathcal{V}_{i-1}$, $0 \leq i_0 < i$ and $f_{i_0 i}(v) = u$. If $v = u$, then $T_u^i \subseteq B(u, \frac{\varepsilon_{i-1}}{2})$ from (2_i). Thus $z \in B(u, \frac{\varepsilon_{i-1}}{2}) \subseteq B(u, \varepsilon_{i_0})$.

Otherwise $z \in T_v^{i+1} \in \mathcal{T}_{i+1}$, where $v \in \mathcal{V}_i$, $0 \leq i_0 < i$ and $f_{i_0 i}(v) = u$. Let $i = i_0 + n$ and $f_{ji}(v) = u_j \in Z_j$ for $j = i_0+1, \ldots, i-1$. Then $u = f_{i_0}(u_{i_0+1})$, $f_j(u_{j+1}) = u_j$ for any j, and $v = f_i(z)$.

From definition of f_n, the choice of ε_n, and (2_j) we obtain:

$$d(u,z) \leq d(u, u_{i_0+1}) + d(u_{i_0+1}, u_{i_0+2}) + \cdots + d(u_{i_0+n-1}, v) + d(v,z) <$$
$$< \frac{\varepsilon_{i_0}}{2} + \frac{\varepsilon_{i_0+1}}{2} + \cdots + \frac{\varepsilon_{i_0+n-1}}{2} + \frac{\varepsilon_i}{2} < \frac{\varepsilon_{i_0}}{2}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^n} + \ldots\right) < \varepsilon_{i_0}.$$

We set $Z = cl\,(\bigcup_{n=0}^{\infty} Z_n)$ and $Z_\infty = \varprojlim\{Z_n, f_n\}$.

Theorem 4.1. *$Z_\infty = \varprojlim\{Z_n, f_n\}$ is homeomorphic to $Z = cl\,(\bigcup_{n=0}^{\infty} Z_n)$.*

Proof. We define $h : Z_\infty \to Z$ by $h(\{z_i\}) = \lim z_i$. From [1, Theorem I] and its proof it follows that h is a homeomorphism if the following conditions are satisfied:

(a) For each $k_0 \in \mathbb{N}$ and each $\varepsilon > 0$, there exists $\delta > 0$ such that if $k_0 < k$, $p, q \in Z_k$ and $d\left(f_{k_0 k}(p), f_{k_0 k}(q)\right) > \varepsilon$, then $d(p,q) > \delta$.

(b) For each $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $\operatorname{diam}\left(\bigcup_{k > k_0} f_{k_0 k}^{-1}(z)\right) < \varepsilon$ for any $z \in Z_{k_0}$.

To prove (a) note that $\lim\limits_{i\to\infty}(\operatorname{mesh}(\mathcal{E}_i)) = 0$. Thus there exists $m > k_0$ with $\operatorname{mesh}(\mathcal{E}_m) < \frac{\varepsilon}{4}$. We have

$$\varepsilon_m < \frac{1}{4}\operatorname{mesh}(\mathcal{V}_m) \leq \frac{1}{4}\operatorname{mesh}(\mathcal{E}_m) < \frac{\varepsilon}{4}$$

For each $k \geq k_0$ the map $f_{k_0 k} : Z_k \to Z_{k_0}$ is uniformly continuous. So, for each $k \in \{k_o, k_0+1, \ldots, m\}$ there exists $\delta_k > 0$ such that if $a, b \in Z_k$ and $d(a,b) \leq \delta_k$, then $d(f_{k_0 k}(a), f_{k_0 k}(b)) \leq 4\varepsilon_m$. Set

$$\delta = \min\{\varepsilon_m, \delta_{k_0}, \delta_{k_0+1}, \ldots, \delta_m\}.$$

Let $p, q \in Z_k$ and $d\left(f_{k_0 k}(p), f_{k_0 k}(q)\right) > \varepsilon$. Then $f_{k_0 k}(p) \neq f_{k_0 k}(q)$.

If $k \in \{k_0, k_0+1, \ldots, m\}$, then $d(f_{k_0 k}(p), f_{k_0 k}(q)) > 4\varepsilon_m$. So $d(p,q) > \delta_k > \delta$.

Suppose that $k > m$. Then $Z_{k_0} \subsetneqq Z_m \subsetneqq Z_k$. We have three cases to consider.

1^{st} *case* : $p, q \in Z_m$. Then $f_{mk}(p) = p$ and $f_{mk}(q) = q$. So, $f_{k_0 m}(p) = f_{k_0 k}(p)$ and $f_{k_0 m}(q) = f_{k_0 k}(q)$. Thus $d\left(f_{k_0 m}(p), f_{k_0 m}(q)\right) > \varepsilon > 4\varepsilon_m$ and, therefore, $d(p,q) > \delta_m \geq \delta$.

2^{nd} *case* : $p, q \in Z_k \setminus Z_m$. Then $f_{mk}(p), f_{mk}(q) \in \mathcal{V}_m$. Thus $d(f_{mk}(p), f_{mk}(q)) \geq \operatorname{mesh}(\mathcal{V}_m) > 4\varepsilon_m$. From (7_m): $d(p, f_{mk}(p)) < \varepsilon_{k_0}$ and $d(q, f_{mk}(q)) < \varepsilon_{k_0}$. Since $\varepsilon_{k_0} < \varepsilon_m$, it follows that

$$d(p,q) \geq d(f_{mk}(p), f_{mk}(q)) - d(q, f_{mk}(q)) - d(p, f_{mk}(p)) > 2\varepsilon_m > \delta.$$

3^d *case* : $p \in Z_m$ and $q \in Z_k \setminus Z_m$. Then $p = f_{mk}(p) \in E_p \in \mathcal{E}_m$ and $f_{mk}(q) = v_q \in \mathcal{V}_m$. Since $p, q \notin Z_{k_0}$, it follows that $E_p \subseteq f^{-1}_{k_0 m}(f_{k_0 k}(p))$ and $v_q \in f^{-1}_{k_0 m}(f_{k_0 k}(p))$. Since $f_{k_0 k}(p) \neq f_{k_0 k}(q)$, $f^{-1}_{k_0 m}(f_{k_0 k}(p)) \cap f^{-1}_{k_0 m}(f_{k_0 k}(q)) = \emptyset$. Hence, $v_q \notin E_p$. From the choice of μ_m it follows that

$$d(v_q, p) > d(v_q, E_p) > \mu_m > 4\varepsilon_m.$$

Since $d(v_q, q) < \varepsilon_m$ from (7_m), we conclude that

$$d(p, q) \geq d(p, v_q) - d(q, v_q) > 4\varepsilon_m - \varepsilon_m > \varepsilon_m > \delta.$$

To prove (b) take any $\varepsilon > 0$. Since $\lim\limits_{i \to \infty} \varepsilon_i = 0$, there exists $k_0 \in \mathbb{N}$ such that $2\varepsilon_{k_0} < \varepsilon$. If $z \in Z_{k_0} \setminus \left(\bigcup_{i \geq k_0} \mathcal{V}_i\right)$, then $\bigcup_{k > k_0} f^{-1}_{k_0 k}(z) = \{z\}$. So (a) holds.

Let $z \in Z_{k_0} \cap \left(\bigcup_{i \geq k_0} \mathcal{V}_i\right)$ and let $i_z \geq k_0$ be the least integer such that $z \in \mathcal{V}_{i_z}$. If $k_0 < k \leq i_z$, then $f^{-1}_{k_0 k}(z) = \{z\}$. Hence, $\bigcup_{k > k_0} f^{-1}_{k_0 k}(z) = \bigcup_{k > i_z} f^{-1}_{i_z k}(z)$.

From the properties (3_k) and (7_k) with $k > i_z$ it follows that $\bigcup_{k > i_z} f^{-1}_{i_z k}(z) \subseteq B(z, \varepsilon_{i_z})$. Thus again

$$\operatorname{diam}\Big(\bigcup_{k > k_0} f^{-1}_{k_0 k}(z)\Big) = \operatorname{diam}\Big(\bigcup_{k > i_z} f^{-1}_{i_z k}(z)\Big) < 2\varepsilon_{i_z} < 2\varepsilon_{k_0} < \varepsilon.$$

□

Theorem 4.2. *Z is a one-dimensional cactoid such that any two cut points of Z can be joined by a simple cyclic chain that is a cactus.*

Proof. Since $Z_\infty = \varprojlim\{Z_n, f_n\}$, where each Z_n is locally connected and each f_n is a monotone surjection, it follows that Z_∞ is a locally connected continuum (see [7, 8.47]). Thus Z is a locally connected continuum from Theorem 4.1.

Let $a, b \in c(Z)$, $a \neq b$. If $a, b \in \bigcup_{i=0}^{\infty} Z_k$, then there exists a cactus Z_k such that $a, b \in Z_k$. Thus a and b can be joined by a simple cyclic chain that is a cactus. It suffices to show that $Z \setminus \bigcup_{i=0}^{\infty} Z_k$ contains no cut points of Z. Suppose, on the contrary, that there exists a cut point $z \in Z \setminus \bigcup_{i=0}^{\infty} Z_k$. Then $Z \setminus \{z\} = O_1 \cup O_2$, where O_1 and O_2 are disjoint, non empty, and open subsets of Z. Since $\bigcup_{i=0}^{\infty} Z_k$ is connected, we may suppose that $\bigcup_{i=0}^{\infty} Z_k \subseteq O_1$. Then $O_2 \cap (\bigcup_{i=0}^{\infty} Z_k) = \emptyset$. Hence, $cl\,(\bigcup_{i=0}^{\infty} Z_k) \neq Z$ which is a contradiction.

Let S be a true cyclic element of Z. Then $E(S) = \emptyset$. Hence, $ord_Z(x) \geq ord_S(x) > 1$ for each $x \in S$. Therefore, $S \cap E(Z) = \emptyset$. If $S \subseteq \bigcup_{i=0}^{\infty} Z_n$, then S is a simple closed curve from construction of Z_n. It suffices to prove that $Z \setminus \bigcup_{i=0}^{\infty} Z_n \subseteq E(Z)$.

Let $e \in Z \setminus \bigcup_{i=0}^{\infty} Z_i$ and $\varepsilon > 0$. It remains to find an open subset U_e of Z such that $e \in U_e \subseteq B(e, \varepsilon)$ and $bd_Z(U_e)$ consists of one point.

The map $h : Z_\infty \to Z$ defined by $h(\{z_i\}_{i=0}^{\infty}) = \lim z_i$ is a homeomorphism from the proof of Theorem 4.1. Let $h^{-1}(e) = \{e_i\}_{i=0}^{\infty}$. Then $f_i(e_{i+1}) = e_i \in Z_i$ for any i. Since $e = \lim e_i \notin \bigcup_{i=0}^{\infty} Z_i$ and each Z_i is compact, it follows that $\{e_i\}_{i=0}^{\infty} \nsubseteq Z_i$ for any i. Therefore, without loss of generality we may suppose that $e_i \neq e_{i+1}$ for any i. Since $f_i(e_{i+1}) = e_i \neq e_{i+1}$, it follows that $e_i \in \mathcal{V}_i$.

There exist $i_0, j_0 \in \mathbb{N}$ such that $e_i \in B(e, \frac{\varepsilon}{2})$ for any $i \geq i_0$ and $\varepsilon_j < \frac{\varepsilon}{2}$ for any $j \geq j_0$. Let $k_0 = \max\{i_0, j_0\}$. Then $e_k \in B(e, \frac{\varepsilon}{2})$ and $\varepsilon_k < \frac{\varepsilon}{2}$ for any $k \geq k_0$.

Let U_e be a component of $Z \setminus \{e_{k_0}\}$ containing e. Since Z is locally connected, U_e is open. Also $bd_Z(U_e) = \{e_{k_0}\}$. It is easy to see that $U_e = \{e\} \cup \left(\bigcup_{k=k_0}^{\infty} T^{k+1}_{e_k}\right)$. Let $z \in U_e$. Then $z \in T^{k+1}_{e_k}$ for some $k \geq k_0$. Therefore $d(z, e_k) < \frac{\varepsilon_{k_0}}{2} < \frac{\varepsilon}{2}$. Thus $d(e, z) \leq d(z, e_k) + d(e, e_k) < \varepsilon$. Hence, $z \in B(e, \varepsilon)$. □

5 The proof of universality of Z

Theorem 5.1. *Z is a universal element in the family of all one-dimensional cactoids X such that any two cut points of X can be joined by a simple cyclic chain that is a cactus.*

Proof. The one-dimensional cactoid X, whose any two cut points can be joined by a simple cyclic chain that is a cactus, is homeomorphic to $X_\infty = \varprojlim\{X_k, g_k\}$, where the inverse sequence $\{X_k, g_k\}_{k=1}^{\infty}$ satisfies the conditions of Theorem 3.5. Also Z is homeomorphic to $Z_\infty = \varprojlim\{Z_k, f_k\}$ by Theorem 4.1. It suffices to find an embedding of X_∞ into Z_∞.

We set $Q(X) = \{t_k\}_{k=1}^{\infty}$ and $Q(Z) = \bigcup_{k=1}^{\infty} \mathcal{V}_k$, where the point t_k satisfies condition (iii) of Theorem 3.5 and $\mathcal{V}_k$ is a set of vertices of cactus Z_k. Note that $X_k \cap Q(X)$ is a countable subset of X_k and $Z_k \cap Q(Z)$ is countable and dense in Z_k for each k.

Observe that X_1 is either a point or a simple closed curve such that there exist a unique point $t_1 \in X_1$ with $|g_1^{-1}(t_1)| > 1$. We also observe that $Z_1 = \overline{v_0 v_1} \cup T^1_{v_0} \cup T^1_{v_1}$, where $T^1_{v_i}$ are triangles. If $X_1 = \{t_1\}$, then $h_1 : X_1 \to Z_1$ with $h_1(t_1) = v_1$ is a homeomorphism. If X_1 is a closed curve, then there exist a homeomorphism $h_1 : X_1 \to T^1_{v_1}$ such that $h_1(t_1) = v_1$ and $h_1(X_1 \cap Q(X)) \subseteq T^1_{v_1} \cap Q(Z)$. We put $n_1 = 1$.

Suppose that $k \in \mathbb{N} \setminus \{0\}$ and for each $j \in 1, ..., k$ we have define an integer n_j and an embedding $h_j : X_j \to Z_{n_j}$ such that:

(1_j) $h_j(X_j \cap Q(X)) \subseteq Z_{n_j} \cap Q(Z)$;

(2_j) the following diagram is commutative for $j > 1$:

$$\begin{tikzcd} X_{j-1} \arrow[d, "h_{j-1}"] & X_j \arrow[l, "g_{j-1}"'] \arrow[d, "h_j"] \\ Z_{n_{j-1}} & Z_{n_j} \arrow[l, "f_{n_{j-1}n_j}"] \end{tikzcd}$$

(3_j) $n_j > n_{j-1}$ for $j > 1$.

We will define an integer n_{k+1} and an embedding $h_{k+1} : X_{k+1} \to Z_{n_{k+1}}$ that satisfy the properties $(1_{k+1}) - (3_{k+1})$.

Consider the monotone retraction $g_k : X_{k+1} \to X_k$ and the embedding $h_k : X_k \to Z_{n_k}$. By Theorem 3.5 there is a unique $t_k \in X_k$ such that $g_k^{-1}(t_k)$ is non degenerate. We denote $h_k(t_k) = z_k$. From (1_k) we have $z_k \in Z_{n_k} \cap (\bigcup_{i=1}^{\infty} \mathcal{V}_i)$. Since $\mathcal{V}_i \subseteq \mathcal{V}_{i+1}$ for all i, there exists $m > 1$ such that $z_k \in \mathcal{V}_{n_k+m}$. Put $n_{k+1} = n_k + m + 1$.

Since $Z_{n_k} \subseteq Z_{n_k+m}$, h_k is also embedding of X_k into Z_{n_k+m}. Observe that $Z_{n_k+m+1} = Z_{n_k+m} \cup \left(\bigcup_{v \in V_{n_k+m}} T_v^{n_k+m}\right)$. Thus z_k is a vertex of some triangle $T_{z_k}^{n_k+m+1} \subseteq Z_{n_k+m+1}$ such that $T_{z_k}^{n_k+m+1} \cap Z_{n_k+m} = \{z_k\}$.

If $g_k^{-1}(t_k) = A$ is a free arc of X_{k+1}, then $A \cap X_k = \{t_k\}$ and t_k is an end point of A. Let E be one of the sides of triangle $T_{z_k}^{n_k+m+1}$ with $z_k \in E$. There exists a homeomorphism $h_A : A \to E$ such that $h_A(t_k) = z_k$ and $h_A(A \cap Q(X)) \subseteq E \cap Q(Z)$, because $E \cap Q(Z)$ is dense in E.

We define a homeomorphism $h_{k+1} : X_{k+1} = X_k \cup A \to Z_{n_{k+1}}$ by

$$h_{k+1}(x) = \begin{cases} h_A(x), & x \in A \\ h_k(x), & x \in X_k \end{cases}$$

If $g_k^{-1}(t_k) = S$ is a closed curve of X_{k+1}, then $S \cap X_k = \{t_k\}$. There exists a homeomorphism $h_S : S \to T_{z_k}^{n_k+m+1}$ such that $h_S(t_k) = z_k$ and $h_S(S \cap Q(X)) \subseteq T_{z_k}^{n_k+m+1} \cap Q(Z)$, because $T_{z_k}^{n_k+m+1} \cap Q(Z)$ is dense in $T_{z_k}^{n_k+m+1}$.

We define a homeomorphism $h_{k+1} : X_{k+1} = X_k \cup S \to Z_{n_{k+1}}$ by

$$h_{k+1}(x) = \begin{cases} h_S(x), & x \in S \\ h_k(x), & x \in X_k \end{cases}$$

From (2_j) and (3_j), $j > 1$, the map $h_\infty : \varprojlim\{X_k, g_k\}_{k=1}^\infty \to \varprojlim\{Z_{n_k}, f_{n_k}\}_{k=1}^\infty$ defined by $h_\infty\left((x_k)_{k=1}^\infty\right) = (f_{n_k}(x_k))_{k=1}^\infty$ is continuous and one-to-one (see [7, 2.22]). Since X is a continuum, h_∞ is embedding. Since inverse sequence $\{Z_{n_k}, f_{n_k}\}_{k=1}^\infty$ is confinal in the sequence $\{Z_k, f_k\}_{k=1}^\infty$, there exists a homeomorphism $H : \varprojlim\{Z_{n_k}, f_{n_k}\}_{k=1}^\infty \to \varprojlim\{Z_k, f_k\}_{k=1}^\infty = Z$. Hence, $H \circ h_\infty$ is an embedding of X into Z. □

6 Conclusions and problems. In this section we refer only to continua consisting of more than one point. A continuum X is called totally regular [8] if for any countable subset Q of X, each $x \in X$, and each $\varepsilon > 0$, there exists an open neighborhood U of x in X such that diam $(U) < \varepsilon$, $bd(U)$ is finite, and $bd(U) \cap Q = \emptyset$. Clearly, any graph is totally regular continuum. Totally regular continua were studied also [11] under the term "continua of finite degree". Since the property of being a totally regular continuum is cyclicly extensible and reducible [11, (4.2)], any cactoid is totally regular.

The order of totally regular continuum X is the ordinal number $ord(X) = \sup\{ord(p, X) : p \in X\}$. Note that [13, (3.2), p. 49] $ord(X) \geq 2$. If $ord(X) = 2$, then X is an arc or a simple closed curve [7, Theorem 9.5]. The cactoid Z constructed in section 4 is a totally regular planar continuum of order ω.

R. D. Buskirk proved that that there exists a universal totally regular continuum [4]. The natural problems arisen are the following:

1. Does there exists a universal one-dimensional cactoid.
2. Does there exists a universal one-dimensional cactoid in the family of one-dimensional cactoids of order $\leq n$, where $n > 2$.
3. Does there exists a universal planar totally regular continuum.
4. Does there exists a universal planar totally regular continuum in the family of totally regular continua of order $\leq n$, where $n > 2$.

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